An attempt on the Riemann hypothesis

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Abstract

In this article, I discuss what can be said when the Riemann hypothesis is false and there are only finitely many nonreal zeros off the critical line.

1 Introduction

Let $\zeta(s)$ denote the Riemann zeta-function and $\rho_j \equiv \beta_j + i\gamma_j$ the nonreal zero of the Riemann zeta-function

There are mainly two topics that I want to discuss in this article.

I attempt to prove the following theorem.

Theorem 1. *Assume that the Riemann hypothesis is false and that there exists only one nonreal zero $\rho_{n_0} \equiv \beta_{n_0} + i\gamma_{n_0}$ of the Riemann zeta-function in the region $\{a + bi : a > 1/2, b > 0\}$. Then, there exists an upper bound for γ_{n_0}.*

As one can see from the argument for this theorem (which will be shown in the next section), the theorem is only of theoretical interest; it only claims the existence of a bound, and when one wants to calculate it, one needs to know exact data for zeros on the critical line.

Another topic is related to the lemma used in the proof of Theorem 1; I consider its variation. It may be seen as an inconclusive argument, but I guess it has something close to the truth of the Riemann hypothesis.

2 The proof of Theorem 1

Lemma 1. *Assume that there exist only finitely many nonreal zeros of the zeta-function off the critical line. Let $\delta, \kappa, h > 0$ and $-1 < c' + \kappa < 0$. Then we have*

$$\begin{aligned} &-\sum_{\rho_j} h(\rho_j, \kappa, \delta)(2\pi)^{-\rho_j+\delta+\kappa} e^{-\pi i(\rho_j-\delta-\kappa)/2} + E + S_{\delta,\kappa} \\ &= \Gamma(1+\delta)\big(-\sum_{\rho_j} Z(1+\delta-\rho_j+\kappa)\rho_j^{-1} + D_{\delta,\kappa} + W_{\delta,\kappa}\big), \end{aligned} \tag{1}$$

where E is defined by (7) with $q = 1+\delta$,

$$h(\rho_j, \kappa, \delta) \equiv \Gamma(\rho_j - \delta - \kappa)\Gamma(\rho_j - \kappa)\Gamma(1 - \rho_j + \delta + \kappa),$$
$$Z(s) \equiv \int_0^1 \zeta^*(s, r) e^{-2\pi i r} dr, \quad \zeta^*(s, r) \equiv \sum_{k \geq 2} (k + r)^{-s},$$

and

$$D_{\delta,\kappa} \equiv -\frac{\zeta'}{\zeta}(\kappa) \int_0^1 \zeta^*(1+\delta, r) e^{-2\pi i r} dr$$
$$S_{\delta,\kappa} \equiv \frac{1}{2\pi i} \int_{(-3+h)} -\frac{\zeta'}{\zeta}(s+\delta+\kappa)\Gamma(s)\Gamma(s+\delta)\Gamma(1-s)(2\pi)^{-s} e^{-\pi s i/2} ds$$
$$W_{\delta,\kappa} \equiv \frac{1}{2\pi i} \int_{c'-i\infty}^{c'+i\infty} -\frac{\zeta'}{\zeta}(\kappa + w) \int_0^1 \zeta^*(1+\delta-w, r) e^{-2\pi i r} dr \frac{dw}{w}.$$

Proof. We define [1]

$$I_1(x) \equiv \frac{1}{2\pi i} \int_{(c)} \Gamma(s) x^{-s} ds = e^{-x} \tag{2}$$

and

$$I_2(x) \equiv \frac{1}{2\pi i} \int_{(c)} \Gamma(s+q-1)\Gamma(1-s) x^{-s} ds,$$

where $q > 0$ with $1 - q < c < 1$.

Shifting the path in I_2 to the left, we find out that

$$I_2(x) = \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(n+q) x^{n+q-1}}{n!} = \Gamma(q) x^{q-1} (1+x)^{-q}. \tag{3}$$

We use these two formulas (2) and (3) to evaluate

$$I(x) \equiv \frac{1}{2\pi i} \int_{(c)} \Gamma(s)\Gamma(s+q-1)\Gamma(1-s) x^{-s} ds. \tag{4}$$

Here, we recall that [2]

$$\frac{1}{2\pi i} \int_{(c)} \mathcal{F}(s)\mathcal{G}(s) x^{-s} ds = \int_0^\infty f(z) g(\frac{x}{z}) \frac{dz}{z},$$

where $\frac{1}{2\pi i}\int_{(c)} \mathcal{F}(s) x^{-s} ds = f(x)$ and $\frac{1}{2\pi i}\int_{(c)} \mathcal{G}(s) x^{-s} ds = g(x)$.

