Parametric Search Report

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This report is a continuation of my article "Parametric Densities" in which I explained how a certain parametric system

$$\begin{aligned}
x &= 8(t^2-4)(5t-4)(3t^2+2)(5t^3-11t^2+8t+2) \\
y &= 7t(t^2-4)(5t^2-8t+6)(5t^3-32t^2+22t+16) \\
z &= 28(2t-3)(3t^2+2)(5t^2-6)(t^2-4t+2) \\
\sqrt{x^2+y^2} &= (t^2-4)(625t^6-2120t^5+3660t^4-2976t^3+612t^2+352t+128) \\
\sqrt{x^2+z^2} &= 4(3t^2+2)(50t^6-150t^5+17t^4+164t^3+292t^2-608t+260) \\
\sqrt{y^2+z^2} &= 7(25t^8-200t^7+584t^6-352t^5-548t^4+96t^3+976t^2-768t+288)
\end{aligned}$$

(due to a French mathematician M. Rignaux) appears to be the most efficient of all known parametric systems for computing body cuboids. This system was given on page 94 of Kraïtchik's *Théorie des Nombres, Volume III* published in 1947. However the system is possibly much older than that, since a cuboid paper by Rignaux appeared in 1918 in the journal *Intermédiaire Math.* (volume 25, page 127). I have not been able to obtain a copy of that paper, but it is described briefly in Volume II of Leonard Eugene Dickson's *History of the Theory of Numbers* (page 501). An internet search found no other papers by Rignaux: indeed I have not even been able to discover his first name — it would seem that "M." stands for "Monsieur"! In his treatise *A Collection of Algebraic Identities* (available online) Tito Piezas III refers to an R. Rignaux who worked on the Cuboid Problem. This must be our Rignaux.

Note that the Rignaux system above holds for all t real or complex, but of course to generate body cuboids with integer edges we compute only real rational $t = h/k$ where h and k are relatively prime integers. Since we have

$$x(-t) \neq \pm x(t)\ ,\quad y(-t) \neq \pm y(t)\ ,\quad z(-t) \neq \pm z(t)$$

both positive and negative $t = h/k$ must be tested. The substitution $t = h/k$ gives the integer formulas

$$\begin{aligned}
X &= 8(h^2-4k^2)(5h-4k)(3h^2+2k^2)(5h^3-11h^2k+8hk^2+2k^3) \\
Y &= 7hk(h^2-4k^2)(5h^2-8hk+6k^2)(5h^3-32h^2k+22hk^2+16k^3) \\
Z &= 28hk(2h-3k)(3h^2+2k^2)(5h^2-6k^2)(h^2-4hk+2k^2)
\end{aligned}$$

where of course only h and k with $\gcd(h,k)=1$ are computed. Each body cuboid (X,Y,Z) generated must be reduced to its primitive by dividing all three edges by $g=\gcd(X,Y,Z)$. Henceforth $(x,y,z)=(X/g,Y/g,Z/g)$ will always denote a primitive body cuboid.

Perhaps the most remarkable property of the Rignaux system is that, with these restrictions, it apparently finds body cuboids without repetitions. I say "apparently" because all the other parametric systems I examined had numerous repetitions visible on the computer screen, while the Rignaux cuboids had no such obvious repetitions. Now a rigorous proof must show that if (h_1, k_1) and (h_2, k_2) generate the same primitive cuboid then $h_1 = h_2$ and $k_1 = k_2$. Since this involves three eighth degree polynomials in two variables, I have no clue how one would prove such a thing. However, a simple program read my master file (described in detail below) and found that the 87 cuboids with body diagonals within 10^{-11} of an integer were all different, as were the 958 within 10^{-10}. This suggests that if repetitions do exist they must be quite rare.

Let $m = |h| + k$ where $h \neq 0$ and $k > 0$. For $m = 2, 3, 4, \cdots, 10^4$ the expected number of cuboids produced is approximately

$$2 \cdot 2 \cdot \frac{6}{\pi^2} \int_0^{10^4} n\, dn = \frac{12}{\pi^2}(10^4)^2 = 1.2158542 \times 10^8$$

(2 for the duals, 2 for $\pm h$, and $6/\pi^2$ is the probability that $\gcd(h, k) = 1$ for random nonzero integers h and k). In fact, the actual number of cuboids produced for $m \leq 10^4$ was 121,589,932 and this took a single computer about two hours, which works out to roughly one million body cuboids per minute!

An identical program ran on each of my computers, but each program checked different ranges of m. Whenever any computer found a cuboid with $\text{abs}(E) < 10^{-7}$ (where E was the difference between $\sqrt{x^2 + y^2 + z^2}$ and the nearest integer) it displayed m, h, k, x, y, z, E and saved the values m, h, k, x, y, z to a hard disk file. Later these files were collected and appended into a master file for future analysis. For $m \leq 2,000,000$ this master file contained data on 973,706 cuboids.

