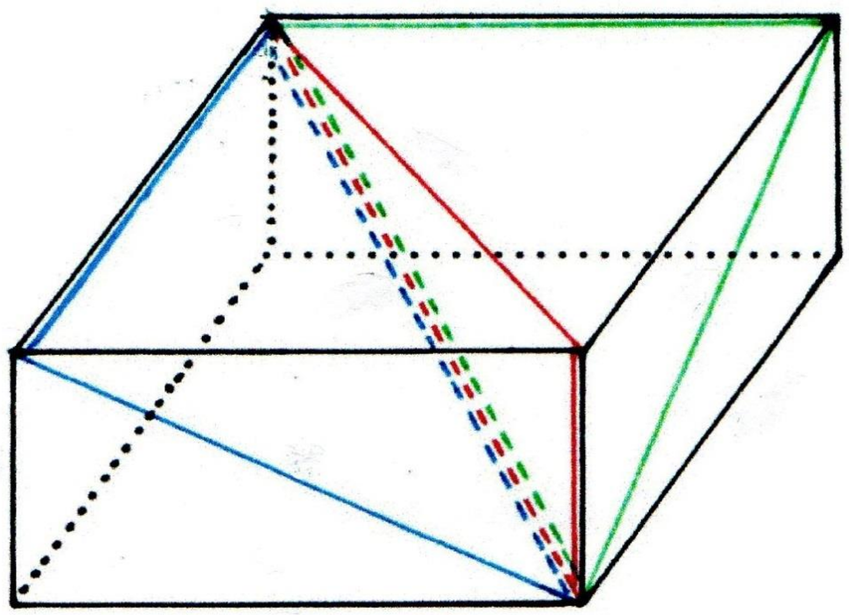

EULER BRICKS

AND

PERFECT CUBOIDS

 $\beta\gamma\beta\alpha\alpha\gamma\alpha\alpha$ 25/10/2017

PYTHAGOREAN TRIPLES AND QUADRUPLES

Does a **Perfect Cuboid** exist? is an unsolved problem in Mathematics.

Definitions

A **Pythagorean Triple** is defined as a right-angled triangle with the two shorter sides $\{x, y\}$ such that $x^2 + y^2 = z^2$ where z is the hypotenuse and all three are integers.

A **Pythagorean Quad** (also known as an **Euler Brick**) is defined as a rectangular parallelepiped (a rectangular box) with side lengths $\{a, b, c\}$ such that $a^2 + b^2 + c^2 = d^2$ where d is the internal space diagonal and all four are integers.

A **Perfect Pythagorean Quad** (also known as a **Perfect Cuboid**), is defined as a Pythagorean Quad where the face diagonals $\{e, f, g\}$ are also integers see FIGURE 1. Does such a Quad exist? As of May 2017, computer searches into the billions for the shortest side a have failed to find a Perfect Pythagorean Quad, nor has it been proved that one does not exist. In the following article it is hoped to prove that no such Cuboid exists. The proof requires a short analysis of Pythagorean Triples.

HISTORY

The first known, and smallest, Euler Brick with side lengths $\{a, b, c\} = (44, 117, 240)$ and face diagonals $\{e, f, g\} = (125, 244, 267)$, was found by Paul Halke (1662 – 1731) in 1719.

Nicholas Saunderson (1682 – 1739), the fourth Lucasian Professor at Cambridge, who was blind from the age of one, found a parametric solution to the Euler Brick. [1]
Given that $\{x, y, z\}$ is a Pythagorean Triple then

$$\{a, b, c\} = \{ y(4x^2 - z^2),\ x(4y^2 - z^2),\ 4xyz\} \text{ and}$$

$$\{e, f, g\} = \{ z^3,\ y(4x^2 + z^2),\ x(4y^2 + z^2)\},$$

e.g.

$$\{x, y, z\} = (3, 4, 5) \text{ produces}$$

$$\{a, b, c\} = \big(4(36 - 25),\ 3(64 - 25),\ (4 \times 3 \times 4 \times 5)\big) = (44, 117, 240)$$

and $$\{e, f, g\} = \big(5^3,\ 4(36 + 25),\ 3(64 + 25)\big) = (125, 244, 267).$$

In the 1770's Euler (1707 – 1783) found a couple of parametrisations and more were found in his papers after his death. It is surprising that Euler made no attempt to solve the Perfect Cuboid problem, but there is nothing in his notes concerning the problem.

Parametrisations will find an infinite number of results but not all, and none of them include the space diagonal d.

There is a great deal of information on the Internet, including at least three proposed proofs; Google Euler Brick or Perfect Cuboid and follow the links.

PYTHAGOREAN TRIPLES

An investigation into Pythagorean Quads requires an analysis of Pythagorean Trip**les**.

Abbreviations (I) = Integer > 0, □ ⇒ (I^2) should be square but not proven.

Pythagoras' Theorem is taken for granted, for a rigorous proof see Euclid 1.XLVII. [2]

A **Pythagorean Triple** (**PT**) is defined as $x^2 + y^2 = z^2$ where x and y are the two shorter sides and z is the hypotenuse of a right-angled triangle and all three are integers.

A **Primitive Pythagorean Triple** (**P^{2}T**) is defined as a **PT** with GCD (x, y) = 1.

Theorem 1 Any integer $x > 2$ with $(0 < w < x)$ and $w \mid x^2$ generates all Pythagorean Triples.

Proof $x^2 + y^2 = z^2$ and without loss of generality assume $x < y$,

then $x^2 = z^2 - y^2 = (z - y)(z + y)$ since $(z - y)(z + y) = z^2 + zy - yz - y^2 = z^2 - y^2$.

Let $z - y = w$ then $z = y + w$ and $x^2 = w(y + w + y) = w(2y + w) = 2yw + w^2$

therefore $y = \dfrac{x^2 - w^2}{2w}$ and $z = y + w = \dfrac{x^2 - w^2}{2w} + \dfrac{2w^2}{2w} = \dfrac{x^2 + w^2}{2w}$. See Table 1.

For $x > 2$, $w \mid x^2$ and for y to be positive, $0 < w < x$.
For $x > 2$ with $(0 < w < x)$ and $w \mid x^2$ generates all Pythagorean Triples. ❑

Notes for Theorem 1. For all Primitive Pythagorean Triples : –
If x is odd, w is odd since the numerator of both y and z must be even to be divisible by 2.
If x is even, w is even since now the numerator of both y and z must be divisible by 4.
If x is odd, w is odd and y is even, if x is even, w is even and y is odd, in both cases z is odd.
For all $x > 2$ there always exists a Triple, if x is odd let $w = 1$, if x is even let $w = 2$.

Theorem 2 **Diophantus'** method for generating Pythagorean Triples $\{X, Y, Z\}$.

Given any u and v with $u < v$ then $X = v^2 - u^2$, $Y = 2uv$ and $Z = v^2 + u^2$.

Proof $X^2 + Y^2 = (v^2 - u^2)^2 + (2uv)^2 = v^4 - 2v^2u^2 + u^4 + 4u^2v^2$

$= v^4 + 2v^2u^2 + u^4 = (v^2 + u^2)^2 = Z^2$. ❑

Theorem 3 Theorems 1 and 2 are equivalent.

Proof From Theorem 2, $(v^2 - u^2)^2 + (2uv)^2 = (v^2 + u^2)^2$, interchanging X and Y and

dividing through by $4u^2$ gives $v^2 + \dfrac{(v^2 - u^2)^2}{4u^2} = \dfrac{(v^2 + u^2)^2}{4u^2}$

and letting $v = x$ and $u = w$ retrieves Theorem 1. ❑

PYTHAGOREAN QUADS

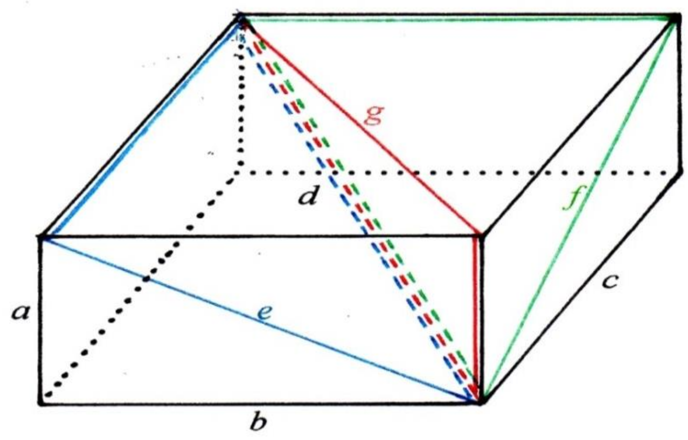

FIGURE 1

A **Pythagorean Quad** (**PQ**) is defined as $a^2 + b^2 + c^2 = d^2$ where a, b and c are the lengths of the three sides and d is the internal space diagonal of a Rectangular Parallelepiped (rectangular box). An Euler Brick is defined when a, b, c and d are integers, see FIGURE 1.

Theorem 4 For an Euler Brick of sides a, b, and c, $a^2 + b^2 + c^2 = d^2$.

Proof By Pythagoras' Theorem $a^2 + b^2 = e^2$ and $e^2 + c^2 = d^2$, use Theorem 1 twice. ❑

There are an infinite number of such Quads, the smallest being {1, 2, 2, 3}. See Appendix 2.

A **Perfect Pythagorean Quad** (**P²Q**) is defined as a **PQ** where the edges, the internal space diagonal and all three face diagonals $\{e, f, g\}$ are also integers. Does such a Quad exist ?

