A Study of the Perfect Cuboid Problem

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Abstract
We develop a procedure to generate face-cuboids. A face-cuboid is a cuboid with only one non-integer face diagonal. We show that it impossible to extend this method to generate a perfect-cuboid. Since the face-cuboid procedure is general, this constitutes a proof that a perfect-cuboid is impossible.

Introduction
All variables in this paper stand for integers unless otherwise specified. We will be developing a parameterization for a general face-cuboid. The resulting solution will not necessarily be primitive. We assume that any common factors can be removed after the cuboid is generated. We will show that the face-cuboid cannot be extended to form a perfect-cuboid.

A perfect cuboid is defined by the following integer equations:
1) \( x^2 + y^2 = p^2 \)
2) \( x^2 + z^2 = q^2 \)
3) \( y^2 + z^2 = r^2 \)
4) \( x^2 + y^2 + z^2 = s^2 \)

Substituting Eqs. 1) to 3) into 4) yields:
5) \( x^2 + r^2 = s^2 \)
6) \( y^2 + q^2 = s^2 \)
7) \( z^2 + p^2 = s^2 \)

We can assume that \( x, q, r, \) and \( s \) are odd, and \( y, z, \) and \( r \) are even. Then we can parameterize these equations by
8) \( x = (a^2 - b^2) \times t_x \)
9) \( r = 2 \times a \times b \times t_x \)
10) \( y = 2 \times c \times d \times t_y \)
11) \( q = (c^2 - d^2) \times t_y \)
12) \( z = 2 \times e \times f \times t_z \)
13) \( p = (e^2 - f^2) \times t_z \)
14) \( s = (a^2 + b^2) \times t_x = (c^2 + d^2) \times t_y = (e^2 + f^2) \times t_z \)

where \((a,b), (c,d), (e,f)\) are co-prime pairs of integers, one odd, one even; and \( t_x, t_y, t_z \) are scale factors.

We see that \( s \) must contain all the factors in Eq. 14). Therefore we can identify
15) \( t_x = (e^2 + d^2) \cdot (e^2 + f^2) \)
16) \( t_y = (a^2 + b^2) \cdot (e^2 + f^2) \)
17) \( t_z = (a^2 + b^2) \cdot (c^2 + d^2) \)
18) \( s = (a^2 + b^2) \cdot (c^2 + d^2) \cdot (e^2 + f^2) \)

An additional scale factor could be added to ‘s’, but it is unnecessary because it would multiply the \( t_x, t_y, t_z \) scale factors as well.

**Solving the cuboid equations**

The parameterization assures that Eqs. 5) to 7) are satisfied. To continue the analysis we solve for \( z^2 \) in Eq. 3), and substitute the parameterizations, Eqs. 8) to 11).

19) \( z^2 = r^2 - y^2 \)
20) \( z^2 = (2 * a * b * t_x)^2 - 2 * c * d * t_y)^2 \)

After including values for \( t_x, t_y, \) and factoring the result we get

21) \( z^2 = 4 \cdot (a^2 \cdot c^2 - b^2 \cdot d^2) \cdot (b^2 \cdot c^2 - a^2 \cdot d^2) \cdot (e^2 + f^2)^2 \)

For Eq. 21) to hold, the middle two factors on the right hand side must be a squared integer. We therefore introduce the following additional parameterization:

22) \( a \cdot c = (g^2 + h^2) \cdot m_1 \cdot \sqrt{n} \)
23) \( b \cdot d = (g^2 - h^2) \cdot m_1 \cdot \sqrt{n} \)
24) \( b \cdot c = (j^2 + k^2) \cdot m_2 \cdot \sqrt{n} \)
25) \( a \cdot d = (j^2 - k^2) \cdot m_2 \cdot \sqrt{n} \)

where \((g, h)\) and \((j, k)\) are coprime pairs of integers; \(m_1\) and \(m_2\) are scale factors; and \(n\) is a non-square integer, common to both factors. Note that \((g, h)\) and \((j, k)\) can be (odd, odd) or (odd, even) pairs since in general the middle two factors can be odd or even.

