Why No Perfect Boxes? (Argument no Euler Brick has integer space diagonals)

We shall argue there are no perfect boxes, in other words that no Euler Brick (rectangular edges a, b, c all integer length, face diagonals d, e, f all integer length) also has an integer-length space diagonal g.

Labelling lengths as in **diagram 1**, this is equivalent to showing there is no right prism (**diagram 2**) whose “floor” and “ceiling” is the right-angled triangle a, b, f, with vertical height c and face diagonals d, e, g, in which all seven lengths are of integer length.

Were there such a prism, its three vertical faces’ diagonals would each be the hypotenuse of a right-angled Pythagorean triangle, each having the prism height c as its second edge, so as to be integer lengths. So we list the squares of the three hypotenuses ($a^2 + c^2$, $b^2 + c^2$, $f^2 + c^2$), and look for a potential vertical height c. (We ignore distinctions between primitive Pythagorean triangles and multiples of primitives in the following.)

Example: Using $15^2 = 225$ for c^2, we test (8^2, 20^2, 36^2), three squared base edge lengths and (17^2, 25^2, 39^2), the squared hypotenuses, squared vertical face diagonals of that would-be prism. Three vertical-plane right-angled triangles (8, 15, 17), (20, 15, 25), and (36, 15, 39) give three face diagonals (17 : 25 : 39) for a potential prism of vertical edge height 15.

Unfortunately, the three would-be edges of the right prism floor 8, 20, 36 don’t even form a triangle, let alone a right-angled triangle.

Do these two conditions rule each other out? Perhaps we obtain three Pythagorean triangles of equal height for the prism faces only if the prism floor is not a Pythagorean right-angled triangle?

How did we obtain those three c-height triangles? We use the fact that strings of consecutive odd numbers sum to squares (most simply, 1+3=4, 1+3+5=9, 1+3+5+7=16 but also later “mid-strings” like (73+{75}+77) = 225). Meanwhile, c^2 has factors (in the case of 15^2 : 3, 9, 5, 15, 25, 45, 75). Nine consecutive odds sum to 225 between 17 and 35, five consecutive odds sum to 225 between 41 & 51, and so on.

So can (i.) a, b, & f obey $a^2 + b^2 = f^2$, while the same a, b, f are each (ii.) one edge of three right-angled triangles all with height = c?

Consider a sequence of “L-numbers” of fixed “hole size” where an L-number is a square with a smaller square removed (the hole). We could represent squares 8^2, 20^2, 36^2, each followed by a mid-string of odds summing to 15^2, as L-numbers where the fixed hole size is 15^2 or 225. There are two questions about general sums of this form.

1) Can 2 L-numbers of hole-size x sum to form another L-number of hole size x?
2) An L-number can clearly also be a square, but can three squares sum as a Pythagorean triple when each of the three is also an L-number of identical hole size?

Answering (1), we easily find 3 L-numbers (hole size 4) summing as follows: 32 (36-4) + 45 (49-4) = 77 (81-4).

To answer (2) we note that differences between given L-numbers match differences between the larger squares in each L-number. Consider two squares u and v that sum to a third square w. In a series of L-numbers hole size x, L1=u, L2=v, and L3=w must be in positions in the L-number sequence such that differences L2-L1 and L3-L2 do not match the differences v-u and w-v on a sequence of perfect squares, so do not sum.

Reasoning the other way, mark out a large enough section of the integer number line, marking all perfect squares and individually labelling all Pythagorean triples in that section. Then we can identify whichever Pythagorean triple we believe is identical to L1, L2, and L3, and see the triple has been shifted along the line by those L-numbers’ hole size to a location with different inter-L differences, and so does not add up.

Therefore, answering no to (2), either L1+L2 = L3 but at least one L-number is not a square, or all of L1, L2, L3 are squares but L1 and L2 do not sum to L3.

This is why three prism floor lengths a, b, f that obey $a^2 + b^2 = f^2$ do not form the first edge of each of three face-diagonal triangles of equal height c. Likewise a set of a, b, f values such as our example 8, 20, 36, which *do* give three vertical triangles of equal heights c = 15, cannot qualify as a prism base of the kind we need.

From this, there is no right prism of the type shown in **diagram 2**, that is no Euler brick with an integer space diagonal.

Mark Griffith