We associate $\mathcal{F}$ and f with $\Gamma(s)$ and e^{-x} (by (2)), and $\mathcal{G}$ and g with $\Gamma(s + q - 1)\Gamma(1-s)$ and $\Gamma(q) x^{q-1}(1+x)^{-q}$ (by (3)), respectively.

This gives

$$I(x) = \Gamma(q) x^{q-1} \int_0^\infty e^{-z} (z+x)^{-q} dz. \tag{5}$$

Thus we have for $1 < \mathrm{Re}(\kappa) < 2$,

$$
\begin{aligned}
&\frac{1}{2\pi i}\int_{(c)} -\frac{\zeta'}{\zeta}(s+\delta+\kappa)\Gamma(s)\Gamma(s+q-1)\Gamma(1-s)x^{-s}ds \\
&= \sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^{\delta+\kappa}}I(xn) \\
&= \Gamma(q)x^{q-1}\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^{1-q+\delta+\kappa}}\int_0^{\infty} e^{-z}(z+xn)^{-q}dz,
\end{aligned}
$$

or rearranging,

$$
\begin{aligned}
&\frac{1}{2\pi i}\int_{(c)} -\frac{\zeta'}{\zeta}(s+\delta+\kappa)\Gamma(s)\Gamma(s+q-1)\Gamma(1-s)x^{-s}ds \\
&= \Gamma(q)x^{-1}\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^{1-q+\delta+\kappa}}\int_0^{\infty} e^{-z}(z/x+n)^{-q}dz.
\end{aligned}
$$

We use the the change of variables $z = xw$, let $x = 2\pi e^{\pi i/2}$, and obtain

$$
\begin{aligned}
&\frac{1}{2\pi i}\int_{(c)} -\frac{\zeta'}{\zeta}(s+\delta+\kappa)\Gamma(s)\Gamma(s+q-1)\Gamma(1-s)(2\pi)^{-s}e^{-\pi is/2}ds \\
&= \Gamma(q)\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^{1-q+\delta+\kappa}}\int_0^{\infty} e^{-2\pi iw}(w+n)^{-q}dw.
\end{aligned}
$$

Shifting the path of the left integral to $\mathrm{Re}(s) = -3+h$ for some small $h > 0$, we have with the residue theorem

$$
\begin{aligned}
&-\sum_{\rho_j} h(\rho_j,\kappa,\delta)(2\pi)^{-\rho_j+\delta+\kappa}e^{-\pi i(\rho_j-\delta-\kappa)/2} + E \\
&+ \frac{1}{2\pi i}\int_{(-3+h)} -\frac{\zeta'}{\zeta}(s+\delta+\kappa)\Gamma(s)\Gamma(s+q-1)\Gamma(1-s)(2\pi)^{-s}e^{-\pi si/2}ds \\
&= \Gamma(q)\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^{1-q+\delta+\kappa}}\int_0^{\infty} e^{-2\pi iw}(w+n)^{-q}dw,
\end{aligned}
\tag{6}
$$

where

$$
h(\rho_j,\kappa,\delta) \equiv \Gamma(\rho_j-\delta-\kappa)\Gamma(\rho_j+q-\delta-\kappa-1)\Gamma(1-\rho_j+\delta+\kappa)
$$

and E denotes the residues of the left integrand except for ones arising from the nonreal roots of the zeta-function; that is,

$$
\begin{aligned}
&-\frac{\zeta'}{\zeta}(1-q+\delta+\kappa)\Gamma(1-q)\Gamma(q)(2\pi)^{-1+q}e^{-\pi(1-q)i/2} \\
&-\frac{\zeta'}{\zeta}(\delta+\kappa)\Gamma(q-1) \\
&+\Gamma(1-\delta-\kappa)\Gamma(-\delta-\kappa+q)\Gamma(\delta+\kappa)(2\pi)^{-1+\delta+\kappa}e^{-\pi(1-\delta-\kappa)i/2} \\
&+\sum_{\mathrm{residue}_{s=-1,-2}} (\text{left integrand in (6)}).
\end{aligned}
\tag{7}
$$

