A Simple Proof of the Collatz Conjecture

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November 23, 2019

1 The Conjecture

Suppose we take an arbitrary positive integer. If the integer is even, we divide it by two. If it is odd, we multiply it by three and add one, giving us a new even integer. We then repeat this process.

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \equiv 0 \pmod 2 \\ 3n+1 & \text{if } n \equiv 1 \pmod 2 \end{cases} \tag{1.0.1}$$

We can stop iterating if we reach 1, as we will quickly find ourselves in a small loop cycling between 1, 4, and 2.

The Collatz Conjecture states that this process will always lead to 1, but a proof has proved elusive for decades.

We will see that not only can this conjecture be confirmed, but that a proof is actually simple and does not require any "new maths"!

2 The Proof

2.1 The Collatz Tree

Consider the root of the "Collatz Tree"; an iteration on a given node leads downward to another node. (We will group 8 dividing to 1 into a single node and ignore the connection between 1 and 4 that creates a loop.) **Figure 1** presents a small portion of this infinite tree.

We can immediately see that the organization of the even numbers are rather trivial. Square numbers line the left side of the tree, and each odd number n begins an infinite vine containing all integers $2^x \times n$. Since we know all even numbers will be divided by two until reaching an odd number, let us dispense with them entirely. The connections between the odd numbers are what truly interest us. If all odd numbers can be shown to be connected to 1, then the conjecture is true.

Upon ruthlessly casting the even numbers into the void of triviality, we are left with an odd-looking tree of odd numbers. **Figure 2** presents again just a small portion of an infinite tree.

The tree in Figure 2 may seem a chaotic mess, but there is in fact an order to it, which we shall soon uncover. If we consider generating the tree from the root up, iterating Collatz in reverse, we can notice that if an odd number is divisible by three, it becomes a leaf node, an end point from which no other node is produced (3, 21, and 69 in Figure 2, for instance). When creating a full Collatz tree in reverse, an even number n may only produce an odd number when $\frac{n-1}{3}$ is an integer. If n is a multiple of three, $\frac{n-1}{3}$ can never produce an integer.

Any node that is not a multiple of three will generate infinite nodes. For instance, 1 will generate all nodes $\frac{2^{2x}-1}{3}$ where x is an integer greater than 1. Similarly, 5 will generate $\frac{5\times 2^{2x-1}-1}{3}$.

2.2 Three Types of Odd Numbers

Let's reorganize our tree so that we do not have nodes requiring infinite children. (Whenever I refer to "children", I am referring to the nodes *above* a given node, as though we are considering the tree in reverse, with nodes "generating" possible preceding nodes.) We will consider three types

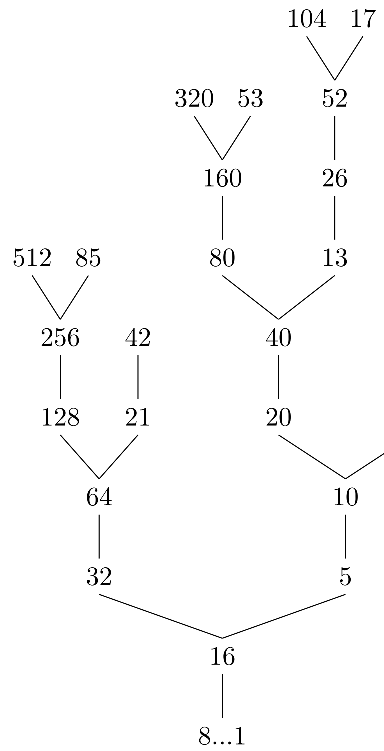

Figure 1: Collatz Tree

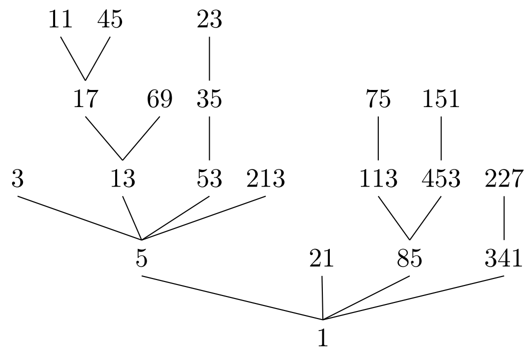

Figure 2: Collatz Odd-Only Tree

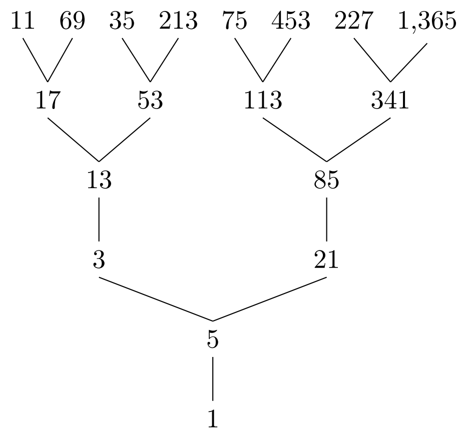

Figure 3: Collatz Odd-Only Variant Tree

of connections between the odd numbers in our tree based on how they iterate through Collatz to reach one another.

