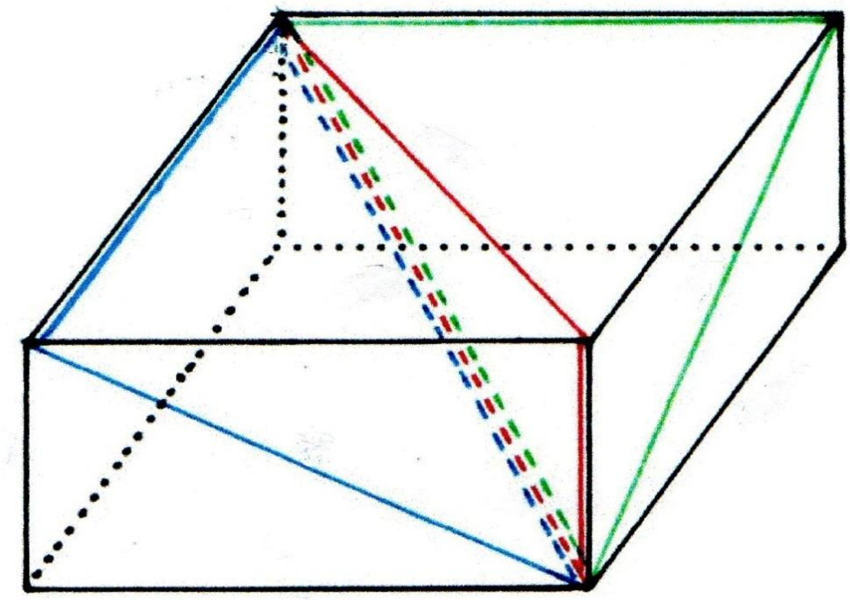

EULER BRICKS

AND THE

PERFECT CUBOID

 18/04/2018

PYTHAGOREAN TRIPLES AND QUADRUPLES

INTRODUCTION

Does a **Perfect Cuboid** exist? is an unsolved problem in Mathematics.

Definitions

A **Pythagorean Triple** is defined as a right-angled triangle with the two shorter sides $\{x, y\}$ such that $x^2 + y^2 = z^2$ where $\{z\}$ is the hypotenuse and all three are integers.

A **Pythagorean Quadruple** (**Pythagorean Quad** for short) is defined as a rectangular parallelepiped (a rectangular box) with side lengths $\{a, b, c\}$ such that $a^2 + b^2 + c^2 = d^2$ where $\{d\}$ is the internal space diagonal and all four are integers, see FIGURE 1.

An **Euler Bric**k is defined as a rectangular parallelepiped where the side lengths $\{a, b, c\}$ and the face diagonals $\{e, f, g\}$ are all integers. If the internal space diagonal $\{d\}$ is also an integer then this is defined as a **Perfect Cuboid**. Does such a Cuboid exist? As of January 2018, computer searches into the billions for the shortest side $\{a\}$ have failed to find a Perfect Cuboid, nor has it been proved that one does not exist. In the following article, it is hoped to prove that no such Cuboid exists. The proof attempts to show that if the internal space diagonal is an integer then one of the face diagonals is not an integer and if all three of the face diagonals are integers then the internal space diagonal is not an integer. The proof requires a short analysis of Pythagorean Triples.

HISTORY

The first known, and smallest, Euler Brick with side lengths $\{a, b, c\} = (44, 117, 240)$ and face diagonals $\{e, f, g\} = (125, 244, 267)$, was found by Paul Halke (1662 – 1731) in 1719.

Nicholas Saunderson (1682 – 1739), the fourth Lucasian Professor at Cambridge, who was blind from the age of one, found a parametric solution to the Euler Brick [1].

Saunderson's parametrisation, given that $\{x, y, z\}$ is a Pythagorean Triple then

$$\{a, b, c\} = \left(y\,(4x^2 - z^2),\ \ x(4y^2 - z^2),\ \ 4x\,y\,z\right)$$

and $$\{e, f, g\} = \left(z^3,\ y\,(4x^2 + z^2),\ \ x(4y^2 + z^2)\right),$$

e.g. $\{x, y, z\} = (3, 4, 5)$ produces

$$\{a, b, c\} = \left(4 \times (36 - 25),\ 3 \times (64 - 25),\ (4 \times 3 \times 4 \times 5)\right) = (44, 117, 240)$$

and $$\{e, f, g\} = \left(5^3,\ 4 \times (36 + 25),\ 3 \times (64 + 25)\right) = (125, 244, 267).$$

In the 1770's, Euler (1707 – 1783) found a couple of parametrisations and more were found in his papers after his death. It is surprising that Euler made no attempt to solve the Perfect Cuboid problem, but there is nothing in his notes concerning the problem.

Parametrisations will find an infinite number of results but not all, and none of them include the space diagonal$\{d\}$.

There is a great deal of information on the Internet, including at least three proposed proofs; Google Euler Brick or Perfect Cuboid and follow the links.

PYTHAGOREAN TRIPLES

An investigation into **Pythagorean Quads** requires an analysis of **Pythagorean Triples**.

Abbreviations (I) = Integer > 0,

I□ ⇒ I^2 should be square but not proven for a perfect cuboid.

Pythagoras' Theorem is taken for granted, for a rigorous proof see **Euclid 1.XLVII** [2].

A **Pythagorean Triple** is defined as $x^2 + y^2 = z^2$ where x and y are the two shorter sides and z is the hypotenuse of a right-angled triangle and all three are integers.

A **Primitive Pythagorean Triple** is defined as a Pythagorean Triple with GCD $(x, y) = 1$.

Theorem 1 Any integer $x > 2$ with $(0 < w < x)$ and $w \mid x^2$ generates all Pythagorean Triples.

Proof $x^2 + y^2 = z^2$ and without loss of generality assume $x < y$,

then $x^2 = z^2 - y^2 = (z - y)(z + y)$ since $(z - y)(z + y) = z^2 + zy - yz - y^2 = z^2 - y^2$.

Let $z - y = w$ then $z = y + w$ and $x^2 = w(y + w + y) = w(2y + w) = 2yw + w^2$

therefore $y = \dfrac{x^2 - w^2}{2w}$ and $z = y + w = \dfrac{x^2 - w^2}{2w} + \dfrac{2w^2}{2w} = \dfrac{x^2 + w^2}{2w}$. See Table 1.

For $x > 2$, $w \mid x^2$ and for y to be positive, $0 < w < x$.
For $x > 2$ with $(0 < w < x)$ and $w \mid x^2$ generates all Pythagorean Triples. ❑

Notes for Theorem 1.

1 If x is even, w odd or even, then y, and hence z, can be fractions with the denominator equal to 2. e.g. for $x = 4$, $y = 15/2$ and $z = 17/2$ with $w = 1$, multiplying by 2 gives the Triple (8, 15, 17). $x = 10$ produces not only (10, 24, 26) with w = 2, but also (10, 21/2, 29/2) with $w = 4$ and (10, 15/2, 25/2) with $w = 5$. To eliminate this problem, ensure that y is an integer by using the instruction `If[IntegerQ[y]` in the software for Pythagorean Triples, and `If[IntegerQ[g]]` or `If[IntegerQ[d]` for Pythagorean Quadruples.

For all Primitive Pythagorean Triples GCD (x, y) = 1 : –

2 For $x > 2$ there always exists a Pythagorean Triple, if x is odd let $w = 1$ and if x is even let $w = 2$.
3 If x is odd, w is odd since the numerator of both y and z must be even to be divisible by 2.
4 If x is even, and w is even then the numerator of both y and z is divisible by 4.
5 If x is odd, w is odd and y is even, if x is even, w is even and y is odd, in both cases z is odd.

Theorem 2 **Diophantus'** method for generating Pythagorean Triples $\{X, Y, Z\}$.

Given any u and v with $u < v$ then $X = v^2 - u^2$, $Y = 2uv$ and $Z = v^2 + u^2$.

Proof $X^2 + Y^2 = (v^2 - u^2)^2 + (2uv)^2 = v^4 - 2v^2u^2 + u^4 + 4u^2v^2$

$= v^4 + 2v^2u^2 + u^4 = (v^2 + u^2)^2 = Z^2$. ❑

It is left to the reader to verify that $u = 1$, $v = 2$ produces the Pythagorean Triple (3, 4, 5)!

PYTHAGOREAN QUADS

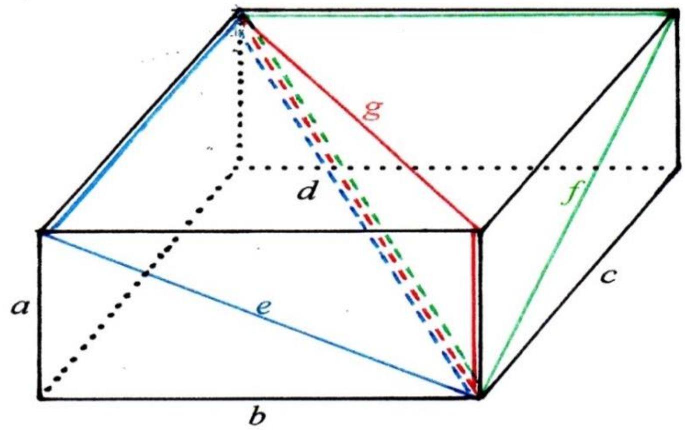

FIGURE 1

A **Pythagorean Quad** is defined as $a^2 + b^2 + c^2 = d^2$ where a, b and c are the lengths of the three sides and d is the internal space diagonal of a Rectangular Parallelepiped (rectangular box) and all four are integers. See FIGURE 1.

