Cuboid Generators

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1. Pythagorean Triangles

More than a thousand years before Pythagoras (570? - 495? BC) ancient Babylonians knew how to construct right triangles with integer sides. An unknown Babylonian scribe left us a cuneiform tablet, Plimpton 322, which lists fifteen right triangles in sexagesimal (base 60) notation, and they appear to have been based on the assumption that

$$(a^2 - b^2, 2ab, a^2 + b^2)$$

is a always a right triangle for all integers $a > b > 0$. Of course we know this is true since

$$(a^2 - b^2)^2 + (2ab)^2 = (a^2 + b^2)^2$$

in the notation of modern algebra. The two integers a and b are called the *generators* of the triangle. If (L, M, N) is any triangle with integer sides and $L^2 + M^2 = N^2$ it is easy to find its generators. First, it must be reduced to its *primitive* $(L/g, M/g, N/g)$ where $g = \gcd(L, M)$. Now only one of the legs L/g and M/g is odd, so if necessary switch L and M so that L/g is the odd leg. Then since $a^2 - b^2 = L/g$ and $a^2 + b^2 = N/g$ we have

$$a = \sqrt{(N + L)/2g} \quad \text{and} \quad b = \sqrt{(N - L)/2g}\ .$$

It is well known that $a + b$ is always odd and that $\gcd(a, b) = 1$.

2. Kraïtchik's Cuboides Rationnels

In 1947 Maurice Kraïtchik published Volume III of his *Théorie des Nombres*. Born in Russia in 1882 he spent much of his adult life in Belgium but emigrated to the United States during World War II; he died in Brussels in 1957. The second half of Volume III is devoted to "cuboides rationnels" and concludes with an extensive table of 241 cuboids with odd edges less than one million, a remarkable achievement for a mathematician with no access to an electronic computer.

Kraïtchik (page 76) defined a *rational cuboid* to be a triple (x, y, z) of rational numbers satisfying $x^2 + y^2 = Z^2$, $x^2 + z^2 = Y^2$, $y^2 + z^2 = X^2$ where X, Y, Z were also rational numbers. Kraïtchik noted immediately that by multiplying through by the least common multiple of the denominators, all the x, y, z, X, Y, Z could be assumed to be integers. The cuboid was *primitive* if $\gcd(x, y, z) = 1$ and he regarded two cuboids to be equivalent if they reduced to the same primitive: thus $(88, 234, 480) \equiv (132, 351, 720)$ since both reduce to the primitive $(44, 117, 240)$.

Kraïtchik defined the *derived cuboid* (a concept first introduced by Euler) of (x, y, z) to be $D_K(x, y, z) = (yz, xy, xz)$ but because

$$D_K^2(x, y, z) = D_K(yz, xy, xz) = (xyxz, yzxy, yzxz) = xyz \cdot (x, y, z) \equiv (x, y, z)$$

I prefer to call this the *dual* of (x, y, z). Note that

$$D_K(x, y, z, X, Y, Z) = (yz, xy, xz, yY, zZ, xX) \ .$$

Kraïtchik introduced the generators (a, b, c, d, e, f) by defining

$$x = 2kab = l(c^2 - d^2) \ , \ y = 2lcd = 2mef \ , \ z = k(a^2 - b^2) = m(e^2 - f^2)$$

and noted that the face diagonals are $X = m(e^2 + f^2)$, $Y = k(a^2 + b^2)$, $Z = l(c^2 + d^2)$. Multiplying x by y by z and canceling lmn we get

$$\frac{a^2 - b^2}{2ab} \cdot \frac{c^2 - d^2}{2cd} = \frac{e^2 - f^2}{2ef} \ ,$$

which Kraïtchik called the "relation fondamentale" for a rational cuboid. Note that

$$(e^2 - f^2)/2ef = z/y = \tan\theta$$

where θ is the acute angle opposite z in the face triangle (y, z, X). Similarly

$$(a^2 - b^2)/2ab = z/x \ \text{ and } \ (c^2 - d^2)/2cd = x/y$$

are tangents of acute angles in the face triangles (x, z, Y) and (x, y, Z). Hence Kraïtchik's "relation fondamentale" is in fact a trigonometric identity on the faces of any cuboid.