A **Primitive Perfect Pythagorean Quad** (**P³Q**) is defined as a **P²Q** with GCD $(a, b, c) = 1$.
By the definition of a **P²T**, one of a, b or c is odd and the other two are even, hence d is odd.

From Figure 1, with e, f and g as the face diagonals

$$a^2 + b^2 + c^2 = d^2 \quad \text{and} \quad a^2 + b^2 = e^2, \quad a^2 + c^2 = f^2, \quad b^2 + c^2 = g^2$$

also $$a^2 + g^2 = d^2, \; b^2 + f^2 = d^2 \; \text{and} \; c^2 + e^2 = d^2$$

i.e. $\{a, b, e\}$, $\{a, c, f\}$, $\{b, c, g\}$, $\{a, g, d\}$, $\{b, f, d\}$ and $\{c, e, d\}$ all require to be Pythagorean Triples for a Perfect Cuboid to exist.

Some of the Triples can be taken in sets of three to generate all of the values of a Pythagorean Quad. Assuming that at least one of the values must be an integer, take that to be the side a.

In general, take a to be the smallest side and $a < b < c$ by making $0 < m < l < a$ when deriving Triples as per Theorem 1.

Consider the set $\{\{a, b, e\}, \{a, c, f\}, \{a, g, d\}\}$. From Theorem 1, with a = (I),

$$b = \frac{a^2 - l^2}{2l} \qquad e = \frac{a^2 + l^2}{2l} \qquad c = \frac{a^2 - m^2}{2m} \qquad f = \frac{a^2 + m^2}{2m}$$

hence $e - b = \dfrac{a^2 + l^2}{2l} - \dfrac{a^2 - l^2}{2l} = l$ and $f - c = \dfrac{a^2 + m^2}{2m} - \dfrac{a^2 - m^2}{2m} = m$,

therefore $e = b + l$ and $f = c + m$, where $0 < m < l < a$ and m and l are different divisors of $a^2 \Rightarrow$ a, b, c, e and f are all integers for some integer a.

In the following, let $\delta^{\square}$ and $\gamma^{\square}$ be as shown below in the brackets under the root signs for d and g respectively, where $\delta^{\square}$ and $\gamma^{\square}$ must be, may be, should be or are square but not proven. δ (delta) and γ (gamma) are the Greek letters for d and g respectively.

Now $$d^2 = c^2 + e^2 = \frac{(a^2 - m^2)^2}{4m^2} + \frac{(a^2 + l^2)^2}{4l^2} = \frac{l^2(a^2 - m^2)^2 + m^2(a^2 + l^2)^2}{4l^2m^2}$$

hence $d = \dfrac{1}{2lm}\sqrt{l^2(a^2 - m^2)^2 + m^2(a^2 + l^2)^2 \;(= \delta^{\square})}$, for d = (I) → Tables 2 & 3.

Also $$d^2 = f^2 + b^2 = \frac{(a^2 + m^2)^2}{4m^2} + \frac{(a^2 - l^2)^2}{4l^2} = \frac{l^2(a^2 + m^2)^2 + m^2(a^2 - l^2)^2}{4l^2m^2}$$

and $d = \dfrac{1}{2lm}\sqrt{l^2(a^2 + m^2)^2 + m^2(a^2 - l^2)^2 \;(= \delta^{\square})}$

$= \dfrac{1}{2lm}\sqrt{(a^4 + l^2m^2)(l^2 + m^2) \;(= \delta^{\square})}$. See Note 2.

Let $(a^4 + l^2m^2) = p$ and $(l^2 + m^2) = q$.

For d = (I) see Table 3 for p and q = (I) and Table 4 for p and $q \neq$ (I) but with common √.

Now $$g^2 = b^2 + c^2 = \frac{(a^2 - l^2)^2}{4l^2} + \frac{(a^2 - m^2)^2}{4m^2} = \frac{m^2(a^2 - l^2)^2 + l^2(a^2 - m^2)^2}{4l^2m^2}$$

hence $g = \dfrac{1}{2lm}\sqrt{l^2(a^2 - m^2)^2 + m^2(a^2 - l^2)^2 \;(= \gamma^{\square})}$, for g = (I) → Table 4.

$= \dfrac{1}{2lm}\sqrt{(a^4 + l^2m^2)(l^2 + m^2) - 4a^2l^2m^2 \;(= \gamma^{\square})}$. See Note 3.

Note 1 $l^2(a^2 \pm m^2)^2 + m^2(a^2 \mp l^2)^2 = \delta^{\square}$ and $l^2(a^2 - m^2)^2 + m^2(a^2 - l^2)^2 = \gamma^{\square}$ and since all three expressions are derived from a^2, b^2, c^2, e^2 and f^2, which are all square by construction, then for a Perfect Pythagorean Quad all three expressions require to be Pythagorean Triples.

Three versions are shown for $\delta^{\square}$ and two for $\gamma^{\square}$ and others will be developed for Theorem 5. Since all versions are numerically equal, if any one version of $\delta^{\square} \neq$ square then $d \neq$ an integer and if any one version of $\gamma^{\square} \neq$ square then $g \neq$ an integer.

Note 2 $l^2(a^2 \pm m^2)^2 + m^2(a^2 \mp l^2)^2 = (a^4 + l^2m^2)(l^2 + m^2)$.

See reference [4] and Appendix 4.

Proof LHS $= l^2(a^2 \pm m^2)^2 + m^2(a^2 \mp l^2)^2$

$= l^2a^4 \pm 2a^2l^2m^2 + l^2m^4 + m^2a^4 \mp 2a^2l^2m^2 + m^2l^4$

$= l^2(a^4 + l^2m^2) + m^2(a^4 + l^2m^2) = (a^4 + l^2m^2)(l^2 + m^2) =$ RHS. ❑

Note 3 $l^2(a^2 - m^2)^2 + m^2(a^2 - l^2)^2 = (a^4 + l^2m^2)(l^2 + m^2) - 4a^2l^2m^2$.

Proof LHS $= l^2(a^2 - m^2)^2 + m^2(a^2 - l^2)^2$

$= l^2a^4 - 2a^2l^2m^2 + l^2m^4 + m^2a^4 - 2a^2l^2m^2 + m^2l^4$

$= l^2(a^4 + l^2m^2) + m^2(a^4 + l^2m^2) - 4a^2l^2m^2$

$= (a^4 + l^2m^2)(l^2 + m^2) - 4a^2l^2m^2 =$ RHS. ❑

Theorem 5 A **Perfect Pythagorean Quad** does not exist, but first a trio of Lemmas.

Lemma 1 The product of two squares is also a square.

Proof $x^2 \times y^2 = (xy)^2$ for all x and y greater than 0. ❑

Lemma 2 The product of two numbers, one square and the other non-square is not a square.

Proof $x^2 \times y^2 = (xy)^2 \Rightarrow x^2 \times y \neq$ square for both x and $y > 1$ and $y \neq$ square. ❑

Lemma 3 $a^4 + l^4 \neq z_l^4$ and $a^4 + m^4 \neq z_m^4$.

That the sum of two fourth powers does not equal a fourth power was proved by Fermat in or about 1637 (by his method of Infinite Descent), but was definitively proved by Professor Wiles in 1995 for all values of n greater than two. In this case $n = 4$.

Proof

Assume for a contradiction, $a^4 + l^4 = z_l^4$ then $(a^2)^2 = (z_l^2)^2 - (l^2)^2 = (z_l^2 - l^2)(z_l^2 + l^2)$ where both $(z_l^2 - l^2)$ and $(z_l^2 + l^2)$ are square by Lemma 1. Now $(z_l^2 + l^2)$ is a Pythagorean Triple where l is less than a and therefore less than z_l. Turning this round, $l^2 + z_l^2$ is a square

and by Theorem 1, $z_l = \dfrac{l^2 - r^2}{2r}$ where r is a divisor of l^2. Substituting this value for z_l in

$(z_l^2 - l^2)$ gives $\dfrac{l^2 - r^2}{2r} - l^2 = \dfrac{l^2 - r^2 - 2l^2r}{2r}$ which is negative hence $(z_l^2 - l^2)(z_l^2 + l^2)$

is also negative which is ridiculous since $a^4 + l^4$ is positive. Hence $a^4 + l^4 \neq$ a fourth power, and similarly $a^4 + m^4 \neq$ a fourth power. Therefore $a^4 + l^4$ and $a^4 + m^4$ are not square, since $z^4 = (z^2)^2 = z^2 \times z^2$. Also $m^2(a^4 + l^4)$ and $l^2(a^4 + m^4)$ are not square, by Lemma 2.

Repeating the argument for $(z_l^2 - l^2)$ gives $\dfrac{l^2 + r^2}{2r} + l^2 = \dfrac{l^2 + r^2 + 2l^2r}{2r} \neq$ square etc. ❑

Theorem 5 A **Perfect Pythagorean Quad** does not exist.

Proof Part 1 Consider $\delta^{\square}$ and $\gamma^{\square}$ the square root parts of d and g respectively.

$$\delta^{\square} = m^2 (a^2 \pm l^2)^2 + l^2 (a^2 \mp m^2)^2 \Rightarrow \text{two Pythagorean Triples} \qquad \text{equation 1}$$

and $$\gamma^{\square} = m^2 (a^2 - l^2)^2 + l^2 (a^2 - m^2)^2 \Rightarrow \text{one Pythagorean Triple} \qquad \text{equation 2.}$$

$$\delta^{\square} - \gamma^{\square} = m^2 (a^2 + l^2)^2 - m^2(a^2 - l^2)^2 = l^2(a^2 + m^2)^2 - l^2(a^2 - m^2)^2 = 4a^2 l^2 m^2,$$

hence $\delta^{\square} = \gamma^{\square} + 4a^2 l^2 m^2$ and $\gamma^{\square} = \delta^{\square} - 4a^2 l^2 m^2$.

