Parametric Densities

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The present article is a continuation of my paper "A Collection of Cuboid Parametric Formulas" (hereafter abbreviated CCPF) which listed eleven systems generating body cuboids: cuboids (x, y, z) for which all the face diagonals $\sqrt{x^2+y^2}$, $\sqrt{x^2+z^2}$, $\sqrt{y^2+z^2}$ are integers, but the body diagonal $\sqrt{x^2+y^2+z^2}$ may not be an integer.

It is well known that no parametric system can generate all primitive body cuboids, but apparently there has never been a study which attempts to answer the question "How many is not all?". One in ten, one in a thousand, one in a million, one in a billion? We shall see that the answer depends on the degree of the parametric system and the number of decimal digits in the integer part of $\sqrt{x^2+y^2+z^2}$.

1. The Saunderson–Euler System

This is the only known system of degree six:

$$\begin{aligned} x &= 8t(t^4-1) \\ y &= 2t(3t^2-1)(t^2-3) \\ z &= (t^2-1)(t^4-14t^2+1) \end{aligned}$$

(the degree of a system is the highest power of t in x, y, or z). Note that we need consider only $t > 0$ since $x(-t) = -x(t)$, $y(-t) = -y(t)$, and $z(-t) = z(t)$. Obviously we always take the absolute value of x, y, and z and we discard any cuboids with $xyz = 0$. Setting $t = h/k$ where h and k are integers, we find

$$\begin{aligned} X &= 8hk(h^4-k^4) \\ Y &= 2hk(3k^2-k^2)(h^2-3k^2) \\ Z &= (h^2-k^2)(h^4-14h^2k^2+k^4) \end{aligned}$$

where $X = xk^6$, $Y = yk^6$, and $Z = zk^6$. Since X, Y, Z are symmetric with respect to h and k, and since $h = k$ implies $X = Z = 0$, we may assume that $0 < h < k$. Also we need only test h and k with $\gcd(h, k) = 1$ since any common divisor of h and k must divide X, Y, and Z. Let W be the integer part of $\sqrt{X^2+Y^2+Z^2}$ and let d be the number of decimal digits in W. With the final assumption that $h+k+d$ is odd, a simple computer search found primitive cuboids (X, Y, Z) with no repetitions — this included the duals (XY, XZ, YZ) reduced to primitives. In the following table $\#(d)$ is the number of primitive cuboids found in which the integer part of the body diagonal had exactly d decimal digits.

d	$\#(d)$	ratio	d	$\#(d)$	ratio	d	$\#(d)$	ratio
3	1	—	11	366	2.103	19	159881	2.146
4	3	3.000	12	777	2.123	20	343440	2.148
5	4	1.333	13	1661	2.138	21	738124	2.149
6	8	2.000	14	3534	2.127	22	1586996	2.150
7	19	2.375	15	7564	2.140	23	3413335	2.151
8	39	2.053	16	16208	2.143	24	7343606	2.151
9	80	2.051	17	34722	2.142	25	15803113	2.152
10	174	2.175	18	74487	2.145	26	34014707	2.152

Note that after some initial instability, the sequence of ratios appears to be converging. [Might the true limit be $10^{1/3} \approx 2.1544$ — look at every third value of $\#(d)$.] Thus we have exact values of $\#(d)$ for $d = 3$ to 26, and for $d > 26$ we have the approximation

$$\#(d) \approx 34014707 \times 2.152^{d-26}.$$

2. Systems of Degree Eight

I am aware of only nine parametric systems of degree eight for body cuboids, and all are listed in my paper CCPF published on Tim Roberts' website. A rather astonishing fact is that several of these nine systems produce exactly the same body cuboids, even though the formulas themselves appear to be quite different. We partition the nine systems as follows:

A	Bremner 1, Bremner 4, Bremner 5, Rignaux 1
B	Bremner 2, Bremner 3
C	Raines 1, Narumiya–Shiga 1
D	Rignaux 2

That is, the systems in each of these four categories produce exactly the same body cuboids, although the number of repeated cuboids varies.