Example 1. Verify Kraïtchik's relation for the cuboid $(44, 117, 240)$. The trick is to get the edges in the proper order. First $x \neq 117$ and $y \neq 117$ since x and y are both even: thus $z = 117$. If $x = 240$ then $60/11 = 240/44 = x/y = (c^2 - d^2)/2cd$ and so $120cd = 11(c^2 - d^2)$; but this is impossible since $c^2 - d^2$ is odd. Therefore $(x, y, z) = (44, 240, 117)$. Now

$$(x, y, Z) = (44, 240, 244) = 4 \cdot (11, 60, 61)$$

and so $c^2 - d^2 = 11$ and $c^2 + d^2 = 61$; thus $(c, d, l) = (6, 5, 4)$. Similarly $(x, z, Y) = 1 \cdot (44, 117, 125)$ gives $(a, b, k) = (11, 2, 1)$ and $(y, z, X) = (240, 117, 267) = 3 \cdot (80, 39, 89)$ gives $(e, f, m) = (8, 5, 3)$. Consequently

$$\frac{a^2 - b^2}{2ab} \cdot \frac{c^2 - d^2}{2cd} = \frac{117}{44} \cdot \frac{11}{60} = \frac{39}{80} = \frac{e^2 - f^2}{2ef}$$

and all is well.

Kraïtchik gave formulas for recovering the three sides:

$$x = lcm(2cd, a^2 - b^2)\ ,\quad y = lcm(2ab, 2ef)\ ,\quad z = lcm(c^2 - d^2, e^2 - f^2)$$

where lcm is the least common multiple. Continuing Example 1 we have

$$x = lcm(44, 11) = 44\ ,\quad y = lcm(60, 80) = 240\ ,\quad z = lcm(117, 39) = 117$$

with the cuboid (x, y, z) already primitive. The generators of (x, y, z) can also be used to find the dual cuboid $(x', y'z') = (yz, xy, xz)$. Note that

$$\frac{x'}{y'} = \frac{yz}{xy} = \frac{z}{x}\ ,\quad \frac{z'}{x'} = \frac{xz}{yz} = \frac{x}{y}\ ,\quad \frac{z'}{y'} = \frac{xz}{xy} = \frac{z}{y}$$

and this means that the three right triangles in the faces of (x, y, z) and those in its dual (x', y', z') reduce to the same three primitive right triangles. Only z/y keeps the same face in the dual, so Kraïtchik calls (y, z, X) the "triangle résultant" while the other two, (x, y, Z) and (x, z, Y), he calls the "triangles composants". Thus Kraïtchik simply switches (a, b) and (c, d) to get $x' = lcm(2ab, c^2 - d^2)$, $y' = lcm(2cd, 2ef)$, $z' = lcm(a^2 - b^2, e^2 - f^2)$ so that $x' = lcm(60, 117) = 2340$, $y' = lcm(44, 80) = 880$, $z' = lcm(11, 39) = 429$ and again (x', y', z') is already primitive.

Example 2. Kraïtchik (page 82) took $(a, b, c, d, e, f) = (6, 5, 4875, 3916, 10413, 10000)$ so that

$$\frac{a^2 - b^2}{2ab} \cdot \frac{c^2 - d^2}{2cd} = \frac{11}{60} \cdot \frac{8791 \cdot 959}{2 \cdot 4875 \cdot 3916} = \frac{11}{2^2 \cdot 3 \cdot 5} \cdot \frac{59 \cdot 149 \cdot 7 \cdot 137}{2 \cdot 3 \cdot 5^3 \cdot 13 \cdot 2^2 \cdot 11 \cdot 89}$$
$$= \frac{137 \cdot 149 \cdot 7 \cdot 59}{2 \cdot 3^2 \cdot 13 \cdot 89 \cdot 2^4 \cdot 5^4} = \frac{20413 \cdot 413}{2 \cdot 10413 \cdot 10000} = \frac{e^2 - f^2}{2ef}$$

and this means that a, b, c, d, e, f should generate a cuboid (x, y, z). Then with the above lcm formulas he obtained

$$x = 2^2 \cdot 3 \cdot 5 \cdot 7 \cdot 59 \cdot 137 \cdot 149 = 505834140$$
$$y = 2^5 \cdot 3^2 \cdot 5^4 \cdot 11 \cdot 13 \cdot 89 = 2290860000$$
$$z = 7 \cdot 11 \cdot 59 \cdot 137 \cdot 149 = 92736259$$

and

$$x' = 2^3 \cdot 3 \cdot 5^3 \cdot 11 \cdot 13 \cdot 89 = 38181000$$
$$y' = 2^5 \cdot 3^2 \cdot 5^4 \cdot 13 \cdot 89 = 20826000$$
$$z' = 7 \cdot 59 \cdot 137 \cdot 149 = 8430569\ .$$