Proof Part 2 Assume that $d = (\mathrm{I}) \Rightarrow \delta^{\square} = \delta^2$

then $$\gamma^{\square} = m^2 (a^2 + l^2)^2 - 2a^2 l^2 m^2 + l^2 (a^2 - m^2)^2 - 2a^2 l^2 m^2$$

$$= m^2 (a^4 + l^4) + l^2 (a^4 - 4\, a^2 m^2 + m^4) \Rightarrow \gamma^{\square} \text{ is not a Pythagorean Triple}$$

since $m^2 (a^4 + l^4) \neq$ square by Lemmas 1, 2 and 3, and $l^2(a^4 - 4\, l^2 m^2 + m^4) \neq$ square since to be square the expression must be of the form $(a^4 - 2\, l^2 m^2 + m^4) = (a^2 - m^2)^2$ and Lemma 2 applies to the product with l^2.

Hence, given that $d = (\mathrm{I})$ then $\gamma^{\square}$ is not the sum of two squares and is not a Pythagorean Triple, a requirement by equation 2. See Example 1 and Tables 2 and 3.

Proof Part 3 Assume that $g = (\mathrm{I}) \Rightarrow \gamma^{\square} = \gamma^2$

then $$\delta^{\square} = \gamma^2 + 4a^2 l^2 m^2 = m^2 (a^2 - l^2)^2 + 2a^2 l^2 m^2 + l^2 (a^2 - m^2)^2 + 2a^2 l^2 m^2$$

$$= m^2(a^4 + l^4) + l^2(a^4 + m^4) \Rightarrow \delta^{\square} \text{ is not a Pythagorean Triple,}$$

since $m^2(a^4 + l^4) \neq$ square and $l^2(a^4 + m^4) \neq$ square by Lemmas 1, 2 and 3.

Hence, given that $g = (\mathrm{I})$ then $\delta^{\square}$ is not the sum of two squares and is not a Pythagorean Triple, a requirement by equation 1. See Example 2 and Table 4.

Parts 2 and 3 imply $\{a, g, d\}$ is not Pythagorean Triple and hence a Perfect Pythagorean Quad does not exist.

Theorem 5 only proves $\{a, g, d\}$ is not a right-angled triangle, but individually $\delta^{\square}$ or $\gamma^{\square}$ may be square by chance (the sum of two non-square numbers may be square e.g. $5 + 11 = 4^2$). In this case, geometrically, the triangle will not be right angled and the parallelepiped will be skewed along either the d or g axes and will not be rectangular. Hence, for a rectangular parallelepiped, one of $\delta^{\square}$ or $\gamma^{\square}$ cannot be a square and therefore one of d or g cannot be an integer. ❑

A similar proof can be applied to triangles $\{c, e, d\}$ and $\{b, f, d\}$ with similar results, consider the sets $\{\{a, b, e\}, \{a, c, f\}, \{c, e, d\}\}$ and $\{\{a, b, e\}, \{a, c, f\}, \{b, f, d\}\}$.

Example 1 From Table 3 $a = 104$; $l = 32$; $m = 8$; $(2\,l\,m) = 512$;

$$\delta^{\square} = m^2(a^4 + l^4) + l^2(a^4 + m^4)$$

$$= 64 \times 118034432 + 1024 \times 116989952$$

$$= (512\sqrt{28817})^2 + (2048\sqrt{28562})^2$$

$$d = \sqrt{64 \times 118034432 + 1024 \times 116989952}/(2\,l\,m) = 697$$

$$\gamma^{\square} = m^2(a^4 + l^4) + l^2(a^4 - 4a^2m^2 + m^4)$$

$$= 64 \times 118034432 + 1024 \times 114221056$$

$$= (512\sqrt{28817})^2 + (2048\sqrt{27886})^2$$

$$g = \sqrt{64 \times 118034432 + 1024 \times 114221056}/(2\,l\,m) = 3\sqrt{52777}$$ ❑

Example 2 From Table 4 $a = 44$; $l = 8$; $m = 4$; $(2\,l\,m) = 64$;

$$\delta^{\square} = m^2(a^4 + l^4) + l^2(a^4 + m^4)$$

$$= 16 \times 3752192 + 64 \times 3748352$$

$$= (64\sqrt{14657})^2 + (128\sqrt{14642})^2$$

$$d = \sqrt{16 \times 3752192 + 64 \times 3748352}/(2\,l\,m) = 5\sqrt{2929}.$$

$$\gamma^{\square} = m^2(a^4 + l^4) + l^2(a^4 - 4a^2m^2 + m^4)$$

$$= 16 \times 3752192 + 64 \times 3624448$$

$$= (64\sqrt{14657})^2 + (128\sqrt{14156})^2$$

$$g = \sqrt{60035072 + 231964672}\,/\,(2\,l\,m) = 267.$$ ❑

References

1 N. Saunderson, The Elements of Algebra, volume 2, Cambridge, 1740.

2 Euclid's Elements, Dover Publications Inc. 2nd edition.

3 Mathematica™, a mathematical programming language by Wolfram Research.

4 For some interesting information on Pythagorean Triples and Quads see Recreations in the Theory of Numbers by Albert H. Beiler (Dover Publications 2nd edition 1964) chapters XIV and XV.

5 Professor Ian Stewart, Cabinet of Mathematical Curiosities page 58. (Profile Books 2008).

APPENDIX 1   PROGRAMS AND TABLES

Although great care has been taken in the preparation of the tables some transcription errors may still exist. The author takes full responsibility for any such errors and offers his apologies. Tables of course can only produce a CONJECTURE and **NOT** a THEOREM.

The tables were produced with software written using Mathematica™ [3].

Program 1   Pythagorean Triples, produces Table 1.

```
For [x = 2, x < 1002, x++,
   k = Table[Select[Divisors[x^2],# > 0 && # < x &]];
  lk = Length[k];
  For[i = 1, i < lk, i++, w = k[[i]];
    y = (x^2 - w^2)/(4 w^2);
    z = Sqrt[x^2 + y^2];
    If[IntegerQ[y] && y > 0 && GCD[x, y, z] < 2 && y > x && w > 2,
     Print["   x = ", x, "   y = ", y, "   w = ", w]
    ]]]
```

Program 2   Pythagorean Quads, produces Tables 2, 3 and 4.

```
For[a = 1, a < 1001, a++,
  k   = Table[Select[Divisors[a^2], # > 0 && # < a &]];
  lk  = Length[k];
  For[i = 2, i <= lk, i++, l = k[[i]];
    For[j = 1, j < i, j++, m = k[[j]];
      b = (a^2 - l^2)/(2*l);   c = (a^2 - m^2)/(2*m);
      e = b + l;               f = c + m;
      g = Sqrt[b^2 + c^2];     d = Sqrt[e^2 + c^2];
      p = Sqrt[a^4 + l^2*m^2]; q = Sqrt[l^2 + m^2];
      If[IntegerQ[x] && GCD[a,b,c]<2 && (a<b<c),       (* 1 *)
        Print[ " a = ", a, " b = ", b, "  c = ", c]
        Print[ " e = ", e, " f = ", f, "  g = ", g]
        Print[ " d = ", d, " l = ", l, "  m = ", m]
        Print[ " p = ", p, " q = ", q]                 (* 2 *)
        Print[]]
   ] ] ]]
```

Put comment brackets `(*      *)` around whichever statements are not required.
e.g. `(*&&(a<b<c)*)`.

`(* 1 *)`   For `IntegerQ[x]` use `IntegerQ[g]` or `IntegerQ[d]` as required.
`(* 2 *)`   Not required for `IntegerQ[g]`.

Notes The `GCD[a,b,c]<2` instruction eliminates all multiples such that *a*, *b*, and *c* are always relatively prime.
The `(a<b<c)` instruction ensures that only one instance of (*a*, *b*, *c*) occurs, thus eliminating occurrences of values beginning with *b* or *c*. e.g.

a	*b*	*c*	*e*	*f*	*g*	*d*	*l*	*m*
44	117	240	125	244	267	$5\sqrt{2929}$	8	4
117	44	240	125	267	244	$5\sqrt{2929}$	81	27
240	44	117	244	267	125	$5\sqrt{2929}$	200	150

TABLE 1 Pythagorean Triples

$w < 3$ eliminates cases where $z - y > 2$

#	x	y	z	w
1	3	4	5	1
2	5	12	13	1
3	7	24	25	1
4	8	15	17	2
5	9	40	41	1
6	11	60	61	1
7	12	35	37	2
8	13	84	85	1
9	15	112	113	1
10	16	63	65	2
11	17	144	145	1
12	19	180	181	1
13	20	99	101	2
14	21	220	221	1
15	23	264	265	1
16	24	143	145	2
17	25	312	313	1
18	27	364	365	1
19	28	195	197	2
20	29	420	421	1
21	31	480	481	1
22	32	255	257	2
23	33	544	545	1
24	35	612	613	1
25	36	323	325	2
26	37	684	685	1
27	39	760	761	1
28	40	399	402	2
29	41	840	841	1
30	43	924	925	1
31	44	483	485	2
32	45	1012	1013	1
33	47	1104	1105	1
34	48	575	577	2
35	49	1200	120	1
36	51	1300	1301	1
37	52	675	677	2
38	53	1404	1405	1
39	55	1512	1513	1
40	56	783	785	2
41	57	1624	1625	1
42	59	1740	1741	1
43	60	899	901	2
44	61	1860	1861	1
45	63	1984	1985	1
46	64	1023	1025	2
47	65	2112	2113	1
48	67	2244	2245	1
49	68	1155	1157	2
50	69	2380	2381	1
51	71	2529	2521	1
52	72	1295	1297	2
53	73	2664	2665	1
54	74	1368	1370	2