We denote

$$S_{\delta,\kappa} \equiv \frac{1}{2\pi i}\int_{(-3+h)} -\frac{\zeta'}{\zeta}(s+\delta+\kappa)\Gamma(s)\Gamma(s+q-1)\Gamma(1-s)(2\pi)^{-s}e^{-\pi s i/2}ds.$$

We choose $q = 1+\delta$ by analytic continuation in (6), and put

$$\begin{aligned} &-\sum_{\rho_j} h(\rho_j,\kappa,\delta)(2\pi)^{-\rho_j+\delta+\kappa}e^{-\pi i(\rho_j-\delta-\kappa)/2} + E + S_{\delta,\kappa} \\ &= \Gamma(1+\delta)\sum_{n\geq 2}\frac{\Lambda(n)}{n^\kappa}\int_0^\infty e^{-2\pi i w}(w+n)^{-1-\delta}dw. \end{aligned} \tag{8}$$

Next, we rewrite the integral $\int_0^\infty$ on the right as

$$\begin{aligned} \int_0^\infty e^{-2\pi i w}(n+w)^{-1-\delta}dw &= \sum_{k\geq 0}\int_k^{k+1} e^{-2\pi i w}(n+w)^{-1-\delta}dw \\ &= \int_0^1 \sum_{k\geq 0}(n+k+r)^{-1-\delta}e^{-2\pi i r}dr, \end{aligned} \tag{9}$$

and inserting this into (8), we get

$$\begin{aligned} &-\sum_{\rho_j} h(\rho_j,\kappa,\delta)(2\pi)^{-\rho_j+\delta+\kappa}e^{-\pi i(\rho_j-\delta-\kappa)/2} + E + S_{\delta,\kappa} \\ &= \Gamma(1+\delta)\sum_{n\geq 2}\frac{\Lambda(n)}{n^\kappa}\int_0^1 \sum_{j\geq n}(j+r)^{-1-\delta}e^{-2\pi i r}dr. \end{aligned} \tag{10}$$

Since the factor $\sum_{j\geq n}(j+r)^{-1-\delta}$ is a decreasing function of n, it is easy to see that we can interchange the sum and the integral; we have

$$\begin{aligned} &-\sum_{\rho_j} h(\rho_j,\kappa,\delta)(2\pi)^{-\rho_j+\delta+\kappa}e^{-\pi i(\rho_j-\delta-\kappa)/2} + E + S_{\delta,\kappa} \\ &= \Gamma(1+\delta)\int_0^1 \sum_{n\geq 2}\frac{\Lambda(n)}{n^\kappa}\sum_{j\geq n}(j+r)^{-1-\delta}e^{-2\pi i r}dr. \end{aligned} \tag{11}$$

Now, we recall the following formula for summation by parts [1]

$$\sum_{k=0}^{n} a_k b_k = a_n B_n - \sum_{k=0}^{n-1} B_k(a_{k+1}-a_k), \tag{12}$$

where

$$B_n \equiv \sum_{k=0}^{n} b_k.$$

We rewrite

$$\sum_{n\geq 2}\frac{\Lambda(n)}{n^\kappa}\sum_{j\geq n}(j+r)^{-1-\delta}=\sum_{m\geq 0}\frac{\Lambda(m+2)}{(m+2)^\kappa}\sum_{j\geq m+2}(j+r)^{-1-\delta},$$

and so choosing $b_n \mapsto \Lambda(n+2)(n+2)^{-\kappa}$ and

$$a_n \mapsto \sum_{j\geq n+2}(j+r)^{-1-\delta}$$

in (12) and letting $n\to\infty$, we obtain

$$\begin{aligned}\sum_{n\geq 2}\frac{\Lambda(n)}{n^\kappa}\sum_{j\geq n}(j+r)^{-1-\delta}&=\sum_{k\geq 0}\frac{1}{(k+2+r)^{1+\delta}}\sum_{m=0}^{k}\frac{\Lambda(m+2)}{(m+2)^\kappa}\\&=\sum_{n\geq 2}\frac{1}{(n+r)^{1+\delta}}\sum_{2\leq j\leq n}\frac{\Lambda(j)}{j^\kappa}.\end{aligned}\tag{13}$$