A **Primitive Pythagorean Quad** is defined when GCD (a, b, c,) = 1. By the definition of a Primitive Pythagorean Triple, one of a, b or c is odd and the other two are even, hence d is odd.

Theorem 3 For a Rectangular Parallelepiped of side lengths a, b, and c, $a^2 + b^2 + c^2 = d^2$.

Proof By Pythagoras' Theorem $a^2 + b^2 = e^2$ and $e^2 + c^2 = d^2$, use Pythagoras twice. ❑

There are an infinite number of such Quads, the smallest being (1, 2, 2, 3). See Appendix 2.

An **Euler Brick** is defined as a Pythagorean Quad where the edges $\{a, b, c\}$ and the three face diagonals $\{e, f, g\}$ are also integers A **Perfect Cuboid** is defined if the internal space diagonal $\{d\}$ is also an integer. Does such a Cuboid exist ?

A **Primitive Perfect Cuboid** would be defined as a Perfect Cuboid with GCD (a, b, c) = 1.

From Figure 1, with e, f and g as the face diagonals and d as the internal space diagonal

$$a^2 + b^2 + c^2 = d^2 \quad \text{and} \quad a^2 + b^2 = e^2, \quad a^2 + c^2 = f^2, \quad b^2 + c^2 = g^2$$

also $$a^2 + g^2 = d^2, \quad b^2 + f^2 = d^2 \quad \text{and} \quad c^2 + e^2 = d^2$$

i.e. $\{a, b, e\}$, $\{a, c, f\}$, $\{b, c, g\}$, $\{a, g, d\}$, $\{b, f, d\}$ and $\{c, e, d\}$ all require to be Pythagorean Triples for a Perfect Cuboid to exist. Some of the Triples can be taken in sets of three to generate all the values of a Perfect Cuboid. Assuming that at least one of the values must be an integer, take that to be the side a. In general, take a to be the smallest side and $a < b < c$ by making $0 < m < l < a$ when deriving Triples as per Theorem 1.

Consider the set $\{\{a, b, e\}, \{a, c, f\}, \{a, g, d\}\}$ which contains all seven unknowns and each subset contains the value a. From Theorem 1, with a an integer

$$b = \frac{a^2 - l^2}{2l} \qquad e = \frac{a^2 + l^2}{2l} \qquad c = \frac{a^2 - m^2}{2m} \qquad f = \frac{a^2 + m^2}{2m}$$

hence $e - b = \dfrac{a^2 + l^2}{2l} - \dfrac{a^2 - l^2}{2l} = l$ and $f - c = \dfrac{a^2 + m^2}{2m} - \dfrac{a^2 - m^2}{2m} = m,$

therefore $e = b + l$ and $f = c + m$, where $0 < m < l < a$ and m and l are different divisors of $a^2 \Rightarrow a, b, c, e$ and f are all integers for some integer a.

In the following, let $\delta^{\square}$ and $\gamma^{\square}$ be as shown below in the brackets under the root signs for d and g respectively, where $\delta^{\square}$ and $\gamma^{\square}$ must be, may be, should be or are square but not proven. δ (delta) and γ (gamma) are the Greek letters for d and g respectively.

Now $$d^2 = c^2 + e^2 = \frac{(a^2 - m^2)^2}{4m^2} + \frac{(a^2 + l^2)^2}{4l^2} = \frac{l^2(a^2 - m^2)^2 + m^2(a^2 + l^2)^2}{4l^2m^2}$$

hence $$d = \frac{1}{2lm}\sqrt{l^2(a^2 - m^2)^2 + m^2(a^2 + l^2)^2 \ (= \delta^{\square})},$$ for $d = (\mathrm{I}) \rightarrow$ Tables 2 & 3.

Also $$d^2 = f^2 + b^2 = \frac{(a^2 + m^2)^2}{4m^2} + \frac{(a^2 - l^2)^2}{4l^2} = \frac{l^2(a^2 + m^2)^2 + m^2(a^2 - l^2)^2}{4l^2m^2}$$

and $$d = \frac{1}{2lm}\sqrt{l^2(a^2 + m^2)^2 + m^2(a^2 - l^2)^2 \ (= \delta^{\square})}$$

$$= \frac{1}{2lm}\sqrt{(a^4 + l^2m^2)(l^2 + m^2) \ (= \delta^{\square})}.$$ See Note 2.

Let $(a^4 + l^2 m^2) = p$ and $(l^2 + m^2) = q$. For integer d see Table 2 for p and q both integers and Table 3 for p and q not integers but with a common $\sqrt{}$.

Now $$g^2 = b^2 + c^2 = \frac{(a^2 - l^2)^2}{4l^2} + \frac{(a^2 - m^2)^2}{4m^2} = \frac{m^2(a^2 - l^2)^2 + l^2(a^2 - m^2)^2}{4l^2m^2}$$

hence $$g = \frac{1}{2lm}\sqrt{l^2(a^2 - m^2)^2 + m^2(a^2 - l^2)^2 \ (= \gamma^{\square})},$$ for $g = (\mathrm{I}) \rightarrow$ Table 4.

$$= \frac{1}{2lm}\sqrt{(a^4 + l^2m^2)(l^2 + m^2) - 4a^2l^2m^2 (= \gamma^{\square})}.$$ See Note 3.

Note 1 $l^2(a^2 \pm m^2)^2 + m^2(a^2 \mp l^2)^2 = \delta^{\square}$ and $l^2(a^2 - m^2)^2 + m^2(a^2 - l^2)^2 = \gamma^{\square}$ and since all three equations are derived from a^2, b^2, c^2, e^2 and f^2, which are all square by construction, then for a Perfect Cuboid to exist, all three equations require to be Pythagorean Triples.

Three versions are shown for $\delta^{\square}$ and two for $\gamma^{\square}$ and others will be developed for Theorem 4. Since all versions are numerically equal, if any one version of $\delta^{\square} \neq$ square then $d \neq$ an integer and if any one version of $\gamma^{\square} \neq$ square then $g \neq$ an integer.

For notes 2 and 3 see reference [3] **Beiler** page 143.

Note 2 $\quad l^2 (a^2 \pm m^2)^2 + m^2(a^2 \mp l^2)^2 = (a^4 + l^2m^2)(l^2 + m^2)$.

Proof LHS $= l^2 (a^2 \pm m^2)^2 + m^2(a^2 \mp l^2)^2$

$= l^2a^4 \pm 2a^2l^2m^2 + l^2m^4 + m^2a^4 \mp 2a^2l^2m^2 + m^2l^4$

$= l^2(a^4 + l^2m^2) + m^2(a^4 + l^2m^2) = (a^4 + l^2m^2)(l^2 + m^2) =$ RHS. ❑

Note 3 $\quad l^2 (a^2 - m^2)^2 + m^2(a^2 - l^2)^2 = (a^4 + l^2m^2)(l^2 + m^2) - 4a^2l^2m^2$.

Proof LHS $= l^2 (a^2 - m^2)^2 + m^2(a^2 - l^2)^2$

$= l^2a^4 - 2a^2l^2m^2 + l^2m^4 + m^2a^4 - 2a^2l^2m^2 + m^2l^4$

$= l^2(a^4 + l^2m^2) + m^2(a^4 + l^2m^2) - 4a^2l^2m^2$

$= (a^4 + l^2m^2)(l^2 + m^2) - 4a^2l^2m^2 =$ RHS. ❑

Theorem 4 A **Perfect Cuboid** does not exist, but first a trio of Lemmas, (Lemmata ?).

Lemma 1 The product of two squares is also a square.

Proof $x^2 \times y^2 = (xy)^2$ for all x and y greater than 0. ❑

Lemma 2 The product of two numbers, one square and the other non-square is not a square.

Proof $x^2 \times y^2 = (xy)^2 \Rightarrow x^2 \times y \neq$ square for both x and $y > 1$ and $y \neq$ square. ❑

Lemma 3 $a^4 + l^4 \neq z_l{}^2$ and $a^4 + m^4 \neq z_m{}^2$ and consequently
$a^4 + l^4 \neq z_l{}^4$ and $a^4 + m^4 \neq z_m{}^4$ since $z^4 = (z^2)^2$.

Proof

That the sum of two fourth powers does not equal a fourth power was proved by **Fermat** in or about 1637 (by his method of Infinite Descent) but was definitively proved by **Professor Sir Andrew Wiles** in 1995 for all values of n greater than two. In this case $n = 4$. ❑

See Appendix 3 for a simple proof of Fermat's Last Theorem for $n = 4$ and multiples thereof, e.g. 8, 12, 16, 20 etc.

Theorem 4 A **Perfect Cuboid** does not exist.