Naturally, one would expect the computations to slow down as the cuboids get larger, and indeed at $m = 2,000,000$ the number of new cuboids produced dropped to about 750,000 per minute. The computation for all $m \leq 1,000,000$ took about three months, but continuing to all $m \leq 2,000,000$ has taken over a year. Two of my twelve computers failed during this time (the cooling fan motors burned out) and in hot weather I ran only four computers at a time because of the heat generated. By the count formula there were roughly

$$\frac{12}{\pi^2} \cdot (2,000,000)^2 \approx 4.863 \times 10^{12}$$

body cuboids produced for $m \leq 2,000,000$. In the following table, all the cuboids in the master file are measured. Curiously, the dual cuboids had on average about 47% more decimal digits than the non-duals, so generally the duals are immensely larger. In the following table C_1 and C_2 are the respective counts for non-duals and duals, and μ_1 and μ_2 are the average number of digits in the integer part of the body diagonals. Note that the counts and means are local not global: for example, the data in the bottom row means that m between 900,001 and 1,000,000 generated $C_1 = 23072$ Rignaux cuboids with body diagonals within 10^{-7} of an integer, and that $\mu_1 = 48.8$ for these 23072 cuboids.

m	C_1	C_2	μ_1	μ_2	m	C_1	C_2	μ_1	μ_2
100000	1220	1249	39.2	57.1	1100000	25433	25597	49.1	72.1
200000	3652	3674	42.4	62.1	1200000	27806	28084	49.4	72.6
300000	6074	6033	44.2	64.7	1300000	30494	30496	49.7	73.1
400000	8439	8358	45.3	66.4	1400000	32928	33036	50.0	73.5
500000	10821	10903	46.2	67.7	1500000	35867	35494	50.2	73.9
600000	13590	13317	46.9	68.8	1600000	37625	37630	50.5	74.2
700000	15745	15594	47.5	69.6	1700000	40149	40294	50.7	74.5
800000	18573	18183	48.0	70.4	1800000	42661	42286	50.9	74.8
900000	20914	20653	48.4	71.0	1900000	44888	45277	51.1	75.1
1000000	23072	23085	48.8	71.6	2000000	47258	47254	51.3	75.4

The sums of the counts in the C_1 and C_2 columns are 487209 and 486497 respectively. These are the total number of Rignaux cuboids and dual cuboids in the master file, and they agree with values found by another program (described below). This table clearly shows that the duals of Rignaux cuboids are vastly larger than the originals. This is not typical of most body cuboids.

Nearly Perfect Cuboids

The next table lists the counts of body cuboids in the master file having body diagonals within 10^{-7} to 10^{-13} of an integer.

$m \leq$	10^{-7}	10^{-8}	10^{-9}	10^{-10}	10^{-11}	10^{-12}	10^{-13}
100000	2469	260	20	4	1	0	0
200000	9795	1001	109	15	2	0	0
300000	21902	2252	224	30	5	2	0
400000	38699	3906	407	47	7	2	0
500000	60423	6082	637	73	12	2	0
600000	87330	8758	916	104	13	2	0
700000	118669	11701	1229	139	16	2	0
800000	155425	15554	1604	180	21	2	0
900000	196992	19776	2007	211	25	3	0
1000000	243149	24328	2434	253	30	3	0
1100000	294179	29541	2940	295	32	3	0
1200000	350069	35111	3445	340	37	4	0
1300000	411059	41160	4038	394	42	4	0
1400000	477023	47728	4758	457	44	5	0
1500000	548384	54889	5483	527	50	5	0
1600000	623639	62390	6297	609	58	6	0
1700000	704082	70558	7138	696	64	6	0
1800000	789029	79066	8004	770	72	7	1
1900000	879194	88020	8898	855	78	7	1
2000000	973706	97403	9819	958	87	7	1

The number 973706 in the bottom line is the total number of cuboids contained in the master file. The counts in the preceding table are generally what might be expected statistically: the counts in the columns for 10^{-7} to 10^{-13} decrease by a factor of ten, more or less, moving from left to right. The probability that an arbitrary body cuboid would land in the right–hand column is obviously $2/10^{13}$. We have seen that about 4.863×10^{12} body cuboids were computed for $m \leq 2,000,000$ and since I conjecture these are all different, the probability that the right–hand column would contain all zeros is

$$\left(1 - \frac{2}{10^{13}}\right)^{4.863 \times 10^{12}} \approx 0.378$$

so there is better than a 62% chance that one or more of these cuboids would have a body diagonal so close to an integer. A similar calculation shows that there is only a 10% chance that a cuboid within 10^{-14} would have been found. Such probabilities cast doubt on the existence of a perfect cuboid, but my reply is that a perfect cuboid either exists or it doesn't, and statistics play no role in resolving the existence question.