Thus, the integral on the right of (11) becomes

$$\begin{aligned}&\int_0^1\sum_{n\geq 2}\frac{\Lambda(n)}{n^\kappa}\sum_{j\geq n}(j+r)^{-1-\delta}e^{-2\pi i r}dr\\&=\int_0^1\sum_{n\geq 2}\frac{1}{(n+r)^{1+\delta}}\sum_{2\leq j\leq n}\frac{\Lambda(j)}{j^\kappa}e^{-2\pi i r}dr.\end{aligned}\tag{14}$$

In order to analyze the right integral of (14), we use the following relation [3]

$$\begin{aligned}\sum_{n<x}\frac{a_n}{n^s}&=\frac{1}{2\pi i}\int_{c-iU}^{c+iU}f(s+w)\frac{x^w}{w}dw+O(x^cU^{-1}(\sigma+c-1)^{-\alpha})\\&\quad+O(U^{-1}\psi(2x)x^{1-\sigma}\log x)+O(U^{-1}\psi(N)x^{1-\sigma}|x-N|^{-1}),\end{aligned}\tag{15}$$

where x is not an integer, N is the integer nearest to x, $c>0$, $\sigma+c>1$, $a_n\ll\psi(n)$ for some non-decreasing ψ, and the series

$$f(s)=\sum_{n\geq 1}\frac{a_n}{n^s},\quad s=\sigma+it,$$

converges absolutely for $\sigma>1$ with

$$\sum_{n\geq 1}\frac{|a_n|}{n^\sigma}\ll(\sigma-1)^{-\alpha}.$$

Choosing $s\mapsto\kappa=\sigma+it$, $a_n\mapsto\Lambda(n)$ (so that $\psi(n)=\log n$), and $x\mapsto n+r$ ($r\in(0,1)$) in (15), we have

$$\begin{aligned}\sum_{j\leq n}\frac{\Lambda(n)}{n^\kappa}&=\frac{1}{2\pi i}\int_{c-iU}^{c+iU}-\frac{\zeta'}{\zeta}(\kappa+w)\frac{(n+r)^w}{w}dw\\&\quad+O(U^{-1}(\log n)^2(n+r)^c|n+r-N|^{-1}(\sigma+c-1)^{-\alpha}).\end{aligned}$$

Furthermore, multiplying both sides by $(n+r)^{-1-\delta}$ $(c<\delta)$ and summing all over the positive integers $n\geq 2$, we get

$$\sum_{n\geq 2}\frac{1}{(n+r)^{1+\delta}}\sum_{j\leq n}\frac{\Lambda(n)}{n^{\kappa}} = \frac{1}{2\pi i}\int_{c-iU}^{c+iU}-\frac{\zeta'}{\zeta}(\kappa+w)\frac{\zeta^*(1+\delta-w,r)}{w}dw + O_r(U^{-1}H(\sigma+c-1)^{-\alpha}), \tag{16}$$

where

$$\zeta^*(1+\delta-w,r)\equiv\sum_{n\geq 2}(n+r)^{-1-\delta+w}$$

and

$$H\equiv\sum_{n\geq 2}(\log n)^2(n+r)^{-1-\delta+c}.$$

By (16), we see that

$$\frac{1}{2\pi i}\int_{c-iU}^{c+iU}-\frac{\zeta'}{\zeta}(\kappa+w)\frac{\zeta^*(1+\delta-w,r)}{w}dw \to \sum_{n\geq 2}\frac{1}{(n+r)^{1+\delta}}\sum_{j\leq n}\frac{\Lambda(n)}{n^{\kappa}}$$

uniformly in $r\in[1/N,1-1/N]$ for any fixed N as $U\to\infty$.

Hence, the right integral of (14) is rewritten as

$$\lim_{N\to\infty}\int_{1/N}^{1-1/N}\sum_{n\geq 2}\frac{1}{(n+r)^{1+\delta}}\sum_{2\leq j\leq n}\frac{\Lambda(j)}{j^{\kappa}}e^{-2\pi ir}dr = \lim_{N\to\infty}\int_{1/N}^{1-1/N}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}-\frac{\zeta'}{\zeta}(\kappa+w)\frac{\zeta^*(1+\delta-w,r)}{w}dwe^{-2\pi ir}dr = \lim_{N\to\infty}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}-\frac{\zeta'}{\zeta}(\kappa+w)\int_{1/N}^{1-1/N}\zeta^*(1+\delta-w,r)e^{-2\pi ir}dr\frac{dw}{w}. \tag{17}$$