Part 1 Consider $\delta^{\square}$ and $\gamma^{\square}$ the square root parts of d and g respectively.

$\delta^{\square} = m^2 (a^2 \pm l^2)^2 + l^2 (a^2 \mp m^2)^2 \Rightarrow$ two Pythagorean Triples equation 1

expanding both equations $\delta^{\square} = a^4 l^2 + a^4 m^2 + l^4 m^2 + l^2 m^4$

and $\gamma^{\square} = m^2 (a^2 - l^2)^2 + l^2 (a^2 - m^2)^2 \Rightarrow$ one Pythagorean Triple equation 2

expanding this equation $\gamma^{\square} = a^4 l^2 + a^4 m^2 + l^4 m^2 + l^2 m^4 - 4a^2 l^2 m^2$.

Now $\delta^{\square} - \gamma^{\square} =$ equation 1 $-$ equation 2 $= 4a^2 l^2 m^2$,

hence $\delta^{\square} = \gamma^{\square} + 4a^2 l^2 m^2$ and $\gamma^{\square} = \delta^{\square} - 4a^2 l^2 m^2$.

Proof Part 2 Assume that $d = (\mathrm{I}) \Rightarrow \delta^{\square} = \delta^2$

then $\gamma^{\square} = m^2 (a^2 + l^2)^2 - 2a^2 l^2 m^2 + l^2 (a^2 - m^2)^2 - 2a^2 l^2 m^2$

$= m^2 (a^4 + l^4) + l^2 (a^4 - 4\, a^2 m^2 + m^4) \Rightarrow \gamma^{\square}$ is not a Pythagorean Triple

since $m^2 (a^4 + l^4) \neq$ a square by Lemmas 1, 2 and 3. However $l^2(a^4 - 4a^2 m^2 + m^4)$ may be square, (let $m = a/2$) but, for a Perfect Cuboid, $\{a, g, d\}$ requires to be a Pythagorean Triple, and $\gamma^{\square}$ is not the sum of two squares because one side of the triple, $m^2 (a^4 + l^4)$, is not a square. Hence, given that $d = (\mathrm{I})$ then $\gamma^{\square}$ is not a Pythagorean Triple, a requirement by equation 2. See Example 1 and Tables 2 and 3.

Proof Part 3 Assume that $g = (\mathrm{I}) \Rightarrow \gamma^{\square} = \gamma^2$

then $\delta^{\square} = \gamma^2 + 4a^2 l^2 m^2 = m^2 (a^2 - l^2)^2 + 2a^2 l^2 m^2 + l^2 (a^2 - m^2)^2 + 2a^2 l^2 m^2$

$= m^2(a^4 + l^4) + l^2(a^4 + m^4) \Rightarrow \delta^{\square}$ is not a Pythagorean Triple

since both $m^2(a^4 + l^4) \neq$ a square and $l^2(a^4 + m^4) \neq$ a square by Lemmas 1, 2 and 3.

Hence, given that $g = (\mathrm{I})$ then $\delta^{\square}$ is not the sum of two squares and is not a Pythagorean Triple, a requirement by equation 1. See Example 2 and Table 4.

Parts 2 and 3 imply that $\{a, g, d\}$ is not a Pythagorean Triple and hence a Perfect Cuboid does not exist.

Theorem 4 only proves that triangle $\{a, g, d\}$ is not a Pythagorean Triple, but individually $\delta^{\square}$ or $\gamma^{\square}$ may be square by chance (see the example below). In this case, geometrically, the triangle will not be right angled and the parallelepiped will be skewed and not rectangular. Hence, for a rectangular parallelepiped, one of $\delta^{\square}$ or $\gamma^{\square}$ cannot be square and therefore one of d or g cannot be an integer. ❑

Example. Consider 6^2 which is the sum of two numbers in 18 different ways, in particular $1^2 + 35$, $2^2 + 2^5$, $3^2 + 3^3$, $4^2 + 20$, $5^2 + 11$, $6 + 30$, $7 + 29$, etc. all sum to 6^2, not one of which is a Pythagorean Triple.
Also, the sum of two squares may not necessarily be square e.g. $5^2 + 6^2 = 61$.

A similar proof can be applied to triangles$\{c, e, d\}$ and $\{b, f, d\}$ with similar results by considering the sets $\{\{a, b, e\}, \{a, c, f\}, \{c, e, d\}\}$ and $\{\{a, b, e\}, \{a, c, f\}, \{b, f, d\}\}$.

Example 1 From Table 3 $a = 104$; $l = 32$; $m = 8$; $(2\,l\,m) = 512$;

$$\delta^{□} = m^2(a^4 + l^4) + l^2(a^4 + m^4)$$
$$= 64 \times 118034432 + 1024 \times 116989952$$
$$= (512\sqrt{28817})^2 + (2048\sqrt{28562})^2$$
$$d = \frac{1}{2lm}\sqrt{64 \times 118034432 + 1024 \times 116989952} = 697$$
$$\gamma^{□} = m^2(a^4 + l^4) + l^2(a^4 - 4a^2m^2 + m^4)$$
$$= 64 \times 118034432 + 1024 \times 114221056$$
$$= (512\sqrt{28817})^2 + (2048\sqrt{27886})^2$$
$$g = \frac{1}{2lm}\sqrt{64 \times 118034432 + 1024 \times 114221056} = 3\sqrt{52777}.$$ ❑

Example 2 From Table 4 $a = 44$; $l = 8$; $m = 4$; $(2\,l\,m) = 64$;

$$\delta^{□} = m^2(a^4 + l^4) + l^2(a^4 + m^4)$$
$$= 16 \times 3752192 + 64 \times 3748352$$
$$= (64\sqrt{14657})^2 + (128\sqrt{14642})^2$$
$$d = \frac{1}{2lm}\sqrt{16 \times 3752192 + 64 \times 3748352} = 5\sqrt{2929}.$$
$$\gamma^{□} = m^2(a^4 + l^4) + l^2(a^4 - 4a^2m^2 + m^4)$$
$$= 16 \times 3752192 + 64 \times 3624448$$
$$= (64\sqrt{14657})^2 + (128\sqrt{14156})^2$$
$$g = \frac{1}{2lm}\sqrt{60035072 + 231964672} = 267.$$ ❑

References

1 N. Saunderson, The Elements of Algebra, volume 2, Cambridge, 1740.

2 Euclid's Elements, Dover Publications Inc. 2nd edition.

3 For some interesting information on Pythagorean Triples and Quads see Recreations in the Theory of Numbers by Albert H. Beiler (Dover Publications 2nd edition 1964) chapters XIV and XV.

4 Mathematica™, a mathematical programming language by Wolfram Research.

5 Professor Ian Stewart, Cabinet of Mathematical Curiosities page 58. Profile Books 2008.

APPENDIX 1 PROGRAMS AND TABLES

Although great care has been taken in the preparation of the tables some transcription errors may still exist. The author takes full responsibility for any such errors and offers his apologies. Tables of course can only produce a CONJECTURE **NOT** a THEOREM.

The tables were produced with software written using Mathematica™ [4].

Program 1 Pythagorean Triples, produces Tables 1a and 1b.

```
For [x = 2, x < 1002, x++,
   k = Table[Select[Divisors[x^2],# > 0 && # < x &]];
  lk = Length[k];
  For[i = 1, i < lk, i++, w = k[[i]];
    y = (x^2 – w^2)/(2 w); z = y + w;
    If[IntegerQ[y] && y > x && GCD[x, y, z] < 2 (*&& w < 3||w > 2*),
     Print["  x = ", x, "   y = ", y, "   z = ", z, "   w = ", w]
    ]]]
```

Select `w < 3` for Table 1a or `w > 2` for Table 1b.

Program 2 Pythagorean Quads, produces Tables 2, 3 and 4.

```
For[a = 1, a < 1001, a++,
  k   = Table[Select[Divisors[a^2], # > 0 && # < a &]];
  lk  = Length[k];
  For[i = 2, i <= lk, i++, l = k[[i]];
    For[j = 1, j < i, j++, m = k[[j]];
      b = (a^2 - l^2)/(2*l);    c = (a^2 - m^2)/(2*m);
      e = b + l;                f = c + m;
      g = Sqrt[b^2 + c^2];      d = Sqrt[e^2 + c^2];
      p = Sqrt[a^4 + l^2*m^2]; q = Sqrt[l^2 + m^2];
      If[IntegerQ[??] && GCD[a,b,c]<2 && a<b<c   (*&& IntegerQ[p]*),
       Print[ " a = ", a, " b = ", b, "  c = ", c]
       Print[ " e = ", e, " f = ", f, "  g = ", g]
       Print[ " d = ", d, " l = ", l, "  m = ", m]
       Print[ " p = ", p, " q = ", q]     (*not required for IntegerQ[g]*)
       Print[]]
   ] ] ]]
```

For `IntegerQ[??]` use `IntegerQ[g]` or `IntegerQ[d]` as required.