$w > 2$ eliminates cases where $z - y < 3$

#	x	y	z	w
1	20	21	29	8
2	28	45	53	8
3	33	56	65	9
4	36	77	85	8
5	39	80	89	9
6	44	117	125	8
7	48	55	73	18
8	51	140	149	9
9	52	165	173	8
10	57	176	185	9
11	60	91	109	18
12	60	221	229	8
13	65	72	97	25
14	68	285	293	8
15	69	260	269	9
16	75	308	317	9
17	76	357	365	8
18	84	187	205	18
19	84	437	445	8
20	85	132	157	25
21	87	416	425	9
22	88	105	137	32
23	92	525	533	8
24	93	476	485	9
25	95	168	193	25
26	96	247	265	18
27	100	621	629	8
28	104	153	185	32
29	105	208	233	25
30	105	608	617	9
31	108	725	733	8
32	111	680	689	9
33	115	252	277	25
34	116	837	845	8
35	119	120	169	49
36	120	209	241	32
37	120	391	409	18
38	123	836	845	9
39	124	957	965	8
40	129	920	929	9
41	132	475	493	18
42	132	1085	1093	8
43	133	156	205	49
44	135	352	377	25
45	136	273	305	32
46	140	271	221	50
47	140	1221	1229	8
48	141	1100	1109	9
49	145	408	433	25
50	147	1196	1205	9
51	148	1365	1373	8
52	152	345	377	32
53	155	468	493	25
54	155	372	403	31 ≠ □ or 2 × □

$x = 105$ the product of three primes $3 \times 5 \times 7$

#	x	y	z	w
1	105	36	111	$75 = 5^2 \times 3$
2	105	56	119	$63 = 7 \times 3^2$
3	105	88	137	$49 = 7^2$
4	105	100	145	$45 = 5 \times 3^2$
5	105	140	175	$35 = 7 \times 5$
6	105	208	233	$25 = 5^2$
7	105	252	273	$21 = 7 \times 3$
8	105	360	375	$15 = 5 \times 3$
9	105	608	617	$9 = 3^2$
10	105	784	791	7
11	105	1100	1105	5
12	105	1836	1839	3
13	105	5512	5513	1

$x = 1001$ the product of three primes $7 \times 11 \times 13$

#	x	y	z	w
1	1001	168	1015	$847 = 11^2 \times 7$
2	1001	468	1105	$637 = 13 \times 7^2$
3	1001	660	1199	$539 = 11 \times 7^2$
4	1001	2880	3049	$169 = 13^2$
5	1001	3432	3575	$143 = 13 \times 11$
6	1001	4080	4201	$121 = 11^2$
7	1001	5460	5551	$91 = 13 \times 7$
8	1001	6468	6545	$77 = 11 \times 7$
9	1001	10200	10249	$49 = 7^2$
10	1001	38532	38545	13
11	1001	45540	45551	11
12	1001	71568	71575	7
13	1001	501000	501001	1

TABLE 2 $\{(a,b,e),(a,c,f),(a,g,d)\}$. Values for d, p and q = (I).

#	*a*	*b*	*c*	*e*	*f*	*g*	*d*	*l*	*m*	*p*	*q*
1	840	448	495	952	975	$\sqrt{445729}$	1073	504	480	745920	696
2	1680	3404	4653	3796	4947	$5\sqrt{1329505}$	6005	392	294	2824752	490
3	1680	1925	2052	2555	2652	$\sqrt{7916329}$	3277	630	600	2847600	870
4	1680	819	3740	1869	4100	$\sqrt{14658361}$	4181	1050	360	2847600	1110
5	2640	2275	2772	3485	3828	$7\sqrt{262441}$	4453	1210	1056	7085760	1606
6	6552	2464	30225	7000	30927	$\sqrt{919621921}$	31025	4536	702	43046640	4590
7	10920	9152	16065	14248	19425	$\sqrt{341843329}$	21473	5096	3360	120469440	6104
8	21840	14651	37620	26299	43500	$\sqrt{1629916201}$	45901	11648	5880	481877760	13048
9	21840	2925	12628	22035	25228	$\sqrt{168022009}$	25397	19110	12600	534315600	22890
10	24480	27776	35343	37024	42993	$35\sqrt{1649497}$	51185	9248	7650	603432000	12002
11	27720	9504	12103	29304	30247	$5\sqrt{9472345}$	31705	19800	18144	848232000	26856
12	27720	1632	68585	27768	73975	$\sqrt{4706565649}$	73993	26136	5390	781205040	26686
13	31920	31185	63232	44625	70832	$\sqrt{4970790049}$	77393	13440	7600	1023993600	15440
14	36960	3528	10175	37128	38335	$\sqrt{115977409}$	38497	33600	28160	1661721600	43840
15	42840	9120	37961	43800	57239	$\sqrt{1524211921}$	57961	34680	19278	1953246960	39678
16	43680	24957	76076	50307	87724	$5\sqrt{256416385}$	91205	25350	11648	1930656000	27898
17	50160	3843	76076	50307	91124	$35\sqrt{4736593}$	91205	46464	15048	2611369728	48840
18	55440	127281	153920	138831	163600	$\sqrt{39891819361}$	207281	11550	9680	3075626400	15070
19	55440	43549	53580	70499	77100	$\sqrt{4767331801}$	88549	26950	23520	3138273600	35770
20	60192	34944	36575	69600	70433	$7\sqrt{52220689}$	78625	34656	33858	3808347840	48450
21	63840	40120	92169	75400	112119	$\sqrt{10104738961}$	119081	35280	19950	4135874400	40530
22	63840	17575	238392	66215	246792	$\sqrt{57139626289}$	247417	48640	8400	4095974400	49360
23	67320	9975	41888	68055	79288	$7\sqrt{37838881}$	79913	58080	37400	5025662400	69080
24	68640	99099	345100	120549	351860	$7\sqrt{2630910649}$	365549	21450	6760	4713680400	22490
25	82800	27692	352275	87308	361875	$7\sqrt{2548255561}$	362933	59616	9600	6879686400	60384
26	97440	51205	420732	110075	431868	$\sqrt{179637367849}$	434893	58870	11136	9517159680	59914
27	110880	229215	340472	254625	358072	$\sqrt{168460699009}$	425153	25410	17600	12302505600	30910
28	131040	136747	342804	189397	366996	$5\sqrt{5448572977}$	391645	52650	24192	17218656000	57942
29	146160	122496	465647	190704	488047	$5\sqrt{9273295945}$	503185	68208	22400	21417312000	71792
30	150480	100947	105196	181203	183604	$35\sqrt{17352241}$	209525	80256	78408	23502327552	112200
31	177072	110925	219604	208947	282100	$\sqrt{60530272441}$	303125	98022	62496	31947330240	116250
32	205632	144976	428175	251600	474993	$\sqrt{204351871201}$	496625	106624	46818	42578161920	116450
33	210672	71725	185196	222547	280500	$\sqrt{39442034041}$	289525	150822	95304	46652261040	178410
34	221760	289068	307195	364332	378875	$\sqrt{177929076649}$	476557	75264	71680	49472532480	103936
35	221760	133300	550221	258740	593229	$\sqrt{320512038841}$	608021	125440	43008	49472532480	132608
36	251328	418304	546975	488000	601953	$\sqrt{474159887041}$	733025	69696	54978	63281877120	88770
37	271320	195975	227392	334695	354008	$\sqrt{90113322289}$	404633	138720	126616	75680915520	187816
38	286440	314545	410688	425425	500712	$\sqrt{267603190369}$	591313	110880	90024	82652834880	142824
39	314160	700749	2003620	767949	2028100	$\sqrt{4505542265401}$	2145749	67200	24480	98710214400	71520
40	317856	611667	804100	689325	864644	$\sqrt{1020713328889}$	1059125	77658	60544	101141779200	98470
41	341880	223839	769600	408639	842120	$\sqrt{642388057921}$	871361	184800	72520	117647745600	198520
42	351120	457691	1649340	576859	1686300	$17\sqrt{10137728329}$	1747309	119168	36960	123363905280	124768
43	364320	434700	1467389	567180	1511939	$7\sqrt{47799889129}$	1573189	132480	44550	132860217600	139770
44	371280	1679040	1693489	1719600	1733711	$13\sqrt{33651362809}$	2413489	40560	40222	137858491680	57122
45	371280	32868	415915	372732	557525	$13\sqrt{1029973921}$	558493	339864	141610	146008973040	368186
46	393120	108225	150568	407745	420968	$\sqrt{34383373249}$	434657	299520	270400	174479385600	403520
47	425040	137655	311168	446775	526768	$\sqrt{115774423249}$	544457	309120	215600	192560121600	376880
48	432432	422125	1055124	604307	1140300	$\sqrt{1291476171001}$	1215925	182182	85176	187640172720	201110
49	443520	44321	1802640	445729	1856400	$\sqrt{3251475320641}$	1856929	401408	53760	197890129920	404992
50	456456	1030617	1830400	1127175	1886456	$\sqrt{4412535560689}$	2149625	96558	56056	208422374160	111650
51	514800	880929	990080	1020321	1115920	$7\sqrt{35842741009}$	1421729	139392	125840	265598910720	187792
52	526680	222464	518073	571736	738777	$5\sqrt{12715594585}$	771545	349272	220704	287903512512	413160
53	572880	345708	493675	669108	756245	$\sqrt{363229026889}$	831517	323400	262570	338998875600	416570
54	572880	221309	1563660	614141	1665300	$\sqrt{2494010269081}$	1679941	392832	101640	330611339520	405768
55	600600	155705	1196352	620455	1338648	$\sqrt{1455502154929}$	1347673	464750	142296	366732366000	486046
56	632016	138787	153900	647075	650484	$\sqrt{42947041369}$	665125	508288	496584	472509607680	710600
57	637560	736593	1226176	974193	2382024	$425\sqrt{11327761}$	1566065	237600	155848	408165912000	284152
58	683760	1522180	4036851	1668700	4094349	$\sqrt{18613197948601}$	4368149	146520	57498	467603634960	157398
59	683760	726869	7505820	997931	7536900	$23\sqrt{107496545209}$	7571869	271062	31080	467603634960	272838
60	683760	614180	1893771	919100	2013429	$\sqrt{3963585672841}$	2105021	304920	119658	468949274640	327558
61	683760	132349	3581820	696451	3646500	$\sqrt{12846950770201}$	3648901	564102	64680	468949274640	567798
62	702576	70707	766700	706125	1039924	$\sqrt{592828369849}$	1042325	635418	273224	523254014640	691670
63	939120	579215	1089792	1103375	1438608	$\sqrt{1523136619489}$	1550833	524160	348816	900698722560	629616

