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On the form of an odd perfect number

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It has been known since the time of Euler that an odd perfect number N (if it exists) must have the form $N = p^{a}Q^{2}$ where p is prime and $p = a = 1$ mod 4 (e.g., [1, pp. 3–33]).

Further, it has been shown that N must equal 1 mod 12, or 9 mod 36 ([3], [2]). However, we can do a little better than this.

From either result it is immediately evident that if 3 divides N, then 3^{k} divides N, where $k = 0$ mod 2.

If $k = 0$, then N must be of the form 1 mod 12.

For any positive integer $N = p1^{k1}\, p2^{k2} \ldots pn^{kn}$ the sum S of all of its divisors (including 1 and N itself), is given by

$$S = (1+p1 + p1^{2} + \ldots + p1^{k1})(1 + p2 + p2^{2} + \ldots + p2^{k2}) \ldots (1 + pn + pn^{2} + \ldots + pn^{kn})$$

If N is perfect, it is equal to the sum of its divisors (excepting itself), so $N = S - N$, so $2N = S$.

Thus, if N is perfect, and 3^{k} is a factor of N, then N is itself divisible by $(1 + 3^{1} + 3^{2} + \ldots + 3^{k})$.

If $k = 2$, then N must be of the form 9 mod 36. Further, since N is perfect, from the above we know that $3^{0} + 3^{1} + 3^{2} = 1+3+9 = 13$ must divide $2N$, and hence $N = 0$ mod 13. Thus, N must satisfy both $N = 9$ mod 36 and $N = 0$ mod 13.

We can therefore deduce that N must equal 117 mod 468.

If $k > 2$, then N is divisible by $3^{4} = 81$. Thus, N must satisfy both $N = 9$ mod 36 and $N = 0$ mod 81.

We can therefore deduce that N must equal 81 mod 324.

Thus, if N is an odd perfect number, it must be of the form $N = 1$ mod 12 or $N = 117$ mod 468 or $N = 81$ mod 324.

Of course, it is possible to further refine the last of these results in a similar way, by considering separately values of k greater than or equal to 4.

References

[1] Dickson, L.E. (2005). *History of the Theory of Numbers*, Vol. 1, Divisibility and Primality. Dover, New York.

[2] Holdener, J.A. (2002). A theorem of Touchard and the form of odd perfect numbers. *AmericanMathematical Monthly* **109,** 661–663.

[3] Touchard, J. (1953). On prime numbers and perfect numbers. *Scripta Mathematica* **19,** 35–39.