Put comment brackets `(*     *)` around whichever statements are not required.

e.g. `(*&&(a<b<c)*).`

Notes The `GCD[a, b, c] < 2` instruction eliminates all multiples such that *a*, *b*, and *c* are always relatively prime.
The `(a < b < c)` instruction ensures that only one instance of (*a*, *b*, *c*) occurs, thus eliminating occurrences of values beginning with *b* or *c*. e.g.

a	*b*	*c*	*e*	*f*	*g*	*d*	*l*	*m*
44	117	240	125	244	267	$5\sqrt{2929}$	8	4
117	44	240	125	267	244	$5\sqrt{2929}$	81	27
240	44	117	244	267	125	$5\sqrt{2929}$	200	150

TABLE 1a PYTHAGOREAN TRIPLES TABLE 1b

$w < 3$ eliminates cases where $z - y > 2$

#	x	y	z	w
1	3	4	5	1
2	5	12	13	1
3	7	24	25	1
4	8	15	17	2
5	9	40	41	1
6	11	60	61	1
7	12	35	37	2
8	13	84	85	1
9	15	112	113	1
10	16	63	65	2
11	17	144	145	1
12	19	180	181	1
13	20	99	101	2
14	21	220	221	1
15	23	264	265	1
16	24	143	145	2
17	25	312	313	1
18	27	364	365	1
19	28	195	197	2
20	29	420	421	1
21	31	480	481	1
22	32	255	257	2
23	33	544	545	1
24	35	612	613	1
25	36	323	325	2
26	37	684	685	1
27	39	760	761	1
28	40	399	402	2
29	41	840	841	1
30	43	924	925	1
31	44	483	485	2
32	45	1012	1013	1
33	47	1104	1105	1
34	48	575	577	2
35	49	1200	120	1
36	51	1300	1301	1
37	52	675	677	2
38	53	1404	1405	1
39	55	1512	1513	1
40	56	783	785	2
41	57	1624	1625	1
42	59	1740	1741	1
43	60	899	901	2
44	61	1860	1861	1
45	63	1984	1985	1
46	64	1023	1025	2
47	65	2112	2113	1
48	67	2244	2245	1
49	68	1155	1157	2
50	69	2380	2381	1
51	71	2529	2521	1
52	72	1295	1297	2
53	73	2664	2665	1
54	74	1368	1370	2

$w > 2$ eliminates cases where $z - y < 3$

#	x	y	z	w
1	20	21	29	8
2	28	45	53	8
3	33	56	65	9
4	36	77	85	8
5	39	80	89	9
6	44	117	125	8
7	48	55	73	18
8	51	140	149	9
9	52	165	173	8
10	57	176	185	9
11	60	91	109	18
12	60	221	229	8
13	65	72	97	25
14	68	285	293	8
15	69	260	269	9
16	75	308	317	9
17	76	357	365	8
18	84	187	205	18
19	84	437	445	8
20	85	132	157	25
21	87	416	425	9
22	88	105	137	32
23	92	525	533	8
24	93	476	485	9
25	95	168	193	25
26	96	247	265	18
27	100	621	629	8
28	104	153	185	32
29	105	208	233	25
30	105	608	617	9
31	108	725	733	8
32	111	680	689	9
33	115	252	277	25
34	116	837	845	8
35	119	120	169	49
36	120	209	241	32
37	120	391	409	18
38	123	836	845	9
39	124	957	965	8
40	129	920	929	9
41	132	475	493	18
42	132	1085	1093	8
43	133	156	205	49
44	135	352	377	25
45	136	273	305	32
46	140	271	221	50
47	140	1221	1229	8
48	141	1100	1109	9
49	145	408	433	25
50	147	1196	1205	9
51	148	1365	1373	8
52	152	345	377	32
53	155	468	493	25
54	156	667	685	18

$x = 105$ the product of three primes $3 \times 5 \times 7$

#	x	y	z	w
1	105	36	111	$75 = 5^2 \times 3$
2	105	56	119	$63 = 7 \times 3^2$
3	105	88	137	$49 = 7^2$
4	105	100	145	$45 = 5 \times 3^2$
5	105	140	175	$35 = 7 \times 5$
6	105	208	233	$25 = 5^2$
7	105	252	273	$21 = 7 \times 3$
8	105	360	375	$15 = 5 \times 3$
9	105	608	617	$9 = 3^2$
10	105	784	791	7
11	105	1100	1105	5
12	105	1836	1839	3
13	105	5512	5513	1

$x = 1001$ the product of three primes $7 \times 11 \times 13$

#	x	y	z	w
1	1001	168	1015	$847 = 11^2 \times 7$
2	1001	468	1105	$637 = 13 \times 7^2$
3	1001	660	1199	$539 = 11 \times 7^2$
4	1001	2880	3049	$169 = 13^2$
5	1001	3432	3575	$143 = 13 \times 11$
6	1001	4080	4201	$121 = 11^2$
7	1001	5460	5551	$91 = 13 \times 7$
8	1001	6468	6545	$77 = 11 \times 7$
9	1001	10200	10249	$49 = 7^2$
10	1001	38532	38545	13
11	1001	45540	45551	11
12	1001	71568	71575	7
13	1001	501000	501001	1

TABLE 2 $\{(a, b, e), (a, c, f), (a, g, d)\}$. Values for d, p and q = **(I).**

#	*a*	*b*	*c*	*e*	*f*	*g*	*d*	*l*	*m*	*p*	*q*
1	840	448	495	952	975	$\sqrt{445729}$	1073	504	480	745920	696
2	1680	3404	4653	3796	4947	$5\sqrt{1329505}$	6005	392	294	2824752	490
3	1680	1925	2052	2555	2652	$\sqrt{7916329}$	3277	630	600	2847600	870
4	1680	819	3740	1869	4100	$\sqrt{14658361}$	4181	1050	360	2847600	1110
5	2640	2275	2772	3485	3828	$7\sqrt{262441}$	4453	1210	1056	7085760	1606
6	6552	2464	30225	7000	30927	$\sqrt{919621921}$	31025	4536	702	43046640	4590
7	10920	9152	16065	14248	19425	$\sqrt{341843329}$	21473	5096	3360	120469440	6104
8	21840	14651	37620	26299	43500	$\sqrt{1629916201}$	45901	11648	5880	481877760	13048
9	21840	2925	12628	22035	25228	$\sqrt{168022009}$	25397	19110	12600	534315600	22890
10	24480	27776	35343	37024	42993	$35\sqrt{1649497}$	51185	9248	7650	603432000	12002
11	27720	9504	12103	29304	30247	$5\sqrt{9472345}$	31705	19800	18144	848232000	26856
12	27720	1632	68585	27768	73975	$\sqrt{4706565649}$	73993	26136	5390	781205040	26686
13	31920	31185	63232	44625	70832	$\sqrt{4970790049}$	77393	13440	7600	1023993600	15440
14	36960	3528	10175	37128	38335	$\sqrt{115977409}$	38497	33600	28160	1661721600	43840
15	42840	9120	37961	43800	57239	$\sqrt{1524211921}$	57961	34680	19278	1953246960	39678
16	43680	24957	76076	50307	87724	$5\sqrt{256416385}$	91205	25350	11648	1930656000	27898
17	50160	3843	76076	50307	91124	$35\sqrt{4736593}$	91205	46464	15048	2611369728	48840
18	55440	127281	153920	138831	163600	$\sqrt{39891819361}$	207281	11550	9680	3075626400	15070
19	55440	43549	53580	70499	77100	$\sqrt{4767331801}$	88549	26950	23520	3138273600	35770
20	60192	34944	36575	69600	70433	$7\sqrt{52220689}$	78625	34656	33858	3808347840	48450
21	63840	40120	92169	75400	112119	$\sqrt{10104738961}$	119081	35280	19950	4135874400	40530
22	63840	17575	238392	66215	246792	$\sqrt{57139626289}$	247417	48640	8400	4095974400	49360
23	67320	9975	41888	68055	79288	$7\sqrt{37838881}$	79913	58080	37400	5025662400	69080
24	68640	99099	345100	120549	351860	$7\sqrt{2630910649}$	365549	21450	6760	4713680400	22490
25	82800	27692	352275	87308	361875	$7\sqrt{2548255561}$	362933	59616	9600	6879686400	60384
26	97440	51205	420732	110075	431868	$\sqrt{179637367849}$	434893	58870	11136	9517159680	59914
27	110880	229215	340472	254625	358072	$\sqrt{168460699009}$	425153	25410	17600	12302505600	30910
28	131040	136747	342804	189397	366996	$5\sqrt{5448572977}$	391645	52650	24192	17218656000	57942
29	146160	122496	465647	190704	488047	$5\sqrt{9273295945}$	503185	68208	22400	21417312000	71792
30	150480	100947	105196	181203	183604	$35\sqrt{17352241}$	209525	80256	78408	23502327552	112200
31	177072	110925	219604	208947	282100	$\sqrt{60530272441}$	303125	98022	62496	31947330240	116250
32	205632	144976	428175	251600	474993	$\sqrt{204351871201}$	496625	106624	46818	42578161920	116450
33	210672	71725	185196	222547	280500	$\sqrt{39442034041}$	289525	150822	95304	46652261040	178410
34	221760	289068	307195	364332	378875	$\sqrt{177929076649}$	476557	75264	71680	49472532480	103936
35	221760	133300	550221	258740	593229	$\sqrt{320512038841}$	608021	125440	43008	49472532480	132608
36	251328	418304	546975	488000	601953	$\sqrt{474159887041}$	733025	69696	54978	63281877120	88770
37	271320	195975	227392	334695	354008	$\sqrt{90113322289}$	404633	138720	126616	75680915520	187816
38	286440	314545	410688	425425	500712	$\sqrt{267603190369}$	591313	110880	90024	82652834880	142824
39	314160	700749	2003620	767949	2028100	$\sqrt{4505542265401}$	2145749	67200	24480	98710214400	71520
40	317856	611667	804100	689325	864644	$\sqrt{1020713328889}$	1059125	77658	60544	101141779200	98470
41	341880	223839	769600	408639	842120	$\sqrt{642388057921}$	871361	184800	72520	117647745600	198520
42	351120	457691	1649340	576859	1686300	$17\sqrt{10137728329}$	1747309	119168	36960	123363905280	124768
43	364320	434700	1467389	567180	1511939	$7\sqrt{47799889129}$	1573189	132480	44550	132860217600	139770
44	371280	1679040	1693489	1719600	1733711	$13\sqrt{33651362809}$	2413489	40560	40222	137858491680	57122
45	371280	32868	415915	372732	557525	$13\sqrt{1029973921}$	558493	339864	141610	146008973040	368186
46	393120	108225	150568	407745	420968	$\sqrt{34383373249}$	434657	299520	270400	174479385600	403520
47	425040	137655	311168	446775	526768	$\sqrt{115774423249}$	544457	309120	215600	192560121600	376880
48	432432	422125	1055124	604307	1140300	$\sqrt{1291476171001}$	1215925	182182	85176	187640172720	201110
49	443520	44321	1802640	445729	1856400	$\sqrt{3251475320641}$	1856929	401408	53760	197890129920	404992
50	456456	1030617	1830400	1127175	1886456	$\sqrt{4412535560689}$	2149625	96558	56056	208422374160	111650
51	514800	880929	990080	1020321	1115920	$7\sqrt{35842741009}$	1421729	139392	125840	265598910720	187792
52	526680	222464	518073	571736	738777	$5\sqrt{12715594585}$	771545	349272	220704	287903512512	413160
53	572880	345708	493675	669108	756245	$\sqrt{363229026889}$	831517	323400	262570	338998875600	416570
54	572880	221309	1563660	614141	1665300	$\sqrt{2494010269081}$	1679941	392832	101640	330611339520	405768
55	600600	155705	1196352	620455	1338648	$\sqrt{1455502154929}$	1347673	464750	142296	366732366000	486046
56	632016	138787	153900	647075	650484	$\sqrt{42947041369}$	665125	508288	496584	472509607680	710600
57	637560	736593	1226176	974193	2382024	$425\sqrt{11327761}$	1566065	237600	155848	408165912000	284152
58	683760	1522180	4036851	1668700	4094349	$\sqrt{18613197948601}$	4368149	146520	57498	467603634960	157398
59	683760	726869	7505820	997931	7536900	$23\sqrt{107496545209}$	7571869	271062	31080	467603634960	272838
60	683760	614180	1893771	919100	2013429	$\sqrt{3963585672841}$	2105021	304920	119658	468949274640	327558
61	683760	132349	3581820	696451	3646500	$\sqrt{12846950770201}$	3648901	564102	64680	468949274640	567798
62	702576	70707	766700	706125	1039924	$\sqrt{592828369849}$	1042325	635418	273224	523254014640	691670
63	939120	579215	1089792	1103375	1438608	$\sqrt{1523136619489}$	1550833	524160	348816	900698722560	629616