The computer searches for these nine systems is slightly different from the Saunderson–Euler system. The system Rignaux 2 is typical:

$$\begin{aligned} x &= 8(t^2-4)(5t-4)(3t^2+2)(5t^3-11t^2+8t+2) \\ y &= 7t(t^2-4)(5t^2-8t+6)(5t^3-32t^2+22t+16) \\ z &= 28(2t-3)(3t^2+2)(5t^2-6)(t^2-4t+2) \end{aligned}$$

but, unlike the Saunderson–Euler system, in the Rignaux 2 system we have

$$x(-t) \neq \pm x(t)\ ,\quad y(-t) \neq \pm y(t)\ ,\quad z(-t) \neq \pm z(t)$$

so that both positive and negative $t = h/k$ must be tested. Specifically, for $m = 2, 3, 4, \cdots$ and for $0 < k < m$ set $h = m - k$ and then $h = k - m$. My computers checked all nine systems of degree eight and the following body cuboid counts were found. As usual, all body cuboids and their duals were reduced to primitives and any repeats were discarded. As before $d = 3, 4, 5, \cdots$ is the number of decimal digits in the integer part of $\sqrt{x^2 + y^2 + z^2}$ and $\#(d)$ is the number of primitives (x, y, z) for which the integer part of $\sqrt{x^2 + y^2 + z^2}$ has exactly d decimal digits.

Body Cuboid Counts $\#(d)$

d	A	ratio	B	ratio	C	ratio	D	ratio
3	5	—	2	—	1	—	3	—
4	4	0.80	5	2.50	1	1.00	5	1.67
5	7	1.75	4	0.80	4	4.00	11	2.20
6	16	2.29	12	3.00	7	1.75	20	1.82
7	25	1.56	18	1.50	10	1.43	34	1.70
8	54	2.16	27	1.50	23	2.30	59	1.74
9	77	1.43	48	1.78	33	1.43	88	1.49
10	148	1.92	91	1.90	66	2.00	172	1.95
11	241	1.63	149	1.64	108	1.64	289	1.68
12	420	1.74	261	1.75	201	1.86	494	1.71
13	737	1.75	447	1.71	355	1.77	872	1.77
14	1290	1.75	789	1.77	636	1.79	1522	1.75
15	2251	1.74	1357	1.72	1117	1.76	2659	1.75

All four ratio sequences appear to be converging to the same common value, roughly 1.75, so it is easy to estimate the counts $\#(d)$ for $d > 15$. For example, the Rignaux 2 system in category D has

$$\#(d) \approx 2659 \times 1.75^{d-15}.$$

As I mentioned above, the number of repeats was different for each of the nine systems, but the remarkable thing about the Rignaux 2 system was that *there were no repeats whatsoever*, making that system superior to all other parametric systems of degree eight.

3. The Degree Twelve System

The Narumiya–Shiga 2 system has only even powers of t so the computer only has to test $t > 0$. The counts were rather disappointing:

d	$\#(d)$	ratio	d	$\#(d)$	ratio	d	$\#(d)$	ratio
3	2	—	10	6	6.000	17	49	1.361
4	0	—	11	4	0.667	18	80	1.633
5	0	—	12	8	2.000	19	109	1.363
6	0	—	13	12	1.500	20	148	1.358
7	0	—	14	20	1.667	21	231	1.566
8	3	—	15	23	1.150	22	326	1.411
9	1	—	16	36	1.565	23	477	1.463

The sequence of ratios is quite erratic, but seems to be converging to some value probably between 1.4 and 1.5 — call it 1.45 as a compromise. Based on this single example, it would seem that the higher the degree the poorer the performance of a parametric system.

4. Total Counts

The computer search in 2015 on the Linux Cluster in Australia yielded 75,868 primitive body cuboids, and analysis of these cuboids produced the following table.