TABLE 3 {(a, b, e), (a, c, f), (a, g, d)}. Values for d = (I) with a common $\sqrt{\ }$ to p and q.

#	*a*	*b*	*c*	*e*	*f*	*g*	*d*	*l*	*m*	*p*	*q*
1	104	153	672	185	680	$3\sqrt{52777}$	697	32	8	$2624\sqrt{17}$	$8\sqrt{17}$
2	117	520	756	533	765	$4\sqrt{52621}$	925	13	9	$4329\sqrt{10}$	$5\sqrt{10}$
3	252	2261	2640	2275	2652	$\sqrt{12081721}$	3485	14	12	$6888\sqrt{85}$	$2\sqrt{85}$
4	333	644	2040	725	2067	$4\sqrt{286021}$	2165	81	27	$35073\sqrt{10}$	$27\sqrt{10}$
5	399	468	4180	615	4199	$68\sqrt{3826}$	4225	147	19	$13965\sqrt{130}$	$13\sqrt{130}$
6	448	264	975	520	1073	$3\sqrt{113369}$	1105	256	98	$25088\sqrt{65}$	$34\sqrt{65}$
7	495	264	952	561	1073	$40\sqrt{610}$	1105	297	121	$42471\sqrt{34}$	$55\sqrt{34}$
8	756	533	3360	925	3444	$\sqrt{11573689}$	3485	392	84	$39984\sqrt{205}$	$28\sqrt{205}$
9	1276	357	6960	1325	7076	$3\sqrt{5396561}$	7085	968	116	$729872\sqrt{5}$	$436\sqrt{5}$
10	1364	14973	21120	15035	21164	$3\sqrt{74471681}$	25925	62	44	$832040\sqrt{5}$	$34\sqrt{5}$
11	1584	1365	14212	2091	14300	$\sqrt{203844169}$	14365	726	88	$75504\sqrt{1105}$	$22\sqrt{1105}$
12	1771	1428	2640	2275	3179	$12\sqrt{62561}$	3485	847	539	$243089\sqrt{170}$	$77\sqrt{170}$
13	1827	1564	8736	2405	8925	$4\sqrt{4922737}$	9061	841	189	$158949\sqrt{442}$	$41\sqrt{442}$
14	1881	1092	1540	2175	2431	$28\sqrt{4546}$	2665	1083	891	$321651\sqrt{130}$	$123\sqrt{130}$
15	1932	2880	4301	3468	4715	$\sqrt{26793001}$	5525	588	414	$405720\sqrt{85}$	$78\sqrt{85}$
16	2200	9879	60480	10121	60520	$3\sqrt{417269449}$	61321	242	40	$513040\sqrt{89}$	$26\sqrt{89}$
17	2352	5236	7011	5740	7395	$\sqrt{76569817}$	9061	504	384	$209664\sqrt{697}$	$24\sqrt{697}$
18	2920	85239	106560	85289	106600	$9\sqrt{229885441}$	136489	50	40	$1331600\sqrt{41}$	$10\sqrt{41}$
19	2944	3567	16800	4625	17056	$3\sqrt{32773721}$	17425	1058	256	$1354240\sqrt{41}$	$170\sqrt{41}$
20	3124	4557	9840	5525	10324	$3\sqrt{13065761}$	11285	968	484	$4369552\sqrt{5}$	$484\sqrt{5}$
21	3276	7293	16400	7995	16724	$\sqrt{322147849}$	18245	702	324	$749736\sqrt{205}$	$54\sqrt{205}$
22	3627	840	1364	3723	3875	$52\sqrt{949}$	3965	2883	2511	$4748301\sqrt{10}$	$1209\sqrt{10}$
23	3927	21420	77836	21777	77935	$4\sqrt{407328706}$	80825	357	99	$624393\sqrt{610}$	$15\sqrt{610}$
24	3960	7616	16095	8584	16575	$149\sqrt{14281}$	18241	968	480	$116160\sqrt{18241}$	$8\sqrt{18241}$
25	4032	6601	8976	7735	9840	$\sqrt{124141777}$	11849	1134	864	$616896\sqrt{697}$	$54\sqrt{697}$
26	4224	3007	46368	5185	46560	$\sqrt{2159033473}$	46657	2178	192	$1812096\sqrt{97}$	$222\sqrt{97}$
27	4284	15587	84912	16165	85020	$\sqrt{7453002313}$	86437	578	108	$62424\sqrt{86437}$	$2\sqrt{86437}$
28	4389	2548	8352	5075	9435	$4\sqrt{4765513}$	9773	2527	1083	$2554797\sqrt{58}$	$361\sqrt{58}$
29	4788	3920	10659	6188	11685	$\sqrt{128980681}$	12325	2268	1026	$2499336\sqrt{85}$	$270\sqrt{85}$
30	4991	1512	10488	5215	11615	$24\sqrt{194938}$	11713	3703	1127	$17859569\sqrt{2}$	$2737\sqrt{2}$
31	5605	2772	7800	6253	9605	$12\sqrt{475861}$	9997	3481	1805	$6283205\sqrt{26}$	$769\sqrt{26}$
32	5852	861	6864	5915	9020	$3\sqrt{5317313}$	9061	5054	2156	$1556632\sqrt{533}$	$238\sqrt{533}$
33	6048	1665	4264	6273	7400	$\sqrt{20953921}$	7585	4608	3136	$51584\sqrt{7585}$	$64\sqrt{7585}$
34	6105	37400	167832	37895	167943	$8\sqrt{461974066}$	172057	495	111	$6391935\sqrt{34}$	$87\sqrt{34}$
35	6336	5460	10373	8364	12155	$\sqrt{137410729}$	13325	2904	1782	$784080\sqrt{2665}$	$66\sqrt{97}$
36	6536	3927	5952	7625	8840	$3\sqrt{5649737}$	9673	3698	2888	$10679824\sqrt{17}$	$1138\sqrt{17}$
37	6601	9840	40920	11849	41449	$120\sqrt{123005}$	42601	2009	529	$30820069\sqrt{2}$	$1469\sqrt{2}$
38	7260	66352	219555	66748	219675	$\sqrt{5260985929}$	229477	396	120	$1528560\sqrt{1189}$	$12\sqrt{1189}$
39	7347	7920	37604	10803	38315	$4\sqrt{92299201}$	39125	2883	711	$17081775\sqrt{10}$	$939\sqrt{10}$
40	7392	34100	92781	34892	93075	$\sqrt{9771123961}$	99125	792	294	$1940400\sqrt{793}$	$30\sqrt{793}$
41	7700	9405	10608	12155	13108	$3\sqrt{22331521}$	16133	2750	2500	$4015000\sqrt{221}$	$250\sqrt{221}$
42	7904	2520	62985	8296	63479	$75\sqrt{706393}$	63529	5776	494	$15167776\sqrt{17}$	$1406\sqrt{17}$
43	8004	38272	45675	39100	46371	$\sqrt{3550951609}$	60125	828	696	$17768880\sqrt{13}$	$300\sqrt{13}$
44	8208	9405	10220	12483	13108	$5\sqrt{7716097}$	16133	3078	2888	$7953552\sqrt{73}$	$494\sqrt{73}$
45	8624	12180	25707	14924	27115	$3\sqrt{89911361}$	29725	2744	1408	$965888\sqrt{5945}$	$40\sqrt{5945}$
46	8736	2405	12852	9061	15540	$\sqrt{170957929}$	15725	6656	2688	$1397760\sqrt{3145}$	$128\sqrt{3145}$
47	9100	9555	15312	13195	17812	$3\sqrt{36195041}$	20213	3640	2500	$15470000\sqrt{29}$	$820\sqrt{29}$
48	9152	13464	19425	16280	21473	$3\sqrt{62067769}$	25345	2816	2048	$6172672\sqrt{185}$	$256\sqrt{185}$
49	9200	65805	229908	66445	230092	$3\sqrt{6354220721}$	239317	640	184	$13218560\sqrt{41}$	$104\sqrt{41}$
50	9492	76320	89131	76908	89635	$\sqrt{13769077561}$	117725	588	504	$9772560\sqrt{85}$	$84\sqrt{85}$
51	10032	1476	8645	10140	13243	$\sqrt{76914601}$	13325	8664	4598	$2096688\sqrt{2665}$	$190\sqrt{2665}$
52	10080	80325	253916	80955	254116	$\sqrt{70925440681}$	266509	630	200	$1537200\sqrt{4369}$	$10\sqrt{4369}$
53	11088	9555	13940	14637	17812	$5\sqrt{11424865}$	20213	5082	3872	$4716096\sqrt{697}$	$242\sqrt{697}$
54	11700	6765	100912	13515	101588	$\sqrt{10228996969}$	101813	6750	676	$4563000\sqrt{901}$	$226\sqrt{901}$
55	11799	95120	859320	95849	859401	$40\sqrt{467174173}$	864649	729	81	$15373881\sqrt{82}$	$81\sqrt{82}$
56	12144	9792	10465	15600	16031	$\sqrt{205399489}$	18785	5808	5566	$4541856\sqrt{1105}$	$242\sqrt{1105}$
57	13536	15080	17577	20264	22185	$\sqrt{536357329}$	26825	5184	4608	$15344640\sqrt{145}$	$576\sqrt{145}$
58	14112	3885	153340	14637	153988	$5\sqrt{941129953}$	154037	10752	648	$7547904\sqrt{697}$	$408\sqrt{697}$
59	14784	32175	60088	35409	61880	$\sqrt{4645798369}$	69745	3234	1792	$827904\sqrt{69745}$	$14\sqrt{697451}$
60	14820	1952	4725	14948	15555	$\sqrt{26135929}$	15677	12996	10830	$33399720\sqrt{61}$	$2166\sqrt{61}$
61	14960	7293	7476	16643	16724	$15\sqrt{484793}$	18245	9350	9248	$25432000\sqrt{89}$	$1394\sqrt{89}$
62	14973	21164	231840	25925	232323	$4\sqrt{3387356281}$	233285	4761	483	$3232719\sqrt{4810}$	$69\sqrt{4810}$
63	15312	2915	4284	15587	15900	$\sqrt{2684981}$	16165	12672	11616	$17005824\sqrt{265}$	$1056\sqrt{265}$

