A Study of the Perfect Cuboid Problem

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Abstract

We develop a procedure to generate face-cuboids. A face-cuboid is a cuboid with only one non-integer face diagonal. We show that it impossible to extend this method to generate a perfect-cuboid. Since the face-cuboid procedure is general, this constitutes a proof that a perfect-cuboid is impossible.

Introduction

All variables in this paper stand for integers unless otherwise specified. We will be developing a parameterization for a general face-cuboid. The resulting solution will not necessarily be primitive. We assume that any common factors can be removed after the cuboid is generated. We will show that the face-cuboid cannot be extended to form a perfect-cuboid.

A perfect cuboid is defined by the following integer equations:

1) $x^2 + y^2 = \mathrm{p}^2$
2) $x^2 + z^2 = q^2$
3) $y^2 + z^2 = r^2$
4) $\mathrm{x}^2 + \mathrm{y}^2 + \mathrm{z}^2 = \mathrm{s}^2$

Substituting Eqs. 1) to 3) into 4) yields:

5) $x^2 + r^2 = \mathrm{s}^2$
6) $y^2 + q^2 = s^2$
7) $z^2 + p^2 = s^2$

We can assume that *x*, *q*, *r*, and *s* are odd, and *y*, *z*, and *r* are even. Then we can parameterize these equations by

8) $x = (a^2 - b^2) * t_x$
9) $r = 2 * a * b * t_x$
10) $y = 2 * c * d * t_y$
11) $q = (c^2 - d^2) * t_y$
12) $z = 2 * e * f * t_z$
13) $p = (e^2 - f^2) * t_z$
14) $s = (a^2 + b^2) * t_x = (c^2 + d^2) * t_y = (e^2 + f^2) * t_z$

where (*a*,*b*), (*c*,*d*), (*e*,*f*) are co-prime pairs of integers, one odd, one even; and t_x, t_y, t_z are scale factors.

We see that *s* must contain all the factors in Eq. 14). Therefore we can identify

15) $t_x = (c^2 + d^2) * (e^2 + f^2)$
16) $t_y = (a^2 + b^2) * (e^2 + f^2)$
17) $t_z = (a^2 + b^2) * (c^2 + d^2)$
18) $s = (a^2 + b^2) * (c^2 + d^2) * (e^2 + f^2)$
An additional scale factor could be added to ‘s’, but it is unnecessary because it would multiply the t_x, t_y, t_z scale factorss as well.

Solving the cuboid equations

The parameterization assures that Eqs. 5) to 7) are satisfied. To continue the analysis we solve for z^2 in Eq. 3), and substitute the parameterizations, Eqs. 8) to 11).
19) $.z^2 = r^2 - \mathrm{y}^2$
20) $z^2 = (2 * a * b * t_x)^2 - 2 * c * d * t_y)^2$
After including values for $t_x, t_y,$ and factoring the result we get
21) $z^2 = 4 * (a^2 * c^2 - b^2 * d^2) * (b^2 * c^2 - a^2 * d^2) * (e^2 + f^2)^2$

For Eq. 21) to hold, the middle two factors on the right hand side must be a squared integer. We therefore introduce the following additional parameterization:
22) $a * c = (g^2 + h^2) * m_1 * \sqrt{n}$
23) $b * d = (g^2 - h^2) * m_1 * \sqrt{n}$
24) $b * c = (j^2 + k^2) * m_2 * \sqrt{n}$
25) $a * d = (j^2 - k^2) * m_2 * \sqrt{n}$
where (g, h) and (j, k) are coprime pairs of integers; $m_1$and m_2 are scale factors; and n is a non-square integer, common to both factors. Note that (g, h) and (j, k) can be (odd, odd) or (odd, even) pairs since in general the middle two factors can be odd or even.