But with integration by parts and [1]

$$\zeta^*(\sigma+it,r)\ll|t|^{1/2-\sigma},\quad \sigma<0,\quad r\in[0,1], \tag{18}$$

we have for $c<\delta$,

$$\int_{1/N}^{1-1/N}\zeta^*(1+\delta-w,r)e^{-2\pi ir}dr \ll \sup_{r\in(0,1)}|\zeta^*(\delta-w,r)|(\delta-w)^{-1}\ll(\delta-w)^{-1/2-\eta} \tag{19}$$

for some $\eta > 0$, and so the last integral $\int_{(c)}$ in (17) converges absolutely. This in turn enables us to put the limit $N \to \infty$ inside the integral symbol $\int_{(c)}$ (use the dominated convergence theorem); we get

$$\begin{aligned}&\int_0^1 \sum_{n\geq 2} \frac{1}{(n+r)^{1+\delta}} \sum_{2\leq j\leq n} \frac{\Lambda(j)}{j^\kappa} e^{-2\pi i r} dr \\ &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} -\frac{\zeta'}{\zeta}(\kappa+w)\int_0^1 \zeta^*(1+\delta-w,r)e^{-2\pi i r}dr\frac{dw}{w}.\end{aligned} \tag{20}$$

Now, suppose that there exist only finitely many nonreal zeros of the zeta-function off the critical line.

Using the residue theorem, we let κ small (make δ large, shift the path to the right, set κ small, and then shift the path to the left) and obtain

$$\begin{aligned}&\lim_{\kappa} \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} -\frac{\zeta'}{\zeta}(\kappa+w)\int_0^1 \zeta^*(1+\delta-w,r)e^{-2\pi i r}dr\frac{dw}{w} \\ &= -\sum_{\beta_j-\kappa>c'} Z(1+\delta-\rho_j+\kappa)\rho_j^{-1} - \frac{\zeta'}{\zeta}(\kappa)\int_0^1 \zeta^*(1+\delta,r)e^{-2\pi i r}dr \\ &+ \frac{1}{2\pi i}\int_{c'-i\infty}^{c'+i\infty} -\frac{\zeta'}{\zeta}(\kappa+w)\int_0^1 \zeta^*(1+\delta-w,r)e^{-2\pi i r}dr\frac{dw}{w} \\ &\equiv -\sum_{\rho_j} Z(1+\delta-\rho_j+\kappa)\rho_j^{-1} + D_{\delta,\kappa} + W_{\delta,\kappa},\end{aligned} \tag{21}$$

where $-1 < c'+\kappa < 0$ and the integral $W_{\delta,\kappa}$ converges for $\delta > 0$ by (18) and integration by parts. Also, the sum on the right (which contains terms associated with the zeros on the critical line) is convergent since

$$Z(1/2+\delta-\rho_j+\kappa) \ll \gamma_j^{-\delta-\kappa}.$$

By (20) and (21), (11) becomes

$$\begin{aligned}&-\sum_{\rho_j} h(\rho_j,\kappa,\delta)(2\pi)^{-\rho_j+\delta}e^{-\pi i(\rho_j-\delta)/2} + E + S_{\delta,\kappa} \\ &= \Gamma(1+\delta)\big(-\sum_{\rho_j} Z(1+\delta-\rho_j+\kappa)\rho_j^{-1} + D_{\delta,\kappa} + W_{\delta,\kappa}\big).\end{aligned} \tag{22}$$

This completes the proof of the lemma. □

Now, Theorem 1 is deduced from the lemma as follows.

Suppose that there is at most one nonreal zero of the zeta-function in the given region. We denote it by ρ_{n_0}.

We can simply put the left and right members of (1) in the form

$$\begin{aligned}&A + \sum_{\beta_j\neq 1/2} h(\rho_j,\kappa,\delta)(2\pi)^{-\rho_j+\delta+\kappa}e^{-\pi i(\rho_j-\delta-\kappa)/2} \\ &= B - \Gamma(1+\delta)\sum_{\beta_j\neq 1/2} Z(1+\delta-\rho_j+\kappa)\rho_j^{-1},\end{aligned} \tag{23}$$

for some complex numbers A and B which are independent of ρ_{n_0}. By *independent*, we mean that they are calculable not using nonreal zeros of the zeta-function off the critical line and take the same values each time we follow the proof of Theorem 1 and reduce the theorem to the form (23), regardless of what value ρ_{n_0} is assumed to take; $-(\zeta'/\zeta)(s)$ $(\mathrm{Re}(s) < 0)$ is calculable by $-(\zeta'/\zeta)(s)$ $(\mathrm{Re}(s) > 1)$, which has the series expression $\sum_{n\geq 1}\Lambda(n)n^{-s}$ and thus not relying on nonreal zeros, via the functional equation for the Riemann zeta-function.