It is clear from the factorizations that both of these cuboids are already primitive. I suspect that Kraïtchik engineered this example as follows: he started with $f = 10000 = 2^4 \cdot 5^4$ and then fiddled with various values for e until he found $e = 10413 = 3^2 \cdot 13 \cdot 89$ with

$$e^2 - f^2 = (e + f)(e - f) = 20413 \cdot 413 = 137 \cdot 149 \cdot 7 \cdot 59\ .$$

Then he set $c+d = 59\cdot149$ and $c-d = 7\cdot137$ so that $c^2-d^2 = e^2-f^2$, $c = 3916 = 2^2\cdot11\cdot89$, $d = 4875 = 3\cdot5^3\cdot13$ and hence

$$\frac{a^2-b^2}{2ab} = \frac{e^2-f^2}{2ef} \div \frac{c^2-d^2}{2cd} = \frac{cd}{ef} = \frac{2^2\cdot11\cdot89\cdot3\cdot5^3\cdot13}{3^2\cdot13\cdot89\cdot2^4\cdot5^4} = \frac{11}{60}$$

which gave $a = 6$ and $b = 5$. Just a hunch.

3. The Raines–Roberts Computer Search

In 2015 Tim Roberts used a 20-year-old search method of mine to find 75,868 different primitive cuboids on the Linux Cluster at the University of Queensland in Australia. The search method is described in detail in the first half of my paper "Stalking the Perfect Cuboid" published on Roberts' website, so only a brief outline is given here. Let

$$q = a^2-b^2,\ r = 2ab,\ s = a^2+b^2,\ t = c^2-b^2,\ u = 2cd,\ v = c^2+d^2$$

so that $q^2+r^2 = s^2$ and $t^2+u^2 = v^2$. In each of the four cases

Case	x	y	z	x^2+y^2	y^2+z^2
I	qu	ru	rt	s^2u^2	r^2v^2
II	qu	qt	rt	q^2v^2	s^2t^2
III	qt	rt	ru	s^2t^2	r^2v^2
IV	qt	qu	ru	q^2v^2	s^2u^2

x^2+y^2 and y^2+z^2 are always perfect squares, so (x,y,z) is a cuboid whenever x^2+z^2 happens to be a perfect square. Of course (x,y,z) must be reduced to its primitive $(x/g, y/g, z/g)$ where $g = \gcd(x,y,z)$. To make sure that (q,r,s) and (t,u,v) were indeed primitive Pythagorean triangles we required that $\gcd(a,b) = \gcd(c,d) = 1$ and that $a+b$ and $c+d$ were always odd. (This avoided a huge number of equivalent cuboids, and made the program run much faster.) Finally we assumed that $a > b > 0$ and that $a \geq c > d > 0$ making the a-loop the dominant loop in the search. Since there were four nested loops, clearly run time was $O(a^4)$. The search on the Linux Cluster was terminated at $a = 15000$ and took about seven weeks. I'm not sure how many nodes on the Cluster were used on any given week — that was Roberts' department!

The very first cuboid found by the search program was the Case I primitive cuboid $(x,y,z) = (275, 240, 252)$ with $(a,b,c,d,e,f) = (8,3,5,2,18,7)$. Because x is odd we have

$$\frac{e^2-f^2}{2ef} = \frac{x}{z} = \frac{qu}{rt} = \frac{a^2-b^2}{2ab}\cdot\frac{2cd}{c^2-d^2}$$

so it would appear that Kraïtchik's relation has failed. But not really — we saw in Example 1 that that the order of the sides x, y, z is crucial — the generators have simply been switched about. The second cuboid found in the search was the Case II cuboid $(x,y,z) = (1100, 1155, 1008)$ with the same generators.[1] If z is odd in Case I then the

[1] Case I and Case II cuboids are duals of each other, as are Case III and IV cuboids. As we saw in Section 2, dual cuboids always have the same generators. I have always used the dual formula $D(x,y,z) = (xy, xz, yz)$ which differs from Kraïtchik's only in the order of the components.