TABLE 4 $\{(a, b, e), (a, c, f), (a, g, d)\}$. Values for g = (I).

#	*a*	*b*	*c*	*e*	*f*	*g*	*d*	*l*	*m*
1	44	117	240	125	244	267	$5\sqrt{2929}$	8	4
2	85	132	720	157	725	732	$\sqrt{543049}$	25	5
3	140	480	693	500	707	843	$13\sqrt{4321}$	20	14
4	160	231	792	281	808	825	$5\sqrt{28249}$	50	16
5	187	1020	1584	1037	1595	1884	$5\sqrt{143377}$	17	11
6	195	748	6336	773	6339	6380	$5\sqrt{1629697}$	25	3
7	240	252	275	348	365	373	$\sqrt{196729}$	96	90
8	429	880	2340	979	2379	2500	$\sqrt{6434041}$	99	39
9	495	4888	8160	4913	8175	9512	$17\sqrt{313921}$	25	15
10	528	5796	6325	5820	6347	8579	$5\sqrt{2955121}$	24	22
11	780	2475	2992	2595	3092	3883	$\sqrt{15686089}$	120	100
12	828	2035	3120	2197	3228	3725	$13\sqrt{86161}$	162	108
13	832	855	2640	1193	2768	2775	$17\sqrt{29041}$	338	128
14	935	17472	25704	17497	25721	31080	$25\sqrt{1546945}$	25	17
15	1008	1100	1155	1492	1533	1595	$\sqrt{3560089}$	392	378
16	1008	1100	12075	1492	12117	12125	$\sqrt{148031689}$	392	42
17	1080	1881	14560	2169	14600	14681	$\sqrt{216698161}$	288	40
18	1105	9360	35904	9425	35921	37104	$\sqrt{1377927841}$	65	17
19	1155	6300	6688	6405	6787	9188	$\sqrt{85753369}$	105	99
20	1188	16016	39195	16060	39213	42341	$5\sqrt{71766865}$	44	18
21	1560	2295	5984	2775	6184	6409	$13\sqrt{257449}$	480	200
22	1575	1672	9120	2297	9255	9272	$\sqrt{88450609}$	625	135
23	1755	4576	6732	4901	6957	8140	$5\sqrt{2773585}$	325	225
24	2079	44080	65472	44129	65505	78928	$5\sqrt{249358057}$	49	33
25	2163	15840	37100	15987	37163	40340	$\sqrt{1631994169}$	147	63
26	2925	3536	11220	4589	11595	11764	$\sqrt{146947321}$	1053	375
27	2964	9152	9405	9620	9861	13123	$5\sqrt{7239937}$	468	456
28	2964	6160	38475	6836	38589	38965	$\sqrt{1527056521}$	676	114
29	3696	9045	121940	9771	121996	122275	$\sqrt{14964836041}$	726	56
30	4368	4901	13860	6565	14532	14701	$5\sqrt{9407953}$	1664	672
31	4599	23760	144832	24201	144905	146768	$5\sqrt{862479865}$	441	73
32	4599	18368	23760	18935	24201	30032	$5\sqrt{36922873}$	567	441
33	4900	17157	23760	17843	24260	29307	$13\sqrt{5224321}$	686	500
34	4928	10725	30780	11803	31172	32595	$\sqrt{1086719209}$	1078	392
35	5320	63063	353760	63287	353800	359337	$\sqrt{129151381969}$	224	40
36	5368	163680	450225	163768	450257	479055	$\sqrt{229522508449}$	88	32
37	5491	41580	46512	41941	46835	62388	$5\sqrt{156896545}$	361	323
38	5643	14160	21476	15243	22205	25724	$5\sqrt{27742705}$	1083	729
39	5643	43680	76076	44043	76285	87724	$5\sqrt{309093745}$	363	209
40	5720	8415	157248	10175	157352	157473	$17\sqrt{85918561}$	1760	104
41	6072	16929	18560	17985	19528	25121	$5\sqrt{26717353}$	1056	968
42	6435	24080	24684	24925	25509	34484	$\sqrt{1230555481}$	845	825
43	7336	274527	480480	274625	480536	553377	$65\sqrt{72492289}$	98	56
44	7560	13728	35321	15672	36121	37895	$5\sqrt{59727385}$	1944	800
45	7579	8820	17472	11629	19045	19572	$5\sqrt{17620177}$	2809	1573
46	7800	23751	29920	24999	30920	38201	$\sqrt{1520156401}$	1248	1000
47	7840	9828	10725	12572	13285	14547	$\sqrt{273080809}$	2744	2560
48	7885	16320	85932	18125	86293	87468	$\sqrt{7712824249}$	1805	361
49	7920	15232	26649	17168	27801	30695	$5\sqrt{40196377}$	1936	1152
50	8415	157248	643720	157473	643775	662648	$\sqrt{439173184129}$	225	55
51	8532	36960	57275	37932	57907	68165	$29\sqrt{5611489}$	972	632
52	8789	10560	17748	13739	19805	20652	$5\sqrt{20150065}$	3179	2057
53	9180	72611	206448	73189	206652	218845	$5\sqrt{1919096257}$	578	204
54	9405	23600	53196	25405	54021	58196	$\sqrt{3475228441}$	1805	825
55	9504	31372	61845	32780	62571	69347	$5\sqrt{195973297}$	1408	726
56	9856	61560	200683	62344	200825	209817	$5\sqrt{1764812569}$	784	242
57	10296	11753	16800	15625	19704	20503	$25\sqrt{842209}$	3872	2904
58	10395	95004	220400	95571	220645	240004	$\sqrt{57709976041}$	567	245
59	10395	63364	327360	64211	327525	33436	$\sqrt{111287622121}$	847	165
60	12915	36720	290444	38925	290731	292756	$\sqrt{85872872761}$	2205	287
61	14112	15400	19305	20888	23913	24695	$\sqrt{808991569}$	5488	4608
62	14500	29568	83475	32932	84725	88557	$17\sqrt{27863641}$	3364	1250
63	14715	148148	267120	148877	267525	305452	$53\sqrt{33292081}$	729	405

APPENDIX 2 ODDS AND ENDS

Theorem A3 For all integers $a > 0$ there exists at least one Pythagorean Quad.

Proof For $a = 1$, $1^2 + 2^2 + 2^2 = 3^2$. For $a = 2$, $2^2 + 3^2 + 6^2 = 7^2$.
For $a = 3$, $3^2 + 6^2 + 22^2 = 23^2$. For $a = 4$, $4^2 + 4^2 + 7^2 = 9^2$.
For $a > 2$, $a^2 + b^2 = e^2$ and $e^2 + c^2 = d^2$ by application of Theorem 1 twice.
e.g. $3^2 + 4^2 = 5^2$ and $5^2 + 12^2 = 13^2$. ❑

For every integer a there exist an infinite number of Pythagorean Quads, multiply every integer solution from A3.1 by all the integers! Conjecture, for all integers $a > 0$ are there an infinite number of solutions that are not multiples ? See Appendix 3.