TABLE 3 $\{(a, b, e), (a, c, f), (a, g, d)\}$. Values for d = **(I)** with a common $\sqrt{\ }$ to p and q.

#	*a*	*b*	*c*	*e*	*f*	*g*	*d*	*l*	*m*	*p*	*q*
1	104	153	672	185	680	$3\sqrt{52777}$	697	32	8	$2624\sqrt{17}$	$8\sqrt{17}$
2	117	520	756	533	765	$4\sqrt{52621}$	925	13	9	$4329\sqrt{10}$	$5\sqrt{10}$
3	252	2261	2640	2275	2652	$\sqrt{12081721}$	3485	14	12	$6888\sqrt{85}$	$2\sqrt{85}$
4	333	644	2040	725	2067	$4\sqrt{286021}$	2165	81	27	$35073\sqrt{10}$	$27\sqrt{10}$
5	399	468	4180	615	4199	$68\sqrt{3826}$	4225	147	19	$13965\sqrt{130}$	$13\sqrt{130}$
6	448	264	975	520	1073	$3\sqrt{113369}$	1105	256	98	$25088\sqrt{65}$	$34\sqrt{65}$
7	495	264	952	561	1073	$40\sqrt{610}$	1105	297	121	$42471\sqrt{34}$	$55\sqrt{34}$
8	756	533	3360	925	3444	$\sqrt{11573689}$	3485	392	84	$39984\sqrt{205}$	$28\sqrt{205}$
9	1276	357	6960	1325	7076	$3\sqrt{5396561}$	7085	968	116	$729872\sqrt{5}$	$436\sqrt{5}$
10	1364	14973	21120	15035	21164	$3\sqrt{74471681}$	25925	62	44	$832040\sqrt{5}$	$34\sqrt{5}$
11	1584	1365	14212	2091	14300	$\sqrt{203844169}$	14365	726	88	$75504\sqrt{1105}$	$22\sqrt{1105}$
12	1771	1428	2640	2275	3179	$12\sqrt{62561}$	3485	847	539	$243089\sqrt{170}$	$77\sqrt{170}$
13	1827	1564	8736	2405	8925	$4\sqrt{4922737}$	9061	841	189	$158949\sqrt{442}$	$41\sqrt{442}$
14	1881	1092	1540	2175	2431	$28\sqrt{4546}$	2665	1083	891	$321651\sqrt{130}$	$123\sqrt{130}$
15	1932	2880	4301	3468	4715	$\sqrt{26793001}$	5525	588	414	$405720\sqrt{85}$	$78\sqrt{85}$
16	2200	9879	60480	10121	60520	$3\sqrt{417269449}$	61321	242	40	$513040\sqrt{89}$	$26\sqrt{89}$
17	2352	5236	7011	5740	7395	$\sqrt{76569817}$	9061	504	384	$209664\sqrt{697}$	$24\sqrt{697}$
18	2920	85239	106560	85289	106600	$9\sqrt{229885441}$	136489	50	40	$1331600\sqrt{41}$	$10\sqrt{41}$
19	2944	3567	16800	4625	17056	$3\sqrt{32773721}$	17425	1058	256	$1354240\sqrt{41}$	$170\sqrt{41}$
20	3124	4557	9840	5525	10324	$3\sqrt{13065761}$	11285	968	484	$4369552\sqrt{5}$	$484\sqrt{5}$
21	3276	7293	16400	7995	16724	$\sqrt{322147849}$	18245	702	324	$749736\sqrt{205}$	$54\sqrt{205}$
22	3627	840	1364	3723	3875	$52\sqrt{949}$	3965	2883	2511	$4748301\sqrt{10}$	$1209\sqrt{10}$
23	3927	21420	77836	21777	77935	$4\sqrt{407328706}$	80825	357	99	$624393\sqrt{610}$	$15\sqrt{610}$
24	3960	7616	16095	8584	16575	$149\sqrt{14281}$	18241	968	480	$116160\sqrt{18241}$	$8\sqrt{18241}$
25	4032	6601	8976	7735	9840	$\sqrt{124141777}$	11849	1134	864	$616896\sqrt{697}$	$54\sqrt{697}$
26	4224	3007	46368	5185	46560	$\sqrt{2159033473}$	46657	2178	192	$1812096\sqrt{97}$	$222\sqrt{97}$
27	4284	15587	84912	16165	85020	$\sqrt{7453002313}$	86437	578	108	$62424\sqrt{86437}$	$2\sqrt{86437}$
28	4389	2548	8352	5075	9435	$4\sqrt{4765513}$	9773	2527	1083	$2554797\sqrt{58}$	$361\sqrt{58}$
29	4788	3920	10659	6188	11685	$\sqrt{128980681}$	12325	2268	1026	$2499336\sqrt{85}$	$270\sqrt{85}$
30	4991	1512	10488	5215	11615	$24\sqrt{194938}$	11713	3703	1127	$17859569\sqrt{2}$	$2737\sqrt{2}$
31	5605	2772	7800	6253	9605	$12\sqrt{475861}$	9997	3481	1805	$6283205\sqrt{26}$	$769\sqrt{26}$
32	5852	861	6864	5915	9020	$3\sqrt{5317313}$	9061	5054	2156	$1556632\sqrt{533}$	$238\sqrt{533}$
33	6048	1665	4264	6273	7400	$\sqrt{20953921}$	7585	4608	3136	$51584\sqrt{7585}$	$64\sqrt{7585}$
34	6105	37400	167832	37895	167943	$8\sqrt{461974066}$	172057	495	111	$6391935\sqrt{34}$	$87\sqrt{34}$
35	6336	5460	10373	8364	12155	$\sqrt{137410729}$	13325	2904	1782	$784080\sqrt{2665}$	$66\sqrt{2665}$
36	6536	3927	5952	7625	8840	$3\sqrt{5649737}$	9673	3698	2888	$10679824\sqrt{17}$	$1138\sqrt{17}$
37	6601	9840	40920	11849	41449	$120\sqrt{123005}$	42601	2009	529	$30820069\sqrt{2}$	$1469\sqrt{2}$
38	7260	66352	219555	66748	219675	$\sqrt{5260985929}$	229477	396	120	$1528560\sqrt{1189}$	$12\sqrt{1189}$
39	7347	7920	37604	10803	38315	$4\sqrt{92299201}$	39125	2883	711	$17081775\sqrt{10}$	$939\sqrt{10}$
40	7392	34100	92781	34892	93075	$\sqrt{9771123961}$	99125	792	294	$1940400\sqrt{793}$	$30\sqrt{793}$
41	7700	9405	10608	12155	13108	$3\sqrt{22331521}$	16133	2750	2500	$4015000\sqrt{221}$	$250\sqrt{221}$
42	7904	2520	62985	8296	63479	$75\sqrt{706393}$	63529	5776	494	$15167776\sqrt{17}$	$1406\sqrt{17}$
43	8004	38272	45675	39100	46371	$\sqrt{3550951609}$	60125	828	696	$17768880\sqrt{13}$	$300\sqrt{13}$
44	8208	9405	10220	12483	13108	$5\sqrt{7716097}$	16133	3078	2888	$7953552\sqrt{73}$	$494\sqrt{73}$
45	8624	12180	25707	14924	27115	$3\sqrt{89911361}$	29725	2744	1408	$965888\sqrt{5945}$	$40\sqrt{5945}$
46	8736	2405	12852	9061	15540	$\sqrt{170957929}$	15725	6656	2688	$1397760\sqrt{3145}$	$128\sqrt{3145}$
47	9100	9555	15312	13195	17812	$3\sqrt{36195041}$	20213	3640	2500	$15470000\sqrt{29}$	$820\sqrt{29}$
48	9152	13464	19425	16280	21473	$3\sqrt{62067769}$	25345	2816	2048	$6172672\sqrt{185}$	$256\sqrt{185}$
49	9200	65805	229908	66445	230092	$3\sqrt{6354220721}$	239317	640	184	$13218560\sqrt{41}$	$104\sqrt{41}$
50	9492	76320	89131	76908	89635	$\sqrt{13769077561}$	117725	588	504	$9772560\sqrt{85}$	$84\sqrt{85}$
51	10032	1476	8645	10140	13243	$\sqrt{76914601}$	13325	8664	4598	$2096688\sqrt{2665}$	$190\sqrt{2665}$
52	10080	80325	253916	80955	254116	$\sqrt{70925440681}$	266509	630	200	$1537200\sqrt{4369}$	$10\sqrt{4369}$
53	11088	9555	13940	14637	17812	$5\sqrt{11424865}$	20213	5082	3872	$4716096\sqrt{697}$	$242\sqrt{697}$
54	11700	6765	100912	13515	101588	$\sqrt{10228996969}$	101813	6750	676	$4563000\sqrt{901}$	$226\sqrt{901}$
55	11799	95120	859320	95849	859401	$40\sqrt{467174173}$	864649	729	81	$15373881\sqrt{82}$	$81\sqrt{82}$
56	12144	9792	10465	15600	16031	$\sqrt{205399489}$	18785	5808	5566	$4541856\sqrt{1105}$	$242\sqrt{1105}$
57	13536	15080	17577	20264	22185	$\sqrt{536357329}$	26825	5184	4608	$15344640\sqrt{145}$	$576\sqrt{145}$
58	14112	3885	153340	14637	153988	$5\sqrt{941129953}$	154037	10752	648	$7547904\sqrt{697}$	$408\sqrt{697}$
59	14784	32175	60088	35409	61880	$\sqrt{4645798369}$	69745	3234	1792	$827904\sqrt{69745}$	$14\sqrt{697451}$
60	14820	1952	4725	14948	15555	$\sqrt{26135929}$	15677	12996	10830	$33399720\sqrt{61}$	$2166\sqrt{61}$
61	14960	7293	7476	16643	16724	$15\sqrt{484793}$	18245	9350	9248	$25432000\sqrt{89}$	$1394\sqrt{89}$
62	14973	21164	231840	25925	232323	$4\sqrt{3387356281}$	233285	4761	483	$3232719\sqrt{4810}$	$69\sqrt{4810}$
63	15312	2915	4284	15587	15900	$\sqrt{2684981}$	16165	12672	11616	$17005824\sqrt{265}$	$1056\sqrt{265}$