Body Diagonal Counts with d Digits

$d =$	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
$\mathcal{D}_1$	5	13	42	159	432	958	1324	1137	600	200	4	0	0	0	0
$\mathcal{D}_2$	0	0	0	0	5	112	746	1489	1549	1050	506	25	0	0	0
$\mathcal{D}_3$	0	0	0	0	0	19	271	1083	1520	1291	841	461	4	0	0
$\mathcal{D}_4$	0	0	0	0	0	8	110	686	1281	1467	1031	681	150	0	0
$\mathcal{D}_5$	0	0	0	0	0	2	57	397	1148	1319	1215	742	422	0	0
$\mathcal{D}_6$	0	0	0	0	0	0	34	266	950	1225	1186	786	596	47	0
$\mathcal{D}_7$	0	0	0	0	0	0	18	188	821	1276	1210	892	628	169	0
$\mathcal{D}_8$	0	0	0	0	0	1	13	127	622	1181	1224	921	607	334	0
$\mathcal{D}_9$	0	0	0	0	0	1	13	92	553	1107	1257	1005	606	458	0
$\mathcal{D}_{10}$	0	0	0	0	0	0	7	92	417	1039	1148	955	678	522	26
$\mathcal{D}_{11}$	0	0	0	0	0	0	7	45	381	952	1115	1066	694	568	78
$\mathcal{D}_{12}$	0	0	0	0	0	0	0	60	321	876	1134	1070	731	558	152
$\mathcal{D}_{13}$	0	0	0	0	0	0	1	30	254	774	1115	973	744	570	233
$\mathcal{D}_{14}$	0	0	0	0	0	0	2	29	205	722	1010	1109	743	543	325
$\mathcal{D}_{15}$	0	0	0	0	0	0	0	34	189	647	1071	1092	826	557	402

Recall that $\mathcal{D}_1$ is the data file generated by $a \leq 1000$, $\mathcal{D}_2$ is the data file generated by $1000 < a \leq 2000$, and so forth. (The Australian search ended at $a = 15000$.) Clearly the columns for $d = 3$ to 8 are complete, while $d = 9$ is nearly complete, so we have the following total counts:

d	$\#(d)$	ratio
3	5	
4	13	2.60
5	42	3.23
6	159	3.79
7	437	2.75
8	1101	2.52
9	> 2603	> 2.36

The counts for $d = 10, 11, 12, \cdots$ are by no means complete and are anyone's guess. In my paper "Stalking the Perfect Cuboid" I attempted to predict the values of $\#(d)$ for $d > 9$, but in retrospect those efforts may have been like reconstructing a brontosaurus skeleton from a few toe bones. We have seen that the ratio sequences for the parametric systems of degree six and eight and twelve are unstable for small d but that they tend to settle down to apparent limit values (namely 2.15 and 1.75 and 1.45 respectively) when $d \geq 15$. Unfortunately we have no such data for total counts, and we can expect none in the foreseeable future, not until computers become billions of times faster. So we can only guess what the true limit R may be. Here are three columns of plausible values of

$$\#(d) = 1101 \times R^{d-8}$$

for $R = 2.4$, $R = 2.5$, and $R = 2.6$ — note that they do not vary as much as one might expect.

Total Count Values $\#(d)$

d	$R = 2.4$	$R = 2.5$	$R = 2.6$
15	5.05×10^5	6.72×10^5	8.84×10^5
20	4.02×10^7	6.56×10^7	1.05×10^8
30	2.55×10^{11}	6.26×10^{11}	1.48×10^{12}
50	1.02×10^{19}	5.69×10^{19}	2.96×10^{20}
70	4.12×10^{26}	5.18×10^{27}	5.89×10^{28}
100	1.05×10^{38}	4.49×10^{39}	1.66×10^{41}

Whichever R you choose, there are certainly vast numbers of primitive body cuboids (x, y, z) for which the integer part of $\sqrt{x^2 + y^2 + z^2}$ has 100 digits!

Let us assume the middle–of–the–road value $R = 2.5$ and let (1) be the Saunderson–Euler system, (2) the Rignaux 2 system, and (3) the Narumiya–Shiga 1 system, so that

$$R_1 = 2.15\ , \quad R_2 = 1.75\ , \quad R_3 = 1.45$$

and $\#_1(d) = 343440 \times R_1^{d-20}$, $\#_2(d) = 2659 \times R_2^{d-15}$, $\#_3(d) = 477 \times R_3^{d-22}$. Then