Next we look at the ratios \(a/b\) and \(c/d\)

26) \( a \over b = \frac{(g^2+h^2) \cdot m_1}{(j^2+k^2) \cdot m_2} = \frac{(j^2-k^2) \cdot m_2}{(j^2+k^2) \cdot m_2} \)

27) \( c \over d = \frac{(g^2+h^2) \cdot m_1}{(j^2-k^2) \cdot m_2} = \frac{(j^2+k^2) \cdot m_2}{(j^2-k^2) \cdot m_2} \)

Cross multiplying the right hand sides of Eqs. 26) and 27) yields, in both cases, the constraint

28) \( (g^4 - h^4) \cdot m_1^2 = (j^4 - k^4) \cdot m_2^2 \)

or

29) \( g^4 \cdot m_1^2 + k^4 \cdot m_2^2 = h^4 \cdot m_1^2 + j^4 \cdot m_2^2 \)

Substituting Eqns. 22) to 25) into 21) yields

30) \( z^2 = 4 \cdot (4 \cdot g^2 \cdot h^2 \cdot m_1^2) \cdot (4 \cdot j^2 \cdot k^2 \cdot m_2^2) \cdot n^2 \cdot (e^2 + f^2)^2 \)

and

31) \( z = 8 \cdot g \cdot h \cdot j \cdot k \cdot m_1 \cdot m_2 \cdot n \cdot (e^2 + f^2) \)

At this point we can form a face cuboid with integers for the three edges \(x, y, z\), and diagonals \(q, r, s\). The face diagonal \(p\) will not necessarily be an integer. From Eqs. 8) to 18) we have
We assume that the parameters $g, h, j, k, l, m_1, m_2$ have values that are consistent with the constraint Eq. 29), and then insert 22) to 25), into 32) to 36).

Inserting the constraint, Eq. 29) into Eq. 41) yields

Eqs. 31), and 37) to 42) define a general face-cuboid, where the face diagonal $p$ is not necessarily an integer. Each of these equations contains the scale factor, $4 \times (e^2 + f^2) \times n$, which can be dropped. The constraint Eq. 29) must be satisfied for these equations to hold.

**Face-Cuboid Example**

For example let $m_1 = 1, m_2 = 4, g = 133, h = 59, j = 79, k = 67$. Then the constraint Eq. 29) is satisfied:

and we have

However, using Eqs. 1) and 7), we find that the face diagonal, $p$, is not an integer.

The solution is primitive with $x, q$ and $s$ odd, and $y, z$, and $r$ even. Note that the value of the constraint is equal to $s$.

**Extending the analysis to the Perfect Cuboid**

First we note that Eqs. 7), 12), 13), and 17) were not used in the derivation of the face-cuboid. In order to continue, we must keep the scale factors in Eqs. 31), and 37) to 42). We need to
evaluate \( p \) and \( z \) from Eqs. 12) and 13). Using Eq. 17) with Eqs. 22) to 25) and making use of the constraint Eq. 29). We then have

\[
\begin{align*}
51) & \quad t_z = 2 \ast (g^4 \ast m_1^2 + h^4 \ast m_1^2 + j^4 \ast m_2^2 + k^4 \ast m_2^2) \ast n = 4 \ast (g^4 \ast m_1^2 + k^4 \ast m_2^2) \ast n \\
52) & \quad z = 8 \ast (g^4 \ast m_1^2 + k^4 \ast m_2^2) \ast e \ast f \ast n \\
53) & \quad p = 4 \ast (g^4 \ast m_1^2 + k^4 \ast m_2^2) \ast (e^2 - f^2) \ast n
\end{align*}
\]

Note that \( (e^2 + f^2) \) is no longer a scale factor, and cannot be dropped. \( 4 \) and \( n \) are still a scale factors.