TABLE 4 $\{(a, b, e), (a, c, f), (a, g, d)\}$. Values for g = **(I).**

#	*a*	*b*	*c*	*e*	*f*	*g*	*d*	*l*	*m*
1	44	117	240	125	244	267	$5\sqrt{2929}$	8	4
2	85	132	720	157	725	732	$\sqrt{543049}$	25	5
3	140	480	693	500	707	843	$13\sqrt{4321}$	20	14
4	160	231	792	281	808	825	$5\sqrt{28249}$	50	16
5	187	1020	1584	1037	1595	1884	$5\sqrt{143377}$	17	11
6	195	748	6336	773	6339	6380	$5\sqrt{1629697}$	25	3
7	240	252	275	348	365	373	$\sqrt{196729}$	96	90
8	429	880	2340	979	2379	2500	$\sqrt{6434041}$	99	39
9	495	4888	8160	4913	8175	9512	$17\sqrt{313921}$	25	15
10	528	5796	6325	5820	6347	8579	$5\sqrt{2955121}$	24	22
11	780	2475	2992	2595	3092	3883	$\sqrt{15686089}$	120	100
12	828	2035	3120	2197	3228	3725	$13\sqrt{86161}$	162	108
13	832	855	2640	1193	2768	2775	$17\sqrt{29041}$	338	128
14	935	17472	25704	17497	25721	31080	$25\sqrt{1546945}$	25	17
15	1008	1100	1155	1492	1533	1595	$\sqrt{3560089}$	392	378
16	1008	1100	12075	1492	12117	12125	$\sqrt{148031689}$	392	42
17	1080	1881	14560	2169	14600	14681	$\sqrt{216698161}$	288	40
18	1105	9360	35904	9425	35921	37104	$\sqrt{1377927841}$	65	17
19	1155	6300	6688	6405	6787	9188	$\sqrt{85753369}$	105	99
20	1188	16016	39195	16060	39213	42341	$5\sqrt{71766865}$	44	18
21	1560	2295	5984	2775	6184	6409	$13\sqrt{257449}$	480	200
22	1575	1672	9120	2297	9255	9272	$\sqrt{88450609}$	625	135
23	1755	4576	6732	4901	6957	8140	$5\sqrt{2773585}$	325	225
24	2079	44080	65472	44129	65505	78928	$5\sqrt{249358057}$	49	33
25	2163	15840	37100	15987	37163	40340	$\sqrt{1631994169}$	147	63
26	2925	3536	11220	4589	11595	11764	$\sqrt{146947321}$	1053	375
27	2964	9152	9405	9620	9861	13123	$5\sqrt{7239937}$	468	456
28	2964	6160	38475	6836	38589	38965	$\sqrt{1527056521}$	676	114
29	3696	9045	121940	9771	121996	122275	$\sqrt{14964836041}$	726	56
30	4368	4901	13860	6565	14532	14701	$5\sqrt{9407953}$	1664	672
31	4599	23760	144832	24201	144905	146768	$5\sqrt{862479865}$	441	73
32	4599	18368	23760	18935	24201	30032	$5\sqrt{36922873}$	567	441
33	4900	17157	23760	17843	24260	29307	$13\sqrt{5224321}$	686	500
34	4928	10725	30780	11803	31172	32595	$\sqrt{1086719209}$	1078	392
35	5320	63063	353760	63287	353800	359337	$\sqrt{129151381969}$	224	40
36	5368	163680	450225	163768	450257	479055	$\sqrt{229522508449}$	88	32
37	5491	41580	46512	41941	46835	62388	$5\sqrt{156896545}$	361	323
38	5643	14160	21476	15243	22205	25724	$5\sqrt{27742705}$	1083	729
39	5643	43680	76076	44043	76285	87724	$5\sqrt{309093745}$	363	209
40	5720	8415	157248	10175	157352	157473	$17\sqrt{85918561}$	1760	104
41	6072	16929	18560	17985	19528	25121	$5\sqrt{26717353}$	1056	968
42	6435	24080	24684	24925	25509	34484	$\sqrt{1230555481}$	845	825
43	7336	274527	480480	274625	480536	553377	$65\sqrt{72492289}$	98	56
44	7560	13728	35321	15672	36121	37895	$5\sqrt{59727385}$	1944	800
45	7579	8820	17472	11629	19045	19572	$5\sqrt{17620177}$	2809	1573
46	7800	23751	29920	24999	30920	38201	$\sqrt{1520156401}$	1248	1000
47	7840	9828	10725	12572	13285	14547	$\sqrt{273080809}$	2744	2560
48	7885	16320	85932	18125	86293	87468	$\sqrt{7712824249}$	1805	361
49	7920	15232	26649	17168	27801	30695	$5\sqrt{40196377}$	1936	1152
50	8415	157248	643720	157473	643775	662648	$\sqrt{439173184129}$	225	55
51	8532	36960	57275	37932	57907	68165	$29\sqrt{5611489}$	972	632
52	8789	10560	17748	13739	19805	20652	$5\sqrt{20150065}$	3179	2057
53	9180	72611	206448	73189	206652	218845	$5\sqrt{1919096257}$	578	204
54	9405	23600	53196	25405	54021	58196	$\sqrt{3475228441}$	1805	825
55	9504	31372	61845	32780	62571	69347	$5\sqrt{195973297}$	1408	726
56	9856	61560	200683	62344	200825	209817	$5\sqrt{1764812569}$	784	242
57	10296	11753	16800	15625	19704	20503	$25\sqrt{842209}$	3872	2904
58	10395	95004	220400	95571	220645	240004	$\sqrt{57709976041}$	567	245
59	10395	63364	327360	64211	327525	33436	$\sqrt{111287622121}$	847	165
60	12915	36720	290444	38925	290731	292756	$\sqrt{85872872761}$	2205	287
61	14112	15400	19305	20888	23913	24695	$\sqrt{808991569}$	5488	4608
62	14500	29568	83475	32932	84725	88557	$17\sqrt{27863641}$	3364	1250
63	14715	148148	267120	148877	267525	305452	$53\sqrt{33292081}$	729	405