Next we look at the ratios a/b and c/d

26) $$\frac{a}{b} = \frac{(g^2+h^2)*m_1}{(j^2+k^2)*m_2} = \frac{(j^2-k^2)*m_2}{(g^2-h^2)*m_1}$$
27) $$\frac{c}{d} = \frac{(g^2+h^2)*m_1}{(j^2-k^2)*m_2} = \frac{(j^2+k^2)*m_2}{(g^2-h^2)*m_1}$$

Cross multiplying the right hand sides of Eqs. 26) and 27) yields, in both cases, the constraint
28) $(g^4 - h^4) * m_1^2 = (j^4 - k^4) * m_2^2$
or
29) $g^4 * m_1^2 + k^4 * m_2^2 = h^4 * m_1^2 + j^4 * m_2^2$
Substituting Eqns. 22) to 25) into 21) yields
30) $z^2 = 4 * (4 * g^2 * h^2 * m_1^2) * (4 * j^2 * k^2 * m_2^2) * n^2 * (e^2 + f^2)^2$
and
31) $z = 8 * g * h * j * k * m_1 * m_2 * n * (e^2 + f^2)$

At this point we can form a face cuboid with integers for the three edges *x*, *y*, *z*, and diagonals *q*, *r*, *s*. The face diagonal *p* will not necessarily be an integer. From Eqs. 8) to 18) we have

32) $x = ((a*c)^2 + (a*d)^2 - (b*c)^2 - (b*d)^2)*(e^2 + f^2)$
33) $r = 2*((a*c)*(b*c) + (a*d)*(b*d))*(e^2 + f^2)$
34) $y = 2*((a*c)*(a*d) + (b*c)*(b*d))*(e^2 + f^2)$
35) $q = ((a*c)^2 - (a*d)^2 + (b*c)^2 - (b*d)^2)*(e^2 + f^2)$
36) $s = ((a*c)^2 + (a*d)^2 + (b*c)^2 + (b*d)^2)*(e^2 + f^2)$

We assume that the parameters g, h, j, k , $m1_1$, m_2 , have values that are consistent with the constraint Eq. 29), and then insert 22) to 25), into 32) to 36).
37) $x = 4*((g*h*m_1)^2 - ((j*k*m_2)^2)*(e^2 + f^2)*n$
38) $r = 4*((g*j)^2 + (h*k)^2)*m_1*m_2*(e^2 + f^2)*n$
39) $y = 4*((g*j)^2 - (h*k)^2)*m_1*m_2*(e^2 + f^2)*n$
40) $q = 4*((g*h*m_1)^2 + ((j*k*m_2)^2)*(e^2 + f^2)*n$
41) $s = 2*(g^4*m_1^2 + h^4*m_1^2 + j^4*m_2^2 + k^4*m_2^2)*(e^2 + f^2)*n$

Inserting the constraint, Eq. 29) into Eq. 41) yields
42) $s = 4*(g^4*m_1^2 + k^4*m_2^2)*(e^2 + f^2)*n$

Eqs. 31), and 37) to 42) define a general face-cuboid, where the face diagonal *p* is not necessarily an integer. Each of these equations contains the scale factor, $4*(e^2 + f^2)*n$, which can be dropped. The constraint Eq. 29) must be satisfied for these equations to hold.

Face-Cuboid Example

For example let $m_1 = 1, m_2 = 4, g = 133, h = 59, j = 79, k = 67$. Then the constraint Eq. 29) is satisfied:
43) $g^4*m_1^2 + k^4*m_2^2 = h^4*m_1^2 + j^4*m_2^2 = 635318657$
and we have
44) $x = 386678175$
45) $y = 379083360$
46) $z = 332273368$
47) $r = 504093032$
48) $q = 509828993$
49) $s = 635318657$
However, using Eqs. 1) and 7), we find that the face diagonal, *p*, is not an integer.
50) $p = \sqrt{s^2 - z^2} = \sqrt{x^2 + y^2} = 15*\sqrt{1303218688223201}$
The solution is primitive with *x*, *q* and *s* odd, and *y*, *z*, and *r* even. Note that the value of the constraint is equal to *s*.

