Why No Perfect Boxes? (Argument no Euler Brick has integer space diagonals)

We shall argue there are no perfect boxes, in other words that no Euler Brick (rectangular edges a, b, c all integer length, face diagonals d, e, f all integer length) also has an integer-length space diagonal g.

Labelling lengths as in **diagram 1**, this is equivalent to showing there is no right prism (**diagram 2**) whose “floor” and “ceiling” is the right-angled triangle a, b, f, with vertical height c and face diagonals d, e, g, in which all seven lengths are of integer length.

Were there such a prism, its three vertical faces’ diagonals would each be the hypotenuse of a right-angled Pythagorean triangle, each having the prism height c as its second edge, so as to be integer lengths. So we list the squares of the three hypotenuses (a^2 + c^2, b^2 + c^2, f^2 + c^2), and look for a potential vertical height c. (We ignore distinctions between primitive Pythagorean triangles and multiples of primitives in the following.)

Example: Using 15^2 = 225 for c^2, we test (8^2, 20^2, 36^2), three squared base edge lengths and (17^2, 25^2, 39^2), the squared hypotenuses, squared vertical face diagonals of that would-be prism. Three vertical-plane right-angled triangles (8, 15, 17), (20, 15, 25), and (36, 15, 39) give three face diagonals (17 : 25 : 39) for a potential prism of vertical edge height 15.

Unfortunately, the three would-be edges of the right prism floor 8, 20, 36 don’t even form a triangle, let alone a right-angled triangle.

Do these two conditions rule each other out? Perhaps we obtain three Pythagorean triangles of equal height for the prism faces only if the prism floor is not a Pythagorean right-angled triangle?

How did we obtain those three c-height triangles? We use the fact that strings of consecutive odd numbers sum to squares (most simply, 1+3=4, 1+3+5=9, 1+3+5+7=16, --). We assume without loss of generality that a < b < f. Meanwhile, c^2 has factors (in the case of 15^2 : 3, 9, 5, 15, 25, 45, 75), and these govern the lengths and locations of strings of consecutive odd numbers summing to 15^2. Nine consecutive odds sum to 225 if chosen between 8^2 and 17^2, five consecutive odds sum to 225 between 20^2 & 25^2, and so on.

Comparing strings of consecutive odds (17+19+21+23+{25}+27+29+31+33), (41+43+{45}+47+49), and (73+{75}+77), shows that those summing to 15^2 have the simple median-odd:number-of-odds relations 25:9, 45:5, 75:3. [The case of even squares simply replaces a median odd with two median odds (1+{3+5=2.4}+7=16).]

So can (i.) a, b, & f obey a^2 + b^2 = f^2, while the same a, b, f are each (ii.) one edge of three right-angled triangles all with height = c? Condition ii. uses our assumption a < b < f, where a^2 and d^2 (then again b^2 and e^2, and f^2 and g^2) each pairwise bracket a distinct stretch of consecutive odds summing to c^2. (c^2-summing strings overlap in some cases.)

If both conditions above are met, it must be that the stretch of odds between a^2 and f^2 sums to b^2 (by i.), and yet also b^2 is *in* that stretch of the number line (by ii.), a contradiction even with a one-number string which is b^2 itself. Above 9-16-25, squares only occur as individual numbers bracketed between neighbouring pairs of much larger squares, such as 225 = 15^2 equalling 113^2 minus 112^2, or 12769 – 12544, not an a < b < f case. This is because the middle square grows faster than the other two squares’ difference when summing and comparing squares of three consecutive numbers: 1+4=9-4; 4+9=16-3; 9+16=25-/+0; 16+25=36+5; 25+36=49+12; 36+49=64+21.

This contradiction is why the three prism floor lengths a, b, f that obey a^2 + b^2 = f^2 do not form the first edge of each of three face-diagonal triangles of equal height c. Likewise a set of a, b, f values such as our example 8, 20, 36, which *do* give three vertical triangles of equal heights c = 15, cannot qualify as a prism base of the kind we need.

From this, there is no right prism of the type shown in **diagram 2**, that is no Euler brick with an integer space diagonal.

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