d	(1) $\#(d) \div \#_1(d)$	(2) $\#(d) \div \#_2(d)$	(3) $\#(d) \div \#_3(d)$
20	1.51×10^2	1.50×10^3	2.15×10^4
30	8.63×10^2	5.32×10^4	4.98×10^6
40	3.90×10^3	1.88×10^6	1.16×10^9
50	1.76×10^4	6.67×10^7	2.68×10^{11}
60	7.97×10^4	2.36×10^9	6.23×10^{13}
70	3.60×10^5	8.36×10^{10}	1.45×10^{16}
80	1.63×10^6	2.96×10^{12}	3.36×10^{18}
90	7.35×10^6	1.05×10^{14}	7.79×10^{20}
100	3.32×10^7	3.71×10^{15}	1.81×10^{23}

As expected (1) achieves the best performance and (3) the worst.

5. Yet Another Search for a Perfect Cuboid

In my "Stalking" paper I described my search a few years ago for a perfect cuboid using eighteen factory-refurbished HP Compaq computers. These are now over fifteen years old and, amazingly, twelve still work. They have 32-bit math processors which are twice as slow as the newer 64-bit desktops but — guess what — Ubasic, my favorite number theory software for over thirty years, will not run on a 64-bit machine!

In 2015 I used three computers to search for the three systems found by Mamuka Meskhishvili, at his request. This search was described in detail in CCPF, and was terminated for the usual reasons: (1) a certain milestone had been reached, and (2) the squares of the bad diagonals became so large that they were statistically unlikely to be a perfect square. Since then, my twelve Compaq computers have been idle, which is a shame.

On the first page of CCPF I sketched the proof that no Saunderson–Euler cuboid is perfect, and it has been proved that their duals also cannot be perfect. Turning to the nine systems in CCPF of degree eight, we have seen that Rignaux 2 is the clear winner. I have a strange feeling about the Rignaux 2 system — call it a hunch, a suspicion, a premonition, whatever. The last table in Section 4 shows that for $40 \leq d \leq 70$ (below we shall see that these are typical sizes for the integer part of $\sqrt{x^2+y^2+z^2}$ in a serious Rignaux 2 search) shows that finds roughly one in 10^6 to one in 10^{11} of all possible primitive body cuboids, and admittedly these are pretty thin slices. But primitive perfect cuboids, assuming they exist at all, may not be unique — for all we know, there may be *nebulas* of them out there somewhere! My search using the three Meskhishvili systems was limited to three Compaq computers running for about two months. I am unaware of any major computer search devoted to a single parametric system. Recall that the Rignaux 2 formulas are

$$\begin{aligned}
X &= 8(h^2-4k^2)(5h-4k)(3h^2+2k^2)(5h^3-11h^2k+8hk^2+2k^3)\\
Y &= 7hk(h^2-4k^2)(5h^2-8hk+6k^2)(5h^3-32h^2k+22hk^2+16k^3)\\
Z &= 28hk(2h-3k)(3h^2+2k^2)(5h^2-6k^2)(h^2-4hk+2k^2)
\end{aligned}$$

where $h \neq 0$ and $k > 0$ are integers. According to the algebraic software DERIVE

$$\begin{aligned}
X^2+Y^2+Z^2 = \; & 390625h^{16} - 2650000h^{15}k + 6650000h^{14}k^2 - 5800000h^{13}k^3\\
& - 7645000h^{12}k^4 + 29520640h^{11}k^5 - 24411040h^{10}k^6\\
& - 70819328h^9k^7 + 222660944h^8k^8 - 253729024h^7k^9\\
& + 130097024h^6k^{10} - 40010752h^5k^{11} + 56170496h^4k^{12}\\
& - 77367296h^3k^{13} + 53580800h^2k^{14} - 20234240hk^{15} + 4326400k^{16}
\end{aligned}$$

and it is noteworthy that $390625 = 5^8$ and $4326400 = 2080^2$ are both perfect squares. DERIVE could not factor this 16^{th} degree polynomial at all, but my version of this software is, like my 32-bit computers, quite old.