APPENDIX 2 ODDS AND ENDS

Theorem A2.1 For all integers $a > 0$ there exists at least one Pythagorean Quad.

Proof For $a = 1$, $1^2 + 2^2 + 2^2 = 3^2$. For $a = 2$, $2^2 + 3^2 + 6^2 = 7^2$.
For $a = 3$, $3^2 + 6^2 + 22^2 = 23^2$. For $a = 4$, $4^2 + 4^2 + 7^2 = 9^2$.
For $a > 2$, $a^2 + b^2 = e^2$ and $e^2 + c^2 = d^2$ by application of Theorem 1 twice.
e.g. $3^2 + 4^2 = 5^2$ and $5^2 + 12^2 = 13^2$. ❑

For every integer a there exist an infinite number of Pythagorean Quads, multiply every integer solution from Theorem A2.1 by all the integers! Conjecture, for all integers $a > 0$ are there an infinite number of solutions that are not multiples ?

Some Interesting Facts

1 $e^2 + f^2 + g^2 = (a^2 + b^2) + (a^2 + c^2) + (b^2 + c^2) = 2(a^2 + b^2 + c^2) = 2d^2$.

2 If $\{a, b, c\}$ is a Pythagorean Quad then $\{ab, ac, bc\} = \{A, B, C\}$ and repeating this gives $\{AB, AC, BC\}$ which is a multiple of $\{a, b, c\}$.

e.g. $3^2 + 4^2 + 12^2 = 13^2 \Rightarrow 12^2 + 36^2 + 48^2 \Rightarrow 144 \times (3^2 + 4^2 + 12^2 = 13^2)$.
Note $12^2 + 36^2 + 48^2 = (12\sqrt{26})^2$ which is not a Pythagorean Quad [3].

3 Given that $\{x, y, z\}$ and $\{X, Y, Z\}$ are both Pythagorean Triples then
$(xX - yY)^2 + (xY + yX)^2 = (zZ)^2$ e.g. (3, 4, 5) and (5, 12, 13) produces
$(3 \times 5 - 4 \times 12)^2 + (3 \times 12 + 4 \times 5)^2 = (5 \times 13)^2 = (33, 56, 65)$ [5].

Some Interesting Examples

In the following $(a^4 + l^2m^2) = p$ and $(l^2 + m^2) = q$, where $p\,q = \delta^2$.

$a = 576972$ $b = 2371005$ $c = 4226096$ $e = 2440197$ $f = 4265300$ $g = 7\sqrt{479215349209}$
$d = 4880005$ $l = 69192$ $m = 39204$ $p = 150700176\sqrt{4880005}$ $q = 36\sqrt{4880005} \Rightarrow 2lmd = p\,q$
which is always true of course, but the example shows it in a glaringly obvious way.

a	**b**	**c**	**e**	**f**	**g**	**d**	**l**	**m**	**p**	**q**
942480	2639811	4850300	2803011	4941020	$\sqrt{30494012205721}$	5601989	163200	90720	888391929600	186720

a	**b**	**c**	**e**	**f**	**g**	**d**	**l**	**m**	**p**	**q**
942480	157157	593676	955493	1113676	614125	$85\sqrt{175144369}$	98336	520200	980556192000	$72\sqrt{175144369}$

a	**b**	**c**	**e**	**f**	**g**	**d**	**l**	**m**
99099	345100	68640	$7\sqrt{2630910649}$	120549	351860	365549	$(e - b)$	51909
99099	68640	345100	120549	$7\sqrt{2630910649}$	351860	365549	51909	$(f - c)$
68640	345100	99099	351860	120549	$7\sqrt{2630910649}$	365549	6760	21450
68640	99099	345100	120549	351860	$7\sqrt{2630910649}$	365549	21450	6760
345100	68640	99099	351860	$7\sqrt{2630910649}$	120549	365549	283220	$(f - c)$
345100	99099	68640	$7\sqrt{2630910649}$	351860	120549	365549.	$(e - b)$	283220

a	**b**	**c**	**e**	**f**	**g**	**d**	**l**	**m**
7800	$\sqrt{211773121}$	18720	16511	20280	23711	24961	$(e - b)$	1560
520	576	$3\sqrt{68761}$	776	943	975	1105	200	$(f - c)$

$14 + 23 + 70 = 107$ $\quad 14 \times 107 = 1498$ $\quad 1498 + 2461 + 7490 = 107^2$
$14^2 + 23^2 + 70^2 = 75^2$ $\quad 23 \times 107 = 2461$ $\quad 1498^2 + 2461^2 + 7490^2 = 8025^2 = (107 \times 75)^2$
$14^3 + 23^3 + 70^3 = 71^3$ $\quad 70 \times 107 = 7490$ $\quad 1498^3 + 2461^3 + 7490^3 = 7597^3 = (107 \times 71)^3$

Apropos of Fermat's Last Theorem, for $n = 3$ and 4 there are the following. **But for $n > 4$?**

For $n = 3$ $3^3 + 4^3 + 5^3 = 6^3$ and $14^3 + 23^3 + 70^3 = 71^3$, (Observe $14^2 + 23^2 + 70^2 = 75^2$).

There is a formula, due to Vieta (1591), submitted by Steven Dutch, University of Wisconsin – Green Bay. This is not a particularly good formula as it produces some negative results, e.g. (756, –15, 744, 945), since the cubes of negative integers are also negative.

Vieta's method. Given any two numbers m and n with $m > n$ then

$$a = n(m^3 + n^3),\ b = m(m^3 - 2n^3),\ c = n(2m^3 - n^3) \text{ and } d = m(m^3 + n^3).$$

e.g. $m = 2$ and $n = 1$ produces $9^3 + 12^3 + 15^3 = 18^3 \Rightarrow 3(3, 4, 5, 6)$.

See Program 3 and Tables 5a and 5b below for further results by other means.

For $n = 4$ $95800^4 + 217519^4 + 414560^4 = 422481^4$ discovered by Roger Frye in 1988

and $2682440^4 + 15365639^4 + 18796760^4 = 20615673^4$ by Noam Elkies

also $630662624^4 + 275156240^4 + 219076465^4 = 638523249^4$ by Allan MacLeod.

For $n = 5$ $27^5 + 84^5 + 110^5 + 133^5 = 144^5$ discovered by Lander and Parkin.
Note that this is a Pythagorean Quintuple, not a Pythagorean Quadruple.

Program 3 $a^n + b^n + c^n = d^n$

```
n = 3; Print["  n = ", n]
For[a = 1, a < 11, a++,
  For[b = a, b < 501, b++,
    For[c = b, c < 1002, c++,
    d = (a^n + b^n + c^n)^(1/n);
       If[IntegerQ[d] && GCD[a, b, c]<2,
Print["   a = ",  a, "   b = ",  b,
      "   c = ",  c, "   d = ",  d]
   ] ] ] ]
```

This program produces tables 5a and 5b below. The code can be used for all n greater than 2. For any $n > 2$, the tables can be extended to any desired size by increasing the limits of a, b, and c. The number of tuples can be increased by increasing the number of `For` and `d` statements.