generators satisfy

$$\frac{2ab}{a^2-b^2}\cdot\frac{c^2-d^2}{2cd}=\frac{e^2-f^2}{2ef}\ ,$$

another switch. Case III cuboids are simpler: $x = qt$ is always odd and so

$$\frac{a^2-b^2}{2ab}\cdot\frac{c^2-d^2}{2cd}=\frac{e^2-f^2}{2ef}\ .$$

In Cases I and II the computer search found a cuboid whenever

$$x^2+z^2=q^2u^2+r^2t^2=4a^2b^2(c^2-d^2)^2+4c^2d^2(a^2-b^2)^2$$

was a perfect square, and in Cases III and IV whenever

$$x^2+z^2=q^2t^2+r^2u^2=(a^2-b^2)^2(c^2-d^2)^2+16a^2b^2c^2d^2$$

was a perfect square. Of course it is much more efficient to compute q and r in the outer loops and t and u in the inner loops and then check whether $(qu)^2+(rt)^2$ or $(qt)^2+(ru)^2$ is a perfect square.

Over the past fifty years, a number of people have computed face cuboids in their search for a perfect cuboid. (A face cuboid has $\sqrt{x^2+y^2+z^2}$ an integer, but one of the face diagonals not an integer.) Recently it occurred to me that the Linux Cluster could easily have found tens of thousands of these face cuboids, no doubt a world record, just by checking $x^2+y^2+z^2$ — after all the q, r, t, u were right there — but after a moment's thought I realized this would not help find a perfect cuboid: the Cluster had already checked x^2+z^2 for all appropriate generators a, b, c, d with $a \le 15000$, so no face cuboid found in this range could possibly be perfect.

Example 3. Let's return to Example 2, this time from a computer perspective. With generators $(a, b, c, d, e, f) = (4875, 3916, 6, 5, 10413, 10000)$ the two dual cuboids were numbers 4617 and 4618 in File 5, our record of all cuboids with $4000 < a \le 5000$, and they were identified as Case III and Case IV respectively. The Case III cuboid was

$$(x, y, z) \equiv (qt, rt, ru) = (92736259, 419991000, 2290860000)$$

where $q = a^2 - b^2 = 8430569$, $r = 2ab = 38181000$, $t = c^2 - d^2 = 11$, and $u = 2cd = 60$. Since $g = \gcd(qt, rt, ru) = 11$ cuboid # 4167 was

$$(x, y, z) = (8430569, 38181000, 208260000)\ .$$

Similarly # 4168 was the Case IV cuboid

$$(x', y', z') \equiv (qt, qu, ru) = (92736259, 5058344140, 2290860000)$$

and this was a primitive cuboid since $\gcd(qt, qu, ru) = 1$. This pair of cuboids really stood out in the list because the generators $(c, d) = (6, 5)$ were so small: the neighboring primitive cuboids in the list all had edges with 11 to 15 digits. In the course of testing cuboid parametric formulas, I suppose I have computed many millions of duals by the usual method

$$xy = 321887554989000\ , \quad xz = 175575029990000\ , \quad yz = 795175060000000$$

with $g = \gcd(xy, xz, yz) = 3471000$ to find

$$(x', y', z') = (xy, xz, yz)/g = (92736259, 5058344140, 2290860000)$$

and my computer never once complained. Computers never use factorization to compute *gcd*'s — instead they use the Euclidean Algorithm, which is super fast even when the integers have hundreds of digits. Finding all the factors of such large numbers can be difficult to impossible.

Deep down in the innermost loop of the Raines-Roberts search program there were billions upon billions of square roots required. Indeed for $a \leq 15000$ the number of square roots inside the d-loop was[2]

$$O(a^4) = 15000^4 \approx 5 \times 10^{16}\ .$$

Since $a+b$ and $c+d$ must be odd, this is reduced by a factor of four; also we have $\gcd(a, b) = \gcd(c, d) = 1$ and the probability that two random positive integers are relatively prime is $6/\pi^2$. Thus the number of calculations is reduced by the factor

$$4 \cdot (\pi^2/6)^2 = \pi^4/9 \approx 10.82\ .$$

For each value of d there are two square roots: the computer had to check whether

$$x^2 + z^2 = (qu)^2 + (rt)^2 = 4a^2b^2(c^2 - d^2)^2 + 4c^2d^2(a^2 - b^2)^2 \qquad (1)$$

or

$$x^2 + z^2 = (qt)^2 + (ru)^2 = (a^2 - b^2)^2(c^2 - d^2)^2 + 16a^2b^2c^2d^2 \qquad (2)$$

was a perfect square. Hence the total number of square roots required for $a \leq 15000$ was about 10^{16} = ten million billion.