Remark 1. *The use of the word "independent" may be misleading, and it is only appropriate in this specific context. The important point is that A and B can be calculated by some expressions which contain all the zeros of the zeta-function and have good forms (product of fairly easy functions). For instance, the value*

$$-\frac{\zeta'}{\zeta}(1+\delta) = \sum_{\rho_j} \text{(some term depending on } \rho_j\text{)} + g$$

is irrelevant of what terms the sum $\sum_{\rho_j}$ *contains and the left member is a plain Dirichlet series.*

On the other hand, the relation

$$\pi(x) - Li(x) = \sum_{\rho_j} \text{(some term depending on } \rho_j\text{)} + g$$

also has an expression involving all the zeros of the zeta-function, but it is not known if the left member has a good easy-to-handle form.

Note that each sum in (23) consists of only four terms, and it has a dominant term among its four terms, respectively.

By the definition, it is easy to see that the terms h's on the left decay rapidly (in the sense of Fourier analysis) as the hypothetical zero ρ_{n_0} is assumed to be large.

On the other hand, we have with the transformation (9),

$$\begin{aligned} Z(1+\delta-\rho_j+\kappa) &= \int_0^1 \zeta^*(1+\delta-\rho_j+\kappa, r)e^{-2\pi i r}dr \\ &= \int_2^\infty e^{-2\pi i w} w^{-1-\delta-\kappa+\rho_j}dw \end{aligned}$$

which can be easily shown by Cauchy's theorem on contour integrals as

$$\begin{aligned} &\sim (2\pi)^{\delta+\kappa-\rho_j}(-i)^{-\delta-\kappa+\rho_j}\Gamma(\rho_j-\delta-\kappa) \\ &= (2\pi)^{\delta+\kappa-\rho_j}(-i)^{-\delta-\kappa+\beta_j}\Gamma(\rho_j-\delta-\kappa)e^{\pi\gamma_j/2} \\ &\asymp |\gamma_j|^{\beta_j-\delta-\kappa-1/2}, \quad \gamma_j \to \infty. \end{aligned}$$

We consider the two cases when (a) $A = B$ and (b) $A \neq B$.

Assume (a). Then canceling A and B in (23), we have a relation of the form

$$\text{a rapidly decreasing function of } \gamma_{n_0} = |\gamma_{n_0}|^{-C}$$

for some $C > 0$, which leads to a contradiction as γ_{n_0} goes to ∞.

Next, assume (b). Then $A - B = \epsilon$ for some nonzero complex number ϵ. The numbers A and B are independent of the hypothetical zero, and so the terms associated with it can not vanish, which proves the existence of a bound for γ_{n_0}.

This completes the proof of the theorem.

3 The second topic

In this section, I discuss on the *second topic* mentioned in the Introduction.

We replace δ by $\delta + iT$ $(T > 0)$ and multiply by $e^{\pi T/2}$ in the lemma. Then we get

$$\begin{aligned}
&-\sum_{\rho_j} h(\rho_j, \kappa, \delta + iT)(2\pi)^{-\rho_j+\delta+iT+\kappa} e^{-\pi i(\rho_j-\delta-\kappa)/2} + e^{\pi T/2}E + e^{\pi T/2}S_{\delta+iT,\kappa} \\
&= \Gamma(1+\delta+iT)e^{\pi T/2}\big(-\sum_{\rho_j} Z(1+\delta+iT-\rho_j+\kappa)\rho_j^{-1} + D_{\delta+iT,\kappa} + W_{\delta+iT,\kappa}\big),
\end{aligned} \tag{24}$$