For $m = 2, 3, 4, \cdots$ take $k = 1$ to $m-1$ and set $h = m-k$ and then $h = k-m$: if $\gcd(h,k) = 1$ then compute (X,Y,Z) and set $X = \text{abs}(X)$, $Y = \text{abs}(Y)$, $Z = \text{abs}(Z)$. Discard the cuboid if $XYZ = 0$; otherwise reduce (X,Y,Z) to its primitive. Compute the dual (XY, XZ, YZ) and reduce to its primitive. For each primitive cuboid (X,Y,Z) found, compute the body diagonal error

$$E = \sqrt{X^2+Y^2+Z^2} - \text{round}(\sqrt{X^2+Y^2+Z^2}) \; .$$

For $m = 2, 3, 4, \cdots, 10^4$ the expected number of different cuboids is approximately

$$2 \cdot 2 \cdot \frac{6}{\pi^2} \int_0^{m^4} m \, dm = \frac{12}{\pi^2}(10^4)^2 = 1.2158542 \times 10^8$$

(2 for the duals, 2 for $\pm h$, $6/\pi^2$ is the probability that $\gcd(h,k) = 1$ for random integers h and k). In fact, the actual number of primitive cuboids found for $m \leq 10^4$ was 121,589,932

and this took a single computer about two hours. This works out to roughly a million different primitive body cuboids per minute! For $m \leq 10^4$ the cuboid counts for abs$(E) < 10^{-5}$, 10^{-6}, 10^{-7}, and 10^{-8} were respectively 2438, 255, 37, and 5.

An identical program runs on each computer but checks different ranges of m: 2 to 10^4, 10^4+1 to $2\cdot 10^4$, $2\cdot 10^4+1$ to $3\cdot 10^4$, and so forth. Whenever each computer finds a cuboid with abs$(E) < 10^{-7}$ it displays m, h, k, X, Y, Z, E and saves these values to a hard disk file. Later these files are collected and appended into a master file for further analysis. At this writing, my computers have tested all $m \leq 10^6$ so the number of different primitive body cuboids found by Rignaux 2 so far is approximately

$$\frac{12}{\pi^2}(10^6)^2 = 1.216 \times 10^{12} \ ,$$

more than a million million primitive body cuboids.

Of course we expect the primitive body cuboids (x, y, z) to get larger as m increases. The values of $d =$ integer part of $\sqrt{x^2+y^2+z^2}$ tended to cluster into Gaussian bells with center μ and standard deviation σ. In the following table the primitive cuboids whose body diagonals were within 10^{-7} of an integer are counted. Curiously the dual cuboids were generally about 47% larger than the non-duals. For example the row labeled 400,000 means that for $300,000 < m \leq 400,000$ there were $c_1 = 8439$ non-dual primitive cuboids with mean $\mu_1 = 45.34$ digits and standard deviation $\sigma_1 = 1.96$ digits; similarly for the same m there were $c_2 = 8358$ dual primitive cuboids with mean $\mu_2 = 66.43$ digits and $\sigma_2 = 2.89$ digits.

m	c_1	μ_1	σ_1	c_2	μ_2	σ_2
100,000	1220	39.16	2.72	1249	57.12	3.97
200,000	3653	42.38	2.11	3674	62.14	3.00
300,000	6074	44.20	1.99	6033	64.71	2.88
400,000	8439	45.34	1.96	8358	66.43	2.89
500,000	10821	46.20	1.96	10903	67.74	2.91
600,000	13590	46.91	1.97	13317	68.82	2.89
700,000	15745	47.46	1.97	15594	69.65	2.86
800,000	18573	47.96	1.98	18183	70.39	2.86
900,000	20915	48.39	1.98	20653	71.03	2.87
1,000,000	23072	48.77	1.97	23086	71.62	2.87

Naturally, one would expect the computations to slow down as the cuboids gets larger. Indeed at $m = 1,000,000$ the number of new primitive cuboids found dropped to about 804,000 per minute compared with the roughly one million per minute when I began this project. The computation for $m \leq 1,000,000$ has taken about four months, with at least ten computers running on any given day. The cuboid counts for abs$(E) < 10^{-7}$ to 10^{-12} are currently 243149, 24328, 2434, 253, 30, and 3. At this rate, reaching $m = 2,000,000$ should take another twelve months, assuming all the Compaqs are still working by then.