A curious fact was that we did not need the *value* of $\sqrt{x^2 + z^2}$, only whether or not it was an integer. Tim Roberts found a C++ function which answered this question Yes or No, and was three times faster than the ordinary square root function; this no doubt saved many hours on the Linux Cluster.

[2] Actually, since $a > b > 0$ and $a \geq c > d > 0$ a more accurate estimate is

$$\sum_{a=1}^{A} \sum_{b=1}^{a-1} \sum_{c=1}^{a} \sum_{d=1}^{c} 1 \approx \int_0^A \int_0^a \int_0^a \int_o^c dd\ dc\ db\ da = A^4/8\ .$$

This would lower the estimate for $A = 15000$ to roughly 1.25×10^{15} = 1.25 million billion square roots.

4. Kraïtchik's 1947 Search Method

On page 80 of Volume III of his *Théorie des Nombres* Kraïtchik cites formulas (1) and (2) and uses one of them to derive Euler's parametric cuboid formulas — but he never used either (1) or (2) to search for cuboids. Instead he used a method mentioned on the first page of my paper "Stalking the Perfect Cuboid". Since every primitive cuboid has one odd edge and two even edges, Kraïtchik's idea was to test each odd number in sequence. For example, the smallest odd number that produces a cuboid is 85, an edge in the primitive cuboid $(85, 132, 720)$. To find this cuboid, Kraïtchik mentions (page 104) that there are only four y-values such that $(85, y, Z)$ is a Pythagorean triangle, namely $y = 132$, 154, 720, and 3612, but he does not say how he got those four numbers. Here's one way to find them: the legs (x, y) in a Pythagorean triangle must satisfy $x = (a^2 - b^2)k$ and $y = 2abk$ so for $x = 85$ we have

k	$a^2 - b^2$	$a + b$	$a - b$	a	b	$2abk$
1	85	85	1	43	42	3612
1	85	17	5	11	6	132
5	17	17	1	9	8	720
17	5	5	1	3	2	154

and there are no other possible values for y. Since $132^2 + 720^2 = 732^2$ we have found the primitive cuboid $(85, 132, 720)$. Note that $C(4, 2) = 6$ and the other five pairs do not check: for example $3612^2 + 132^2 = 13063968$ is not a perfect square.

Kraïtchik also shows how to test $x = 1155 = 3 \cdot 5 \cdot 7 \cdot 11$. Because there are four prime factors, he finds $(3^4 - 1)/2 = 40$ candidates for y, and all $C(40, 2) = 780$ pairs must be checked! In fact two of these 780 actually do check:

$$1100^2 + 1008^2 = 1492^2 \quad \text{and} \quad 6300^2 + 6688^2 = 9811^2$$

while the other 778 pairs fail; thus there are exactly two primitive cuboids with odd edge 1155, namely $(1155, 1100, 1008)$ and $(1155, 6300, 6688)$.

A few years ago I wrote a computer program in Ubasic which performed the Kraïtchik search. When $x = 3, 5, 7, 9, \cdots$ was small the program was lightning fast, but as you might expect, it became awfully slow as x grew larger. In fact, I eventually got an error message that there were too many y candidates (more than 30000) and the memory had overflowed — the program had to remember all the y candidates because it had to test them in pairs — so I terminated the project. I suppose I could have stored the y candidates on the hard disk, but even then the computer would still have to check all $C(30000, 2) = 449985000$ pairs — all that to test a single large odd number!

Kraïtchik concluded Volume III with a ten-page table of the 241 primitive cuboids he found with odd edge less than one million. There were 1, 11, 39, 100, 214 cuboids with odd edge less than 10^2, 10^3, 10^4, 10^5, 10^6 respectively. The Raines-Roberts counts for 10^2 to 10^7 were 1, 11, 39, 120, 416, and 1165 respectively. Kraïtchik never claimed his table was complete.