The Best Cuboid So Far

The body cuboid with body diagonal closest to an integer for $m \leq 2,000,000$ was generated by $(h, k) = (1750459, 40089)$ with $m = 1790548$. The three edges (x, y, z) had 53, 53, and 52 decimal digits respectively, and the fractional part of its body diagonal was $0.9999999999999780234\cdots$ (thirteen consecutive 9's after the decimal point). This body cuboid was Rignaux (that is, not a dual) and its three edges were generated by

$$\begin{aligned}
(a, b) &= (1019575747150421681 84, 79007470533110259015) \\
(c, d) &= (26271384567412085183, 25489393678760542046) \\
(e, f) &= (4266570128498463373596601, 2492484868784285981538030)\,.
\end{aligned}$$

That is, if $g = \gcd(x, y)$ then $x/g = 2ab$, $y/g = a^2 - b^2$, $\sqrt{x^2 + y^2}/g = a^2 + b^2$, and similarly for the faces xz and yz. Of course $\gcd(x, y, z) = 1$ since (x, y, z) is primitive, but to my surprise

$$\begin{aligned}
\gcd(x, y) &= 3057678198997 \\
\gcd(x, z) &= 36782137551540 \\
\gcd(y, z) &= 7\,.
\end{aligned}$$

For me this tiny value for $\gcd(y, z)$ really set off some alarms. A simple program checked the master file, which contained 973706 cuboids; of these 487209 were Rignaux cuboids, and *every single one* of these had $\gcd(y, z) = 1$, 7, 9, or 63. On the other hand, each of the 486497 dual cuboids in the master file had values of $\gcd(x, y)$, $\gcd(x, z)$, and $\gcd(y, z)$ with roughly the same number of decimal digits, as one would expect when dealing with arbitrary body cuboids. This is why the generators (e, f) are so large for Rignaux cuboids.

Some New Density Results

The generators (a, b, c, d, e, f) of a body cuboid contain a lot of information about the cuboid. My search method used $a > b$ and $a \geq c > d$ to find boxes (X, Y, Z) with $X^2 + Y^2$ and $X^2 + Z^2$ always perfect squares, so that whenever $Y^2 + Z^2$ was also a perfect square, the box (X, Y, Z) was a body cuboid: this was immediately reduced to its primitive (x, y, z). To check that this was a cuboid not previously found, the generators (e, f) for the yz-face were computed: if $e < a$ then (x, y, z) was not new and hence was discarded. Therefore $e \geq a$ in my search method. Note that in the above example e is *much* larger than a and this is typical of Rignaux cuboids.

In his 2015 computer search on the Linux Cluster Supercomputer in Queensland, Tim Roberts stopped at $a = 15000$ after finding 75868 body cuboids. Information on these cuboids was collected in fifteen data files, one for each block of 1000 values of a, and these fifteen files contained 4874, 5482, 5490, 5414, 5302, 5090, 5202, 5030, 5092, 4884, 4906, 4902, 4694, 4688, and 4818 body cuboids respectively: note that $75868/15 \approx 5058$ is the mean of these fifteen numbers. In my report following the seven–week computation on the Linux Cluster, I expended much effort trying to find some sort of pattern to these fifteen numbers, hypothesizing many fancy logarithm formulas. I have compared this effort to the reconstruction of a brontosaurus skeleton starting from two toe bones. The obvious pattern is that each block contained 5000 cuboids, more or less, so that the number of cuboids generated by $a \leq A$ ought to be very roughly five times A. Thus for the winning cuboid described above, the number of cuboids with smaller a-generators should be about $5 \times 101957574715042168184 \approx 5.098 \times 10^{20}$. On the other hand, the winning cuboid had $m = 1790548$ and by the count formula the number of Rignaux cuboids (excluding the duals) found at this point was only

$$\frac{6}{\pi^2} \cdot 1790548^2 \approx 1.949 \times 10^{12}$$

which is a great deal smaller: the ratio is

$$\frac{5.098 \times 10^{20}}{1.949 \times 10^{12}} \approx 2.616 \times 10^8$$

which means that for each primitive Rignaux cuboid in this range there ought to be roughly 260 million primitive non–Rignaux cuboids. A similar calculation shows that for $m = 2,000,000$ this ratio increases to about 340 million. For the duals the situation is horribly worse: for $m = 2,000,000$ the ratio is $2.65 \times 10^{14} = 265$ million million. This is because the duals are vastly larger than the Rignaux cuboids.

The preceding results have convinced me that computing the duals is a waste of time in the search for a perfect cuboid. The duals are a ridiculously thin slice of the cuboids of comparable size and, after all, they are not true Rignaux cuboids. The original master data file stored (m, h, k, x, y, z) for which $\sqrt{x^2 + y^2 + z^2}$ was within 10^{-7} of an integer. Now I have a new slimmer master file containing only (h, k) and no duals. Since it no longer computes duals, the revised SAVE program finds about one million Rignaux cuboids per minute and the quest to $m = 3,000,000$ will go a lot faster.