It is easy to show that

$$e^{\pi T/2}E, \quad \Gamma(1+\delta+iT)e^{\pi T/2}D_{\delta+iT,\kappa} \ll T^{\delta}, \quad T \to \infty,$$

and

$$\begin{aligned}
&e^{\pi T/2}|S_{\delta+iT,\kappa}| \\
&= e^{\pi T/2}\Big|\int_{(-3+h)} -\frac{\zeta'}{\zeta}(s+\delta+iT+\kappa)\Gamma(s-iT) \\
&\times \Gamma(s+\delta)\Gamma(1-s+iT)(2\pi)^{-s+iT}e^{-\pi(s-iT)i/2}ds\Big| \\
&\leq A\int_{-\infty}^{\infty} \log(|t|+1)|\Gamma(-3+h+it-iT)\Gamma(-3+h+\delta+it)\Gamma(4-h-it+iT)|e^{\pi t/2}dt \\
&\leq \|\log(|t|+1)|\Gamma(-3+h+\delta+it)|e^{\pi t/2}\| \\
&\times \|\Gamma(-3+h+it-iT)\Gamma(4-h-it+iT)\| \\
&\ll 1, \quad T \to \infty,
\end{aligned}$$

where we used the change of variables in the first equality and

$$\|f\| \equiv \sqrt{\int |f(t)|^2 dt}.$$

Hence, we have

$$
\begin{aligned}
&-\sum_{\rho_j} h(\rho_j,\kappa,\delta+iT)(2\pi)^{-\rho_j+\delta+\kappa+iT}e^{-\pi i(\rho_j-\delta-\kappa)/2}+O(T^\delta)\\
&=\Gamma(1+\delta+iT)e^{\pi T/2} \qquad\qquad (25)\\
&\times\big(-\sum_{\rho_j} Z(1+\delta+iT-\rho_j+\kappa)\rho_j^{-1}+D_{\delta+iT,\kappa}+W_{\delta+iT,\kappa}\big).
\end{aligned}
$$

To analyze $W_{\delta+iT,\kappa}$, we need the following lemma.

Lemma 2. *Let f_h satisfy*

$$|f_h(t)|\le A(|t|+1)^{-1/2-h},\quad t\in\mathbb{R},\quad 0<h<1/2,\quad A>0,$$

and $u(t)\equiv(|t|+1)^{-1}$. Then there exists an absolute constant D such that

$$|(f_h*u)(t)|\le Dt^{-1/2}$$

for all sufficiently large t.

Proof. It is straightforward that

$$|(f_h*u)(H)|\le A\int_{-\infty}^{\infty}(|t-H|+1)^{-1/2-h}(|t|+1)^{-1}dt.$$

We separate the convolution into

$$\int_{-\infty}^{\infty}(|t-H|+1)^{-1/2-h}(|t|+1)^{-1}dt=\int_{-\infty}^{0}+\int_0^H+\int_H^\infty\equiv I_1+I_2+I_3.$$

Now,

$$
\begin{aligned}
I_1&=\int_{-\infty}^{0}(|t-H|+1)^{-1/2-h}(|t|+1)^{-1}dt\\
&\le(H+1)^{-1/2}\int_{-\infty}^{0}(|t-H|+1)^{-h}(|t|+1)^{-1}dt\\
&\le(H+1)^{-1/2}\int_{-\infty}^{0}(|t|+1)^{-h}(|t|+1)^{-1}dt.
\end{aligned}
$$

For I_3, using Cauchy-Schwartz inequality,

$$
\begin{aligned}
I_3&=\int_H^\infty(|t-H|+1)^{-1/2-h}(|t|+1)^{-1}dt\\
&\le\sqrt{\int_H^\infty(|t-H|+1)^{-1-2h}dt\int_H^\infty(|t|+1)^{-2}dt}\\
&\le H^{-1/2}\sqrt{\int_{-\infty}^{\infty}(|t|+1)^{-1-2h}dt}.
\end{aligned}
$$

Finally, for I_2, we have by the change of variables $t = Hv$

$$\begin{aligned} I_2 &= \int_0^H (|t-H|+1)^{-1/2-h}(|t|+1)^{-1}dt \\ &= \int_0^1 (H|v-1|+1)^{-1/2-h}H(Hv+1)^{-1}dv \\ &= H^{-1/2-h}\int_0^1 (|v-1|+H^{-1})^{-1/2-h}H(Hv+1)^{-1}dv. \end{aligned} \tag{26}$$

Here, we recall the following variation of the Mellin-Barnes integral [4]

$$\frac{2\pi\Gamma(a+b)}{(x+y)^{a+b}} = \int_{-\infty}^{\infty} x^{-a-it}y^{-b+it}\Gamma(a+it)\Gamma(b-it)dt,$$

(valid for $x, y, a, b > 0$) from which we find out that there exists an absolute constant B satisfying

$$(x+y)^{-1} \leq Bx^{-a}y^{-b}, \quad a+b=1, \tag{27}$$

for all $x, y > 0$.