5. Some Relations on a Perfect Cuboid

Let us pretend that we actually possess a perfect cuboid. That is, let (x, y, z) be a primitive perfect cuboid with

$$x^2 + y^2 = Z^2,\ x^2 + z^2 = Y^2,\ y^2 + z^2 = X^2,\ x^2 + y^2 + z^2 = W^2.$$

We may assume that x is odd so that Y, Z, W are also odd and y, z, X are even. If necessary we switch y and z so that $y/z = (e^2 - f^2)/2ef$. Thus if we set

$$x/y = (a^2 - b^2)/2ab \ \text{ and } \ x/z = (c^2 - d^2)/2cd$$

then Kraïtchik's fundamental relation takes the form

$$\frac{c^2 - d^2}{2cd} = \frac{x}{z} = \frac{x}{y} \cdot \frac{y}{z} = \frac{a^2 - b^2}{2ab} \cdot \frac{e^2 - f^2}{2ef} \ . \quad (3)$$

In addition we have

$$\frac{y}{z} = \frac{yx}{xz} = \frac{2ab}{a^2 - b^2} \cdot \frac{c^2 - d^2}{2cd} \ , \quad \frac{Z}{y} = \frac{a^2 + b^2}{2ab} \ , \quad \frac{Z}{x} = \frac{a^2 + b^2}{a^2 - b^2} \ ,$$

$$\frac{Y}{z} = \frac{c^2 + d^2}{2cd} \ , \quad \frac{Y}{x} = \frac{c^2 + d^2}{c^2 - d^2} \ , \quad \frac{X}{z} = \frac{e^2 + f^2}{2ef} \ , \quad \frac{X}{y} = \frac{e^2 + f^2}{e^2 - f^2} \ .$$

Now a perfect cuboid also has three other Pythagorean triangles: since

$$x^2 + X^2 = W^2 \ , \quad y^2 + Y^2 = W^2 \ , \quad z^2 + Z^2 = W^2$$

there are generators $\alpha, \beta, \gamma, \delta, \epsilon, \eta$ such that

$$\frac{x}{X} = \frac{\alpha^2 - \beta^2}{2\alpha\beta} \ , \quad \frac{Y}{y} = \frac{\gamma^2 - \delta^2}{2\gamma\delta} \ , \quad \frac{Z}{z} = \frac{\epsilon^2 - \eta^2}{2\epsilon\eta} \ ,$$

$$\frac{W}{X} = \frac{\alpha^2 + \beta^2}{2\alpha\beta} \ , \quad \frac{W}{y} = \frac{\gamma^2 + \delta^2}{2\gamma\delta} \ , \quad \frac{W}{z} = \frac{\epsilon^2 + \eta^2}{2\epsilon\eta} \ ,$$

$$\frac{W}{x} = \frac{\alpha^2 + \beta^2}{\alpha^2 - \beta^2} \ , \quad \frac{W}{Y} = \frac{\gamma^2 + \delta^2}{\gamma^2 - \delta^2} \ , \quad \frac{W}{Z} = \frac{\epsilon^2 + \eta^2}{\epsilon^2 - \eta^2} \ ,$$

and these give still more relations:

$$\frac{X}{Y} = \frac{X}{y} \cdot \frac{y}{Y} = \frac{e^2 + f^2}{e^2 - f^2} \cdot \frac{2\gamma\delta}{\gamma^2 - \delta^2} \ , \qquad \frac{a^2 - b^2}{2ab} = \frac{x}{y} = \frac{x}{W} \cdot \frac{W}{y} = \frac{\alpha^2 - \beta^2}{\alpha^2 + \beta^2} \cdot \frac{\gamma^2 + \delta^2}{2\gamma\delta} \ ,$$

$$\frac{X}{Z} = \frac{X}{x} \cdot \frac{x}{Z} = \frac{2\alpha\beta}{\alpha^2 - \beta^2} \cdot \frac{a^2 - b^2}{a^2 + b^2} \ , \qquad \frac{c^2 - d^2}{2cd} = \frac{x}{z} = \frac{x}{W} \cdot \frac{W}{z} = \frac{\alpha^2 - \beta^2}{\alpha^2 + \beta^2} \cdot \frac{\epsilon^2 + \eta^2}{2\epsilon\eta} \ ,$$

$$\frac{Y}{Z} = \frac{Y}{z} \cdot \frac{z}{Z} = \frac{c^2 + d^2}{2cd} \cdot \frac{2\epsilon\eta}{\epsilon^2 - \eta^2} \ , \qquad \frac{e^2 - f^2}{2ef} = \frac{y}{z} = \frac{y}{W} \cdot \frac{W}{z} = \frac{2\gamma\delta}{\gamma^2 + \delta^2} \cdot \frac{\epsilon^2 + \eta^2}{2\epsilon\eta} \ .$$