Extending the analysis to the Perfect Cuboid

First we note that Eqs. 7), 12), 13), and 17) were not used in the derivation of the face-cuboid . In order to continue, we must keep the scale factors in Eqs. 31), and 37) to 42). We need to evaluate p and z from Eqs. 12) and 13). Using Eq. 17) with Eqs. 22) to 25) and making use of the constraint Eq. 29). We then have

51) $t_z = 2*(g^4*m_1^2 + h^4*m_1^2 + j^4*m_2^2 + k^4*m_2^2)*n = 4*(g^4*m_1^2 + k^4*m_2^2)*n$

52) $z = 8*(g^4*m_1^2 + k^4*m_2^2)*e*f*n$

53) $p = 4*(g^4*m_1^2 + k^4*m_2^2)*(e^2 - f^2)*n$

Note that $(e^2 + f^2)$ is no longer a scale factor, and cannot be dropped. 4 and *n* are still a scale factors.

Next we equate *z* from Eqs. 31) and 52)

54) $z = 8*g*h*j*k*m_1*m_2*n*(e^2+f^2) = 8*(g^4*m_1^2 + k^4*m_2^2)*e*f*n$

Notice that Eq.54) is quadratic in *e* and *f*. Solving for e/f we find

55) $$\frac{e}{f} = \frac{g^4*m_1^2+k^4*m_2^2 \pm \sqrt{\left(g^4*m_1^2+k^4*m_2^2\right)^2 - (2*g*h*j*k*m_1*m_2)^2}}{2*g*h*j*k*m_1*m_2}$$

In order for this to give a rational result, we require the argument of the square root is a square. It can be a square only if

56) $h*j = g*k$

It is important to note that Eq. 56) is an additional constraint, along with Eq. 29), that is required for a perfect cuboid;

Then Eq. 55) becomes

57) $$\frac{e}{f} = \frac{(g^4*m_1^2+k^4*m_2^2) \pm \left(g^4*m_1^2 - k^4*m_2^2\right)}{2*(g*k)^2*m_1*m_2} = \frac{g^2*m_1}{k^2*m_2} \;\; or = \frac{k^2*m_2}{g^2*m_1}$$

If we take the plus sign, then we can identify

58) $e = g^2*m1*m3$

59) $f = k^2*m2*m3$

where *m3* is an additional scale factor. Taking the minus sign simply reverses the definition if *e* and *f*.

At this point if we can find parameters that satisfy both constraints, Eqs. 29) and 56), then we have a solution for a perfect cuboid. However, we will find that the only parameters that satisfy both constraints will produce only trivial solutions in the sense that at least one edge or one diagonal of the cuboid is zero.

Compatibility of the constraints

The two constraints, Eqs. 29) and 56) when combined, are not compatible with a perfect cuboid..

We square Eq. 56) and multiply both sides by $2*(m_1*m_2)$.

60) $2*g^2*k^2*m_1*m_2 = 2*h^2*j^2*m_1*m_2$

Add Eq. 60) to 29), or subtract it from 29) resulting in
61) $(g^2 * m_1 \pm k^2 * m_2)^2 = (h^2 * m_1 \pm j^2 * m_2)^2$

Take the square root
62) $(g^2 * m_1 \pm k^2 * m_2) = \pm(h^2 * m_1 \pm j^2 * m_2)$

Transpose terms and factor out m_1 and m_2 resulting in 4 combinations.
63) $(g^2 - h^2) * m_1 = \pm(j^2 - k^2) * m_2$
64) $(g^2 + h^2) * m_1 = \pm(j^2 + k^2) * m_2$

Substitute these into Eq. 26) resulting in
65) $a/b = \pm 1$

Then $a = \pm b$, and from Eq. 8)
66) $x = (a^2 - b^2) * t_x = 0$

Thus we see that the combined constraints result in an unacceptable trivial solution, $x = 0$.

We conclude that there are no non-trivial solutions with the constraints Eqns. 29) and 56). Therefore a perfect cuboid cannot be obtained from the face-cuboid formulation.

Conclusion

We have shown that an analysis of the perfect cuboid equations leads to a formulation of a face-cuboid where just one face diagonal is not an integer. The formulation is general, and can generate all possible face-cuboids up to a scale factor. Extending the formulation to a perfect cuboid yields only trivial solutions. Thus we conclude that a perfect cuboid is not possible.

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