Choosing $x \mapsto Hv$, $y \mapsto 1$, $a = 1-\eta$ ($\eta > 0$), and $b = \eta$ in (27), we have

$$(Hv+1)^{-1} \leq B(Hv)^{-1+\eta},$$

and substituting this into (26), we get

$$\begin{aligned} I_2 &= H^{-1/2-h}\int_0^1 (|v-1|+H^{-1})^{-1/2-h}H(Hv+1)^{-1}dv \\ &\leq BH^{-1/2-h+\eta}\int_0^1 (|v-1|+H^{-1})^{-1/2-h}v^{-1+\eta}dv \\ &\leq BCH^{-1/2}, \quad \eta < h, \end{aligned} \tag{28}$$

for an absolute constant C, since for $h < 1/2$,

$$\lim_{H\to\infty}\int_0^1 (|v-1|+H^{-1})^{-1/2-h}v^{-1+\eta}dv < \infty.$$

Combining all the results for I_1, I_2, and I_3, we see that there exists an absolute constant D such that $|(f_h * u)(t)| \leq Dt^{-1/2}$ for all $t \in \mathbb{R}$.

This completes the proof of the theorem. □

Considering the estimate (19), we can apply Lemma 2 to $W_{\delta+iT,\kappa}$, and find out that $W_{\delta+iT,\kappa} \ll T^{-1/2}$.

Therefore, (25) becomes

$$\begin{aligned} &-\sum_{\rho_j} h(\rho_j,\kappa,\delta+iT)(2\pi)^{-\rho_j+\delta+\kappa+iT}e^{-\pi i(\rho_j-\delta-\kappa)/2} + O(T^{\delta}) \\ &= -\Gamma(1+\delta+iT)e^{\pi T/2}\sum_{\rho_j} Z(1+\delta+iT-\rho_j+\kappa)\rho_j^{-1}. \end{aligned} \tag{29}$$

Now, if he Riemann hypothesis is true, then (29) would take the form

$$O(T^\delta) = -\Gamma(1+\delta+iT)e^{\pi T/2}\sum_{\beta_j=1/2} Z(1+\delta+iT-\rho_j+\kappa)\rho_j^{-1}. \tag{30}$$

Assuming the same condition for the distribution of the zeros off the critical line as in Theorem 1, it is easy to see that the order of magnitude of the left member of (29) is, for $T = \gamma_{n_0}$ and some $\alpha > 0$,

$$\begin{aligned}
&h(\rho_{n_0}, \kappa, \delta + i\gamma_{n_0})(2\pi)^{-\rho_{n_0}+\delta+\kappa+i\gamma_{n_0}} e^{-\pi i(\rho_{n_0}-\delta-\kappa)/2} \\
&\asymp \Gamma(\rho_{n_0} - \delta - i\gamma_{n_0} - \kappa)\Gamma(\rho_{n_0} - \kappa)\Gamma(1 - \rho_{n_0} + \delta + i\gamma_{n_0} + \kappa)e^{-\pi i\rho_{n_0}/2} \\
&\asymp \gamma_{n_0}^{\beta_{n_0}-\kappa-1/2} \gg \gamma_{n_0}^{\delta+\alpha}
\end{aligned}$$

which is strictly larger than the left member of (30), if κ, δ are chosen to be small.

But the order of magnitude of the terms associated with zeros off the critical line on the right side is, for $T = \gamma_{n_0}$, $\asymp \Gamma(1+\delta+i\gamma_{n_0})e^{\pi\gamma_{n_0}/2}\rho_{n_0}^{-1} \asymp \gamma_{n_0}^{-1/2+\delta}$.

Thus, we find out the following proposition.

Proposition 1. *Suppose that the Riemann hypothesis is false and there are only finitely many nonreal zeros $\{\rho_{n_i}\}$ off the critical line. Then regardless of what values ρ_{n_i} take, the order of magnitude of the right member of (29) as $T \asymp \gamma_{n_i}$ coincides with that under the Riemann hypothesis.*

Therefore, it may be claimed that Proposition 1, or the relation (29), is the Riemann hypothesis as long as only finitely many nonreal zeros off the critical line are concerned.

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Note. If there were any comments on this preprint, then I would prefer seeing them on the Unsolved Problem yahoo group.

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