<u>Proof of Legendre´s conjecture for sufficiently high values</u>

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Abstract

In this paper we will try to prove Legendre´s conjecture.

Legendre´s conjecture is that

between any 2 Squares N^2 and (N+1)^2,

there is at least one prime number.

The proof uses very simple properties of all 2N intervals between 2 squares N^2 and (N+1)^2

Definitions

Definition 1 : Coprime gap

A coprime gap for the number N is defined as a gap of some length, where all the numbers in the gap are divisible with the primes less than or equal to N.

In this proof we are looking specifically at coprime gaps of length 2N, where all the numbers in the gap are divisible with the primes
less than or equal to N.
Notice that it is neccesary, but not sufficient, that 2N coprime gaps exists, if Legendre´s conjecture is false.
This is because between two squares N^2 and (N+1)^2,
there exist exactly 2N numbers.
If one of these numbers is not divisible by any of the primes less than or equal to N, then it must be prime, because the smallest such number that is not a prime is definetely equal to (P_(n))^2 where P_(n) is the prime following N.

Definition 2 : Distribution n gap

Let us consider a 2N gap, where the number of numbers divisible with primes less than or equal to N are the same as from 1 to 2N
Let us call this type of 2N gap for a *distribution 0 gap*
For example from 1 to 11*2=22, there are the 3s, 3,6,9,12,15,18,21
there are also 2s, 2,4,6,8,10,12,14,16,18,20
and 5s 5,10,15,20
the 7s 7,14,21
the 11s 11,22
A *distribution 1 gap* is a number gap, where there exists one more number divisible with a certain prime.
Within the same distance of 22, it is possible that there are, for example one more number divisible with 3.
For example if 3 is divisible at the 1-point

We have that 3 is divisible at the following points :1,4,7,10,13,16,19,22

This counts one more number divisible with 3.

There could also perhaps be one more number divisible with 3 and 5, which then would be a *distribution 2*.

Generally we have that a *distribution n gap* is a 2N gap, where there is one more number divisible with a prime n times. This can happen at most pi(N)-2 times in a 2N gap.

Definition 3 : A+B+C+D...

The number of numbers divisible with primes less than or equal to N in a 2N gap can be written as

A+B/2+C/3+D/4...etc

where A is the numbers which have only one primefactor less than or equal to N (the A-numbers),

B/2 is the number of numbers with 2 distinct primefactors from 2 to N (The B-numbers)

etc...

If we look at A+B+C+D...

This number is equal to the number of numbers below 2N that are divisible by 2,.

plus the number of numbers below 2N that are divisible by 3... and so on, until we reach the prime less than or equal to N.

We have that the lowest value of A+B+C+D... is for a distribution 0.

This value is constant for any distribution 0.

If we define $A_n + B_n + C_n ...$

as the value of A+B+C for a distribution n then We have

$$A_n + B_n + C_n ... = A_0 + B_0 + C_0 ... + n$$

We have that $A_0 + B_0 + C_0 ...$

can be written as :

$$A_0 + B_0 + C_0 ... = \frac{2N - r_1}{2} + \frac{2N - r_2}{3} + \frac{2N - r_3}{5} + \frac{2N - r_4}{7} + \frac{2N - r_5}{11} .... \frac{2N - r_n}{p_n}$$

where r1,r2,r3...rn is the rests.

Furthermore we have that $A_0 + \frac{B_0}{2} + \frac{C_0}{3} ...$

which is all the non coprimes in a distribution 0 gap,

can be written as :

$$A_0 + \frac{B_0}{2} + \frac{C_0}{3} ... = \frac{2N - r_1}{2} + \frac{2N - r_2}{3} + \frac{2N - r_3}{5} + \frac{2N - r_4}{7} + \frac{2N - r_5}{11} .... \frac{2N - r_n}{p_n} - z$$

where

$$z = \frac{B_0}{2} + \frac{2C_0}{3} ...$$

A-numbers

Since we are going to use the term A-number in the proof, let us clarify precisely what they are.