Table 5.a $n = 3$ $a = 1$

#	a	b	c	d
1	1	6	8	9
2	1	71	138	144
3	1	135	138	172
4	1	242	720	729
5	1	372	426	505
6	1	426	486	577
7	1	566	823	904
8	1	575	2292	2304
9	1	791	812	1010
10	1	1938	2820	3097
11	1	1943	6702	6756
12	1	2196	5984	6081
13	1	2676	3230	3753
14	1	3086	21588	21609
15	1	3318	16806	15489
16	1	3453	24965	24987
17	1	4607	36840	36864
18	1	7251	49409	49461
19	1	7676	11903	12884
20	1	10230	37887	38134

Table 5.b $n = 3$ $a > 1$

#	a	b	c	d
1	2	17	40	41
2	3	4	5	6
3	3	10	18	19
4	3	34	114	115
5	3	36	37	46
6	3	121	131	159
7	3	214	309	340
8	3	245	340	378
9	4	17	22	25
10	4	57	248	249
11	4	303	482	519
12	5	76	123	132
13	5	86	460	461
14	5	163	164	206
15	5	216	436	453
16	5	232	307	346
17	6	32	33	41
18	6	121	768	769
19	6	127	180	199
20	6	179	216	251

APPENDIX 3 FERMAT'S LAST THEOREM

When Fermat wrote a note in the margin of his copy of Bachet's Arithmetica to the effect that he had a marvellous proof that $x^n + y^n \neq z^n$ where n is any integer greater than 2, perhaps he was thinking along the following lines. The proof is for $n = 4$ and multiples thereof.

Define a **Pythagorean Triple** (PT) as $x^2 + y^2 = z_p^2$, and all three are integers.

Define a **Diophantine Triple** (DT) as $y^2 - x^2 = z_d^2$, and all three are integers.

Theorem A3.1 Any integer $x > 2$ with $(1 \leq w_p < x) \mid x^2$ generates all Pythagorean Triples.

Proof $x^2 + y^2 = z_p^2$ and without loss of generality assume $x < y$.

Then $x^2 = z_p^2 - y^2 = (z_p - y)(z_p + y)$ since $(z_p - y)(z_p + y) = z_p^2 + z_p y - z_p z - y^2 = z_p^2 - y^2$

Let $z_p - y = w_p$ then $z_p = y + w_p$ and $x^2 = w_p (y + w_p + y)$

therefore $x^2 = w_p (2y + w_p) = 2w_p + w_p^2$ hence $2yw_p = x^2 - w_p^2$

therefore $y = \dfrac{x^2 - w_p^2}{2w_p}$ and $z_p = y + w = \dfrac{x^2 - w_p^2}{2w_p} + \dfrac{2\,w_p^2}{2w_p} = \dfrac{x^2 + w_p^2}{2w_p}$

For $x > 2$, $w_p \mid x^2$ and for y to be positive, $1 \leq w_p < x$. ❑

Theorem A3.2 Any integer $x > 2$ with $(1 \leq w_d < x) \mid x^2$ generates all Diophantine Triples.

Proof $y^2 - x^2 = z_d^2$. For z_d to be positive and without loss of generality assume $x < y$.

Then $x^2 = y^2 - z_d^2 = (y - z_d)(y + z_d)$ since $(y - z_d)(y + z_d) = y^2 + yz_d - z_d y - z_d^2 = y^2 - z_d^2$

Let $y - z_d = w_d$ then $y = z_d + w_d$ and $x^2 = w_d (z_d + w_d + z_d)$

hence $x^2 = w_d (2z_d + w_d) = 2z_d w_d + w_d^2$ therefore $2z_d w_d = x^2 - w_d^2$

and $z_d = \dfrac{x^2 - w_d^2}{2w_d}$ and $y = z_d + w_d = \dfrac{x^2 - w_d^2}{2w_d} + \dfrac{2\,w_d^2}{2w_d} = \dfrac{x^2 + w_d^2}{2w_d}$ ❑

Note For Theorem A3.3 the x and y in Theorem A3.1 are equal to the x and y in Theorem A3.2.

Theorem A3.3. Any integers $x > 2$ and w such that $(1 \leq w < x) \mid x^2$ cannot generate both a Pythagorean Triple and a Diophantine Triple simultaneously.

Proof.

Assume 1 that $x^2 + y^2 = z_p^2$ is a PT, then to generate a DT requires that $y^2 - x^2 = z_d^2$, hence

$$z_d^2 = \frac{(x^2 - \mathrm{w}_p^2)^2}{4\,\mathrm{w}_p^2} - x^2 = \frac{(x^2 - \mathrm{w}_p^2)^2 - 4w_p^2 x^2}{4\,\mathrm{w}_p^2} \qquad w_p = z_p - y \text{ from Theorem 1}$$

and $z_d = \dfrac{1}{2w_p}\sqrt{x^4 - 6x^2 w_p^2 + w_p^4} \Rightarrow z_d \neq$ an integer. To be square the expression under the $\sqrt{\ }$ must be of the form $x^4 - 2x^2 w_p^2 + w_p^4 = (x^2 - w_p^2)^2$ by the Binomial Theorem, i.e. the coefficients of the expansion of $(x^2 - w_p^2)^2$ are given by the 2nd row of Pascal's Triangle. ❑

Assume 2 that $y^2 - x^2 = z_d^2$ is a DT, then to generate a PT requires that $x^2 + y^2 = z_p^2$ hence

$$z_p^2 = \frac{(x^2 + w_d^2)^2}{4w_d^2} + x^2 = \frac{(x^2 + w_d^2)^2 + 4w_d^2x^2}{4\,w_d^2} \qquad w = y - z_d \text{ from Theorem 2}$$

and $z_p = \frac{1}{2w_d}\sqrt{x^4 + 6x^2w_d^2 + w_d^4} \Rightarrow z_p \neq$ an integer. To be square the expression under the $\sqrt{\ }$ must be of the form $x^4 + 2x^2w_d^2 + w_d^4 = (x^2 + w_d^2)^2$ by the Binomial Theorem, i.e. the coefficients of the expansion of $(x^2 + w_d^2)^2$ are given by the 2nd row of Pascal's Triangle. ❑

Theorem A3.4 Fermat's Last Theorem for $x^4 + y^4 \neq$ a fourth power.

First a couple of Lemmas (Lemmata?).

Lemma 1 The product of two squares is also a square.

Proof $x^2 \times y^2 = (x\,y)^2$ for all x and y greater than 0. ❑

Lemma 2 The product of two numbers, a square and a non-square is not a square.

Proof $x^2 \times y^2 = (x\,y)^2 \Rightarrow x^2 \times y \neq$ square for both x and $y > 1$ and $y \neq$ square. ❑

Fermat's Last Theorem, proof for $n = 4$ i.e. $x^4 + y^4 \neq z^4$.

The proof is for x or y not being a fourth power, hence the equation $x^4 + y^4 = z^4$ is not true.

Assume for a contradiction, $x^4 + y^4 = z^4$ say, then $(x^2)^2 = (z^2)^2 - (y^2)^2 = (z^2 - y^2)(z^2 + y^2)$ where both $(z^2 - y^2)$ and $(z^2 + y^2)$ require to be square by Lemma 1. But by Theorems A3.1, A3.2 and A3.3, if $(z^2 - y^2) =$ a square then $(z^2 + y^2) \neq$ a square and vice-versa, hence $(z^2 - y^2)(z^2 + y^2) \neq$ a square, by Lemma 2. Therefore $x^4 \neq$ square and $\neq$ to a fourth power since $x^4 = (x^2)^2$. Therefore $x^4 \neq z^4 - y^4 \Rightarrow x^4 + y^4 \neq z^4$.
Note, $x + y^4$ may $= z^4$ but in this case x is not a fourth power, e.g. $175 + 3^4 = 4^4$ ❑

Theorem A3.5 Fermat's Last Theorem for $4n$, i.e. $x^{4n} + y^{4n} \neq z^{4n}$ for all $n > 1$.

The proof is by induction on n.

For $n = 1 \Rightarrow x^4 + y^4 = z^4 \Rightarrow x^4 = z^4 - y^4 = (z^2 - y^2)(z^2 + y^2) \neq x^4$
by Theorems A3.1, 2 and 3, already proven by Theorem A3.4.

For $4n$ $x^{4n} + y^{4n} = z^{4n} \Rightarrow x^{4n} = z^{4n} - y^{4n} = (z^{2n})^2 - (y^{2n})^2 = (z^{2n} - y^{2n})(z^{2n} + y^{2n})$

$= (Z^2 - Y^2)(Z^2 + Y^2)$ where $Z = z^n$ and $Y = y^n$

but $(Z^2 - Y^2)(Z^2 + Y^2) \neq$ a fourth power by Theorems A3.1, 2, 3 and 4.

For $4(n + 1)$ $x^{4(n+1)} + y^{4(n+1)} = z^{4(n+1)} \Rightarrow x^{4(n+1)} = z^{4(n+1)} - y^{4(n+1)}$

$= (z^{2(n+1)})^2 - (y^{2(n+1)})^2 = ((z^{2(n+1)}) - (y^{2(n+1)}))((z^{2(n+1)}) + (y^{2(n+1)}))$

$= (Z_1^2 - Y_1^2)(Z_1^2 + Y_1^2)$ where $Z_1 = z^{(n+1)}$ and $Y_1 = y^{(n+1)}$

but $(Z_1^2 - Y_1^2)(Z_1^2 + Y_1^2) \neq$ a fourth power by Theorems A3.1, 2, 3 and 4. ❑