Some Interesting Facts

1 $e^2 + f^2 + g^2 = (a^2 + b^2) + (a^2 + c^2) + (b^2 + c^2) = 2(a^2 + b^2 + c^2) = 2d^2$.

2 If $\{a, b, c,\}$ forms a Pythagorean Quad then $\{ab, ac, bc\} = \{A, B, C\}$ and repeating this gives $\{AB, AC, BC\}$ which is a multiple of $\{a, b, c\}$.

e.g. $3^2 + 4^2 + 12^2 = 13^2 \Rightarrow 12^2 + 36^2 + 48^2 \Rightarrow 144 \times (3^2 + 4^2 + 12^2 = 13^2)$.
Note $12^2 + 36^2 + 48^2 = (12\sqrt{26})^2$ which is not a Pythagorean Quad. [4]

3 Given that $\{x, y, z\}$ and $\{X, Y, Z\}$ are both Pythagorean Triples then
$(xX - yY)^2 + (xY + yX)^2 = (zZ)^2$ e.g. $\{3, 4, 5\}$ and $\{5, 12, 13\}$ produces
$(3{\times}5 - 4{\times}12)^2 + (3{\times}12 + 4{\times}5)^2 = (5{\times}13)^2 = \{33, 56, 65\}$. [5]

Some Interesting Examples

In the following $(a^4 + l^2m^2) = p$ and $(l^2 + m^2) = q$, where $p\,q = \delta^2$.

$a = 576972$ $b = 2371005$ $c = 4226096$ $e = 2440197$ $f = 4265300$ $g = 7\sqrt{479215349209}$
$d = 4880005$ $l = 69192$ $m = 39204$ $p = 150700176\sqrt{4880005}$ $q = 36\sqrt{4880005} \Rightarrow 2lmd = p\,q$
which is always true of course, but the example shows it in a glaringly obvious way.

a	***b***	***c***	***e***	***f***	***g***	***d***	***l***	***m***	***p***	***q***
942480	2639811	4850300	2803011	4941020	$\sqrt{30494012205721}$	5601989	163200	90720	888391929600	186720

a	***b***	***c***	***e***	***f***	***g***	***d***	***l***	***m***	***p***	***q***
942480	157157	593676	955493	1113676	614125	$85\sqrt{175144369}$	98336	520200	980556192000	$72\sqrt{175144369}$

a	***b***	***c***	***e***	***f***	***g***	***d***	***l***	***m***
68640	345100	99099	351860	120549	$7\sqrt{2630910649}$	365549	21450	20449
99099	345100	68640	$7\sqrt{2630910649}$	120549	351860	365549	51909	20449

a	***b***	***c***	***e***	***f***	***g***	***d***	***l***	***m***
7800	$\sqrt{211773121}$	18720	16511	20280	23711	24961	$16511 - \sqrt{211773121}$	1560
520	576	$3\sqrt{68761}$	76	943	975	1105	200	$943 - 3\sqrt{68761}$

a	***b***	***c***	***e***	***f***	***g***	***d***	***l***	***m***
60	$4\sqrt{-209}$	63	16	87	25	65	40	24
60	63	$4\sqrt{-209}$	87	16	25	65	40	24

Apropos of Fermat's Last Theorem, for $n = 3$ and 4 there are the following. **But for $n > 4$?**

For $n = 3$ $3^3 + 4^3 + 5^3 = 6^3$ and $14^3 + 23^3 + 70^3 = 71^3$, (observe $14^2 + 23^2 + 70^2 = 75^2$)

There is a formula due to Vieta (1591) submitted by Steven Dutch, University of Wisconsin – Green Bay. This is not a particularly good formula as it produces some negative results, e.g. (–15, 744, 756, 945, since the cubes of negative integers are also negative).

Given any two numbers m and n with $m > n$ then

$$a = m(m^3 - 2n^3),\ b = n(2m^3 - n^3),\ c = n(m^3 + n^3)\ \text{and}\ d = m(m^3 + n^3).$$

e.g. $m = 2$ and $n = 1$ produces $12^3 + 15^3 + 9^3 = 18^3 \Rightarrow 3(4, 5, 3, 6)$.

See Program 3 and Tables 6.1 and 6.2 below for other results.

For $n = 4$ $95800^4 + 217519^4 + 414560^4 = 422481^4$ discovered by Roger Frye in 1988

and $2682440^4 + 15365639^4 + 18796760^4 = 20615673^4$ by Noam Elkies

also $630662624^4 + 275156240^4 + 219076465^4 = 638523249^4$ by Allan MacLeod.

For $n = 5$ $27^5 + 84^5 + 110^5 + 133^5 = 144^5$ discovered by Lander and Parkin.
Note that this is a Pythagorean Quintuple not a Pythagorean Quadruple.

Program 3 $a^n + b^n + c^n = d^n$

```
n = 3; Print["  n = ", n]
For[a = 1, a < 11, a++,
  For[b = a, b < 501, b++,
    For[c = b, c < 1002, c++,
    d = (a^n + b^n + c^n)^(1/n);
       If[IntegerQ[d] && GCD[a, b, c]<2,
Print["   a = ",  a, "   b = ",  b,
      "   c = ",  c, "   d = ",  d]
   ] ] ] ]
```

This program produces tables 5.1 and 5.2 below. The code can be used for all n greater than 3. For any n, the tables can be extended to any desired size by increasing the limits of a, b, and c. The number of tuples can be increased by increasing the number of `For` statements and `d` statement.

Table 5.1 $n = 3$ $a = 1$

#	a	b	c	d
1	1	6	8	9
2	1	71	138	144
3	1	135	138	172
4	1	242	720	729
5	1	372	426	505
6	1	426	486	577
7	1	566	823	904
8	1	575	2292	2304
9	1	791	812	1010
10	1	1938	2820	3097
11	1	1943	6702	6756
12	1	2196	5984	6081
13	1	2676	3230	3753
14	1	3086	21588	21609
15	1	3318	16806	15489
16	1	3453	24965	24987
17	1	4607	36840	36864
18	1	7251	49409	49461
19	1	7676	11903	12884
20	1	10230	37887	38134

Table 5.2 $n = 3$ $a > 1$

#	a	b	c	d
1	2	17	40	41
2	3	4	5	6
3	3	10	18	19
4	3	34	114	115
5	3	36	37	46
6	3	121	131	159
7	3	214	309	340
8	3	245	340	378
9	4	17	22	25
10	4	57	248	249
11	4	303	482	519
12	5	76	123	132
13	5	86	460	461
14	5	163	164	206
15	5	216	436	453
16	5	232	307	346
17	6	32	33	41
18	6	121	768	769
19	6	127	180	199
20	6	179	216	251

APPENDIX 3

Conjecture For every $\boldsymbol{a} =$ (I) there exist an infinite number of Pythagorean Quads, excluding multiples.