An A-number is a number, which have exactly one prime factor less than equal to P_n. It may have other prime factors given, that they are higher than P_n.

Example on A-numbers when P_n=7:

7*11, 7*7*7, 2*2*2, 2,3,5, 2*13

Main proof

All 2N intervals between N^2 and (N+1)^2 has the following property :

1) All composite numbers in the interval, except for the A-numbers have at least 3 primefactors.

Proof of 1)

If we multiply any 2 primes together, which are less than or equal to N, then we always get a number which is less than N^2+1.

So if a number between N^2 and (N+1)^2 to has only 2 primefactors,

it must have a primefactor greater than N.

These numbers are always semiprimes and A-numbers.

q.e.d

Notice that if a number between N^2 and (N+1)^2 has a smallest prime factor which is greater than (N+1)^(2/3)

then the number must be an A-number, since the 2 other possible prime factors must both be greater than (N+1)^(2/3), so the total product gives a number which is greater than (N+1)^2, which is too large.

So the number must be an A-number.

Now this observation lead to a proof of Legendre conjecture, as we shall see.

We can write the number of numbers divisible with primes less than or equal to N in a 2N gap in the following way :

$$\left(\frac{2N-r_1}{2}+\frac{2N-r_2}{3}+\frac{2N-r_3}{5}+\frac{2N-r_4}{7}+\frac{2N-r_5}{11}....\frac{2N-r_n}{p_n}\right)+d(N)-z=2N-Cop$$

Where d(N) is the distribution n for the 2N gap and Cop is the number of number, which are coprime with the primes from 2 to N in the gap.

In the case of a 2N coprime gap we have that :

1)

$$\left(\frac{2N-r_1}{2}+\frac{2N-r_2}{3}+\frac{2N-r_3}{5}+\frac{2N-r_4}{7}+\frac{2N-r_5}{11}....\frac{2N-r_n}{p_n}+d(N)\right)-z=2N$$

Now we have found earlier, that all numbers, which have a smallest prime factor, which is greater than (N+1)^(2/3) are all A-numbers.

The consequence of this is that we can write this :

$$\left(\frac{2N-r_1}{2}+\frac{2N-r_2}{3}+\frac{2N-r_3}{5}+\frac{2N-r_4}{7}+\frac{2N-r_5}{11}....\frac{2N-r_n}{p_n}+d(N)\right)-z=2N$$

in another way.

We can write it like this :

2)

$$\left(\frac{2N-r_1}{2}+\frac{2N-r_2}{3}+....\frac{2N-r_n}{p_x}+d(N)\right)+A_x-z_2=2N$$

where p_x is the prime just before (N+1)^(2/3)

and A_x is the number of A-numbers divisible with a prime greater than or equal to (N+1)^(2/3).

A close approximation for 1) and 2) is :

1)

$$2N\cdot\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+....\frac{1}{p_n}\right)+d(n)-z=2N$$

2)

$$2N\cdot\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+....\frac{1}{p_x}\right)+A_x+d(n)-z_2=2N$$

Now since both

$\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+....\frac{1}{p_n}\right)$ and $\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+....\frac{1}{p_x}\right)$

are constants for a given N,

We can write it, the following way :

1) 2N*(K_n)+d(N)-z=2N

2) 2N*(K_x)+A_x+d(N)-z2=2N

Now we insert

2N*(K_n)+d(N)-z=2N

into the 2N in 2)

and get :

2)

(2N*(K_n)+d(N)-z)*(K_x)+A_x+d(N)-z2=2N

To make this a bit more easy to read, we write it like this, by substitution :

X=2N, Y=K_n, d=d(n), z1=z, z2=z2, A_x=A, W=K_(x)

(X*Y+d-z1)*W+A+d-z2=X

Which implies :