Next we equate \( z \) from Eqs. 31) and 52)

\[
54) \quad z = 8 \ast g \ast h \ast f \ast j \ast k \ast m_1 \ast m_2 \ast n \ast (e^2 + f^2) = 8 \ast (g^4 \ast m_1^2 + k^4 \ast m_2^2) \ast e \ast f \ast n
\]

Notice that Eq.54) is quadratic in \( e \) and \( f \). Solving for \( e/f \) we find

\[
55) \quad \frac{e}{f} = \frac{g^4 \ast m_1^2 + k^4 \ast m_2^2 \pm \sqrt{(g^4 \ast m_1^2 + k^4 \ast m_2^2)^2 - (2 \ast g \ast h \ast f \ast j \ast k \ast m_1 \ast m_2)^2}}{2 \ast g \ast h \ast j \ast k \ast m_1 \ast m_2}
\]

In order for this to give a rational result, we require the argument of the square root is a square. It can be a square only if

\[
56) \quad h \ast j = g \ast k
\]

It is important to note that Eq. 56) is an additional constraint, along with Eq. 29), that is required for a perfect cuboid;

Then Eq. 55) becomes

\[
57) \quad \frac{e}{f} = \frac{(g^4 \ast m_1^2 + k^4 \ast m_2^2) \pm (g^4 \ast m_1^2 - k^4 \ast m_2^2)}{2 \ast (g \ast k)^2 \ast m_1 \ast m_2} = \frac{g^2 \ast m_1}{k^2 \ast m_2} \text{ or } \frac{k^2 \ast m_2}{g^2 \ast m_1}
\]

If we take the plus sign, then we can identify

\[
58) \quad e = g^2 \ast m_1 \ast m_3 \\
59) \quad f = k^2 \ast m_2 \ast m_3
\]

where \( m_3 \) is an additional scale factor. Taking the minus sign simply reverses the definition if \( e \) and \( f \).

At this point if we can find parameters that satisfy both constraints, Eqs. 29) and 56), then we have a solution for a perfect cuboid. However, we will find that the only parameters that satisfy both constraints will produce only trivial solutions in the sense that at least one edge or one diagonal of the cuboid is zero.

**Compatibility of the constraints**

The two constraints, Eqs. 29) and 56) when combined, are not compatible with a perfect cuboid.

We square Eq. 56) and multiply both sides by \( 2 \ast (m_1 \ast m_2) \).

\[
60) \quad 2 \ast g^2 \ast k^2 \ast m_1 \ast m_2 = 2 \ast h^2 \ast j^2 \ast m_1 \ast m_2
\]
Add Eq. 60) to 29), or subtract it from 29) resulting in
61) \[(g^2 \pm k^2) m_1 = (h^2 \pm j^2) m_2\]

Take the square root
62) \[(g^2 \pm k^2) m_1 \pm j^2 m_2 = \pm(h^2 \pm j^2) m_2\]

Transpose terms and factor out \(m_1\) and \(m_2\) resulting in 4 combinations.
63) \[(g^2 - h^2) m_1 = \pm(j^2 - k^2) m_2\]
64) \[(g^2 + h^2) m_1 = \pm(j^2 + k^2) m_2\]

Substitute these into Eq. 26) resulting in
65) \[a/b = \pm 1\]

Then \(a = \pm b\), and from Eq. 8)
66) \[x = (a^2 - b^2) t_x = 0\]

Thus we see that the combined constraints result in an unacceptable trivial solution, \(x = 0\).

We conclude that there are no non-trivial solutions with the constraints Eqns. 29) and 56). Therefore a perfect cuboid cannot be obtained from the face-cuboid formulation.

**Conclusion**

We have shown that an analysis of the perfect cuboid equations leads to a formulation of a face-cuboid where just one face diagonal is not an integer. The formulation is general, and can generate all possible face-cuboids up to a scale factor. Extending the formulation to a perfect cuboid yields only trivial solutions. Thus we conclude that a perfect cuboid is not possible.

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