#	*a*	*b*	*c*	*d*	*a*	*b*	*c*	*d*	*a*	*b*	*c*	*d*	*a*	*b*	*c*	*d*	*a*	*b*	*c*	*d*	*a*	*b*	*c*	*d*
1	1	2	2	3	2	3	6	7	3	4	12	13	4	4	7	9	5	6	30	31	6	6	7	11
2	1	4	8	9	2	5	14	15	3	6	22	23	4	5	20	21	5	8	44	45	6	6	17	19
3	1	6	18	19	2	6	9	11	3	8	36	37	4	7	32	33	5	10	62	63	6	7	42	43
4	1	8	32	33	2	9	42	43	3	10	54	55	4	8	19	21	5	12	84	85	6	9	58	59
5	1	10	50	51	2	10	11	15	3	12	76	77	4	9	48	49	5	14	110	111	6	10	15	19
6	1	12	12	17	2	10	25	27	3	14	18	23	4	11	68	69	5	16	140	141	6	10	33	35
7	1	12	72	73	2	11	62	63	3	14	102	103	4	12	39	41	5	18	174	175	6	11	78	79
8	1	14	98	99	2	12	36	38	3	16	24	29	4	13	16	21	5	20	212	213	6	13	18	23
9	1	16	128	129	2	13	86	87	3	16	132	133	4	13	92	93	5	22	254	255	6	13	102	103
10	1	18	30	35	2	14	23	27	3	18	166	167	4	15	120	121	5	24	300	301	6	14	27	31
11	1	18	162	163	2	14	49	51	3	20	204	205	4	16	67	69	5	26	350	351	6	14	57	59
12	1	20	200	201	2	15	114	115	3	22	246	247	4	17	28	33	5	28	404	405	6	15	130	131
13	1	22	46	51	2	17	146	147	3	24	28	37	4	17	152	153	5	30	462	463	6	17	30	35
14	1	22	242	243	2	18	39	43	3	24	56	61	4	19	188	189	5	32	524	525	6	17	162	163
15	1	24	288	289	2	18	81	83	3	24	292	293	4	20	103	105	5	34	590	591	6	18	43	47
16	1	26	238	239	2	19	34	39	3	26	66	71	4	21	228	229	5	36	660	661	6	18	89	91
17	1	28	76	81	2	19	182	183	3	26	342	343	4	23	52	57	5	38	50	63	6	19	198	199
18	1	28	392	393	2	21	222	223	3	28	396	397	4	23	272	273	5	38	734	735	6	21	22	31
19	1	30	450	451	2	22	59	63	3	30	46	55	4	24	33	41	5	40	56	69	6	21	238	239
20	1	32	100	105	2	22	121	123	3	30	454	455	4	24	147	149	5	48	60	77	6	22	129	131
21	1	34	38	51	2	23	266	267	3	34	114	119	4	25	320	321	5	54	78	95	6	23	54	59
22	1	38	142	147	2	25	314	315	3	36	68	77	4	27	372	373	5	64	152	165	6	23	282	283
23	1	42	174	179	2	26	29	39	3	36	128	133	4	28	35	45	5	66	162	175	6	25	330	331
24	1	44	68	81	2	26	83	87	3	42	94	103	4	28	199	201	5	70	86	111	6	26	87	91
25	7	8	56	57	8	8	31	33	9	10	90	91	10	10	23	27	11	12	24	29	12	12	71	73
26	7	10	74	75	8	9	12	17	9	12	20	25	10	10	49	51	11	12	132	133	12	13	156	157
27	7	12	96	97	8	9	72	73	9	12	112	113	10	11	110	111	11	14	158	159	12	15	16	25
28	7	14	22	27	8	11	16	21	9	14	138	139	10	13	134	135	11	16	188	189	12	15	184	185
29	7	14	122	123	8	11	92	93	9	16	168	169	10	14	35	39	11	18	42	49	12	16	21	29
30	7	16	28	33	8	12	51	53	9	18	38	43	10	14	73	75	11	18	222	223	12	16	99	101
31	7	16	152	153	8	13	116	117	9	18	202	203	10	15	162	163	11	20	260	261	12	17	216	217
32	7	18	186	187	8	15	144	145	9	20	240	241	10	17	194	195	11	22	58	63	12	19	48	53
33	7	20	224	225	8	16	79	81	9	22	54	59	10	18	51	55	11	22	302	303	12	19	252	253
34	7	22	266	267	8	17	176	177	9	22	282	283	10	18	105	107	11	24	348	349	12	20	135	137
35	7	24	60	65	8	19	40	45	9	24	32	41	10	19	230	231	11	26	398	399	12	21	28	37
36	7	24	312	313	8	20	25	33	9	24	328	329	10	21	270	271	11	28	88	93	12	21	56	61
37	7	26	70	75	8	20	115	117	9	26	378	379	10	22	71	75	11	28	452	453	12	21	292	293
38	7	26	362	363	8	21	48	53	9	28	84	89	10	22	145	147	11	32	112	117	12	23	336	337
39	7	28	416	417	8	21	252	253	9	28	432	433	10	23	314	315	11	36	48	61	12	24	31	41
40	7	30	30	43	8	23	296	297	9	30	50	59	10	25	362	363	11	38	154	159	12	24	41	49
41	7	30	474	475	8	24	159	161	9	30	490	491	10	26	95	99	11	42	66	79	12	24	179	181
42	7	34	118	123	8	25	344	345	9	32	36	39	1o	26	193	195	11	42	186	191	12	25	384	385
43	7	36	132	137	8	27	396	397	9	32	108	113	10	27	414	415	11	44	52	69	12	27	436	437
44	7	40	40	57	8	28	49	57	9	36	228	231	10	29	470	471	11	48	240	245	12	28	231	233
45	7	44	196	201	8	28	211	213	9	38	150	155	10	30	123	127	11	52	280	285	12	29	492	493
46	7	46	214	219	8	29	88	93	9	42	98	107	10	30	249	251	11	58	94	111	12	31	108	113
47	7	48	84	97	8	29	452	453	9	42	182	187	10	34	313	315	11	58	346	351	12	32	69	77
48	7	54	294	299	8	31	100	105	9	46	78	91	10	37	50	63	11	62	146	149	12	32	291	293
49	13	14	34	39	14	14	47	51	15	16	240	241	16	16	127	129	17	18	306	307	18	18	79	83
50	13	14	182	183	14	14	97	99	15	18	26	35	16	17	52	57	17	20	20	33	18	18	161	163
51	13	16	40	45	14	15	210	211	15	18	274	275	16	17	272	273	17	20	344	345	18	19	66	71
52	13	16	212	213	14	17	46	51	15	20	312	313	16	19	308	309	17	22	386	387	18	19	342	343
53	13	18	246	247	14	17	242	243	15	22	354	355	16	20	37	45	17	24	84	89	18	21	38	47
54	13	20	284	285	14	18	21	31	15	24	40	49	16	20	163	165	17	24	432	433	18	21	73	79
55	13	22	326	327	14	18	63	67	15	24	400	401	16	23	76	81	17	26	94	99	18	21	382	383
56	13	24	372	373	14	18	129	131	15	26	450	451	16	23	392	393	17	26	482	483	18	22	99	103
57	13	26	422	423	14	19	278	279	15	28	504	505	16	24	207	209	17	28	536	537	18	22	201	203
58	13	28	476	477	14	21	318	319	15	30	58	67	16	25	440	441	17	30	594	595	18	23	426	427
59	13	34	130	135	14	22	29	39	15	30	110	115	16	27	96	101	17	32	44	47	18	24	224	226
60	13	36	144	149	14	22	83	87	15	30	186	189	16	28	47	57	17	32	656	657	18	25	30	43
61	13	44	208	213	14	22	169	171	15	30	562	563	16	28	61	69	17	34	142	146	18	25	474	475
62	13	46	226	231	14	23	70	75	15	32	624	625	16	28	259	261	17	34	722	723	18	26	45	55
63	13	50	70	87	14	23	362	363	15	34	690	691	16	29	548	549	17	36	156	161	18	26	123	127
64	13	52	76	93	14	25	410	411	15	36	52	65	16	31	608	609	17	36	792	793	18	26	249	251
65	13	54	306	311	14	26	107	111	15	36	80	89	16	32	59	69	17	38	866	867	18	27	34	47
66	13	56	328	333	14	26	217	219	15	36	760	761	16	32	319	321	17	40	944	945	18	27	526	527
67	13	64	424	455	14	27	90	95	15	38	834	835	16	33	132	137	17	42	1026	1027	18	28	276	278
68	13	66	78	103	14	27	462	463	15	40	912	913	16	33	672	673	17	44	220	225	18	29	114	119
69	13	66	450	455	14	29	518	519	15	42	50	67	16	35	740	741	17	44	1112	1113	18	30	151	155
70	13	76	88	117	14	30	135	139	15	42	70	83	16	36	93	101	17	46	86	99	18	30	305	307
71	13	84	204	221	14	30	273	275	15	42	106	115	16	36	387	389	17	46	238	243	18	31	126	131

APPENDIX 4 SUMS OF TWO SQUARES

Attempting to simplify some expressions using Mathemetica™ [3] the author came across the following identity

$$l^2(a^2+m^2)^2+m^2(a^2-l^2)^2 = (a^4+l^2m^2)(l^2+m^2). \qquad [4]$$

Proof LHS $= l^2(a^2+m^2)^2+m^2(a^2-l^2)^2$

$= l^2a^4+2a^2l^2m^2+l^2m^4+m^2a^4 - 2a^2l^2m^2+m^2l^4$

$= l^2(a^4+l^2m^2)+m^2(a^4+l^2m^2) = (a^4+l^2m^2)(l^2+m^2) =$ RHS. ❑

and the following are easy to prove using the proof above with suitable changes of sign.

$$l^2(a^2-m^2)^2+m^2(a^2+l^2)^2 = (a^4+l^2m^2)(l^2+m^2)$$

$$l^2(a^2+m^2)^2-m^2(a^2+l^2)^2 = (a^4-l^2m^2)(l^2-m^2)$$

$$l^2(a^2-m^2)^2-m^2(a^2-l^2)^2 = (a^4-l^2m^2)(l^2-m^2).$$

Obviously, there are eight combinations of the three positive and negative signs in each identity, only four of which simplify as above. The other four, which do not simplify, are:

$$l^2(a^2+m^2)^2+m^2(a^2+l^2)^2 = (a^4+l^2m^2)(l^2+m^2)+4a^2l^2m^2$$

$$l^2(a^2+m^2)^2-m^2(a^2-l^2)^2 = (a^4-l^2m^2)(l^2-m^2)-4a^2l^2m^2$$

$$l^2(a^2-m^2)^2+m^2(a^2-l^2)^2 = (a^4+l^2m^2)(l^2+m^2)-4a^2l^2m^2$$

$$l^2(a^2-m^2)^2-m^2(a^2+l^2)^2 = (a^4-l^2m^2)(l^2-m^2)+4a^2l^2m^2.$$

Also $(a^2-l^2)^2 + 4a^2l^2 = (a^2+l^2)^2$

$(a^2+l^2)^2 - 4a^2l^2 = (a^2-l^2)^2$

$(a^2+l^2)^2 + 4a^2l^2 = (a^2+6a^2l^2+l^4)$

$(a^2-l^2)^2 - 4a^2l^2 = (a^2-6a^2l^2+l^4)$

and replacing l by m produces four more.

Also $(x-y)^2+(x+y)^2 = 2z^2 \Rightarrow 2x^2+2y^2 = 2z^2 \Rightarrow x^2+y^2 = z^2 =$ PT.

For the history of sums of 2, 4 and 8 squares see the article by C. D. Hollings in the | *b s h m* | Bulletin, volume 21 number 2, dated 2006, pages 111 to 118.
| *b s h m* | = British Society for the History of Mathematics.