For the Greek generators I could not find an identity similar to (3) — it would appear there is no such fundamental relation. However, there are many more identities between the Latin and Greek generators. A typical example is

$$\frac{2\alpha\beta}{\alpha^2-\beta^2} = \frac{X}{x} = \frac{X}{y}\cdot\frac{y}{Y}\cdot\frac{Y}{x} = \frac{e^2+f^2}{e^2-f^2}\cdot\frac{2\gamma\delta}{\gamma^2-\delta^2}\cdot\frac{c^2+d^2}{c^2-d^2}$$

which is equivalent to

$$\frac{c^2+d^2}{c^2-d^2}\cdot\frac{e^2+f^2}{e^2-f^2} = \frac{2\alpha\beta}{\alpha^2-\beta^2}\cdot\frac{\gamma^2-\delta^2}{2\gamma\delta}$$

and I found ten more in this form. I also found eight identities of the form

$$\frac{a^2-b^2}{a^2+b^2}\cdot\frac{2cd}{c^2+d^2} = \frac{\alpha^2-\beta^2}{\alpha^2+\beta^2}\cdot\frac{\gamma^2+\delta^2}{\gamma^2-\delta^2}\cdot\frac{2\epsilon\eta}{\epsilon^2-\eta^2}$$

but perhaps my favorite is the singular identity

$$\frac{a^2+b^2}{a^2-b^2}\cdot\frac{c^2+d^2}{2cd}\cdot\frac{e^2+f^2}{e^2-f^2} = \frac{2\alpha\beta}{\alpha^2-\beta^2}\cdot\frac{\alpha^2-\delta^2}{2\gamma\delta}\cdot\frac{\epsilon^2-\eta^2}{2\epsilon\eta}$$

which involves all twelve Latin and Greek generators.

It may seem a bit strange to pursue such identities when a perfect cuboid might not even exist. However, we are not merely playing with the empty set: all these identities are valid for any primitive cuboid *if we do not require the Greek generators to be integers.* If we write the identity $(\alpha^2-\beta^2)/2\alpha\beta = x/X$ as $X\alpha^2 - 2x\alpha\beta - \beta^2 X = 0$ then by the quadratic formula

$$\alpha/\beta = \frac{2x+\sqrt{4x^2+4X^2}}{2X} = \frac{x+W}{X}$$

(note that we do not use $\pm$ because α and β are positive) and similarly we find

$$\frac{\gamma}{\delta} = \frac{Y+W}{y} \quad \text{and} \quad \frac{\epsilon}{\eta} = \frac{Z+W}{z}$$

so α/β, γ/δ, ϵ/η are all irrational since W is irrational. On the other hand $a/b = (x+Z)/y$, $c/d = (x+Y)/z$, and $e/f = (y+X)/z$ are all rational.

Example 4. Consider the primitive cuboid

$$(x, y, z, X, Y, Z) = (117, 44, 240, 244, 267, 125)$$

which has generators $(a, b, c, d, e, f) = (11, 2, 8, 5, 6, 5)$ and $W = \sqrt{73225} = 5\cdot\sqrt{29\cdot 101}$. Then

$$\frac{a}{b} = \frac{117+125}{44} = \frac{11}{2}\ , \quad \frac{c}{d} = \frac{117+267}{240} = \frac{8}{5}\ , \quad \frac{e}{f} = \frac{44+244}{240} = \frac{6}{5}$$

as expected while $\alpha/\beta = (117 + W)/244$, $\gamma/\delta = (267 + W)/44$, and $\epsilon/\eta = (125 + W)/240$ are not rational. Indeed a little more work finds

$$\alpha = \sqrt{36671}\ , \quad \beta = 7\sqrt{2 \cdot 373}\ , \quad \gamma = \sqrt{2 \cdot 19 \cdot 967}\ ,$$

$$\delta = \sqrt{36479}\ , \quad \epsilon = 3 \cdot 5\sqrt{163}\ , \quad \eta = 5\sqrt{2 \cdot 17 \cdot 43}$$

so none of the Greek generators are rational. I wonder if this is true for all primitive non-perfect cuboids — could *some* of their Greek generators be integers?

Concluding Remark. Many people who have searched for the elusive perfect cuboid become convinced that when the body diagonals $\sqrt{x^2 + y^2 + z^2}$ grow too large they have almost no chance of being an integer, and so a perfect cuboid just does not exist. I find that argument plausible on even days of the month. But on odd days I have this nagging feeling that a perfect cuboid is hiding out there somewhere.