X*(W*Y-1)=z2-A-d+(z1-d)*W

And

$$X=\frac{z_2+\left(z_1-d\right)\cdot W-A-d}{WY-1}$$

$\Updownarrow$

$$X=\frac{z_2+\left(z_1-d\right)\cdot W-A-d}{WY-1}$$

$\Updownarrow$

$$\frac{X}{W}=\frac{z_2+\left(z_1-d\right)\cdot W-A-d}{Y-\frac{1}{W}}$$

$\Updownarrow$

$$\frac{X}{W}>\frac{z_2+\left(z_1-d\right)\cdot W-A-d}{Y}$$

$\Updownarrow$

$$X\cdot\frac{Y}{W}>z_2+\left(z_1-d\right)\cdot W-A-d$$

Remember X=2N

and

$$z_2=2N\cdot\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+....\frac{1}{p_x}-1\right)+A_x+d(n)$$

$$z_1=2N\cdot\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+....\frac{1}{p_n}-1\right)+d(n)$$

$$Y=\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+....\frac{1}{p_n}\right)$$

$$W=\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+....\frac{1}{p_x}\right)$$

So we should have this in reduced form :

$$2N\cdot\frac{\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+...\frac{1}{p_n}\right)}{\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+...\frac{1}{p_x}\right)}>2N\cdot\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+....\frac{1}{p_x}-1\right)+\left(2N\cdot\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+....\frac{1}{p_n}-1\right)\right)\cdot\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+....\frac{1}{p_x}\right)$$

We have that :

$$\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{5}...+\frac{1}{p_x}\right)$$

converge to ln(ln(p_(x))+M *)

where M is the Meissel–Mertens constant.

So an upperbound for

$$\left(\frac{1}{p_x}+\frac{1}{p_{x+1}}+\frac{1}{p_{x+2}}...+\frac{1}{p_n}\right)$$

is

$$\left(\frac{1}{p_x}+\frac{1}{p_{x+1}}+\frac{1}{p_{x+2}}...+\frac{1}{p_n}\right)<\left(\ln\ln(p_n)+M\right)-\left(\ln\ln(p_x)+M\right)$$

Generally we have that :

$$\ln\left(\ln\left(X\right)\right)-\ln\left(\ln\left(X^{\frac{2}{3}}\right)\right)=\ln\left(\frac{\ln\left(X\right)}{\ln\left(X^{\frac{2}{3}}\right)}\right)=\ln\left(\frac{\ln\left(X\right)}{\ln\left(X\right)\cdot\frac{2}{3}}\right)=\ln\left(\frac{3}{2}\right)$$

Which is a constant.

So $\dfrac{\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\ldots\frac{1}{p_n}\right)}{\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\ldots\frac{1}{p_x}\right)}$ converge to 1.

So $\dfrac{\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\ldots\frac{1}{p_n}\right)}{\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\ldots\frac{1}{p_x}\right)}$ eventually becomes smaller than 1.5.

So we should have that :

$$2N\cdot 1.5>2N\cdot\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\ldots\frac{1}{p_x}-1\right)+\left(2N\cdot\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\ldots\frac{1}{p_n}-1\right)\right)\cdot\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\ldots\frac{1}{p_x}\right)$$

We can see that this definitely becomes false when

$$\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{5}...+\frac{1}{p_x}\right)>2$$

We should´nt forget that

$$\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{5}...+\frac{1}{p_x}\right)$$

is an approximation.

But it is a very close approximation.

If the approximation value is A(p)

and the real value is R(p)

then A(p)>R(p)/2

Since (2N*1+2N*1*2)=6N

which is the value where A(p)>6N

Then R(p)>6N/2=3N=1.5*2N

We get that this becomes false also for the real value

when

$$\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{5}...+\frac{1}{p_x}\right)>2$$

So Legendre conjecture is definitely true for all N>P_n where

$$\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{5}...+\frac{1}{p_x}\right)>2$$

and P_x is the highest prime less than (N+1)^(2/3)

and P_n is the largest prime less than or equal to N.

Q.E.D

*) Source: wikipedia