First, let's consider 17 iterating to 13 as follows: $17 \to 52 \to 26 \to 13$. Combining the three steps we take, we get $\frac{3n+1}{4}$. What about 7 iterating to 11? Only two steps: $7 \to 22 \to 11$. Combining these two steps, we get $\frac{3n+1}{2}$.

What about our third type of odd numbers? Consider again how any odd node that is not divisible by three will generate infinite nodes. Instead of connecting these nodes to the node that generated them, let's connect them to the node that *precedes* them. For instance, 21 will connect with 5 instead of 1. Likewise, 85 will connect to 21, and so on.

But how can we iterate through Collatz from 21 to 5? We must take a step in reverse! $21 \to 64 \to 32 \to 16 \Rightarrow 5$. Note that for the final step we reverse through Collatz from 16 to 5. Combining these four steps, we get $\frac{n-1}{4}$.

Thus we have a Collatz odd-number-only variant, where n is a positive odd integer:

$$f(n) = \begin{cases} \frac{n-1}{4} & \text{if } n \equiv 1 \pmod 4 \text{ and } n \equiv 5 \pmod 8 \\ \frac{3n+1}{4} & \text{if } n \equiv 1 \pmod 4 \text{ and } n \equiv 1 \pmod 8 \\ \frac{3n+1}{2} & \text{if } n \equiv 3 \pmod 4 \end{cases} \tag{2.2.1}$$

The binary tree this produces is presented in **Figure 3**.

The tree in **Figure 3** is symmetrical, but still seems to be riddled with disorder. However, it still represents the Collatz tree. If we can show that all odd numbers in this variant lead to 1, the conjecture will be proven.

Our main challenge at this point is that pesky $\frac{3n+1}{2}$ formula which inflates n; the other two formulas shrink n. The conjecture would be trivial to prove if n decreased with all three of our formulas. In fact, let us consider a set of formulas that are sure to continually shrink n with each iteration.

2.3 A Simpler Tree

Let us consider a simpler tree generated from the following, where n is again limited to positive odd integers:

$$f(n) = \begin{cases} \frac{n-1}{4} & \text{if } n \equiv 1 \pmod 4 \text{ and } n \equiv 5 \pmod 8 \\ \frac{n+3}{4} & \text{if } n \equiv 1 \pmod 4 \text{ and } n \equiv 1 \pmod 8 \\ \frac{n-1}{2} & \text{if } n \equiv 3 \pmod 4 \end{cases} \tag{2.3.1}$$

This produces a ternary tree; every node branches to three other nodes (though we will ignore the root node's connection to itself). This tree is presented in **Figure 4**.

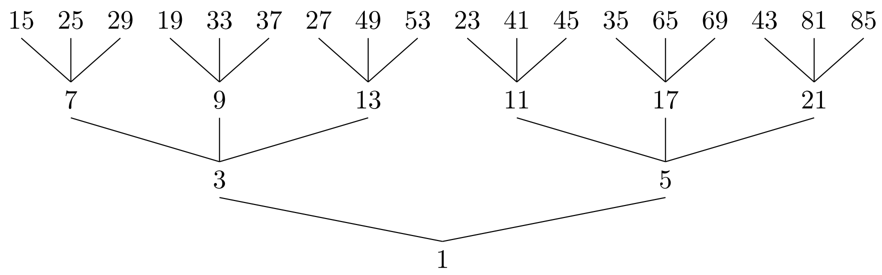

Figure 4: The Simple Tree

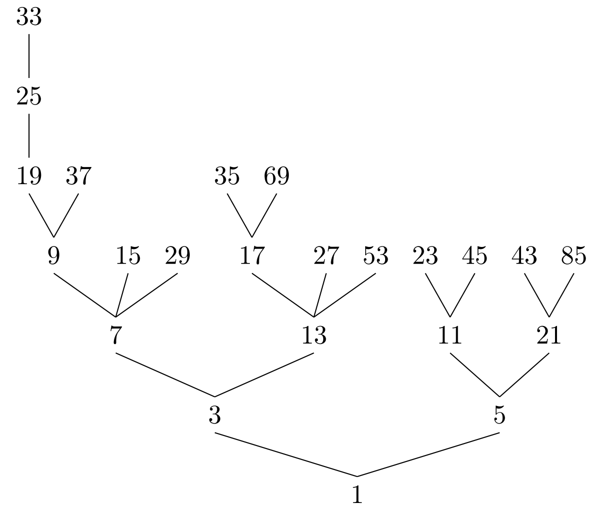

Figure 5: The Simple Tree Shuffled

Given the three formulas in 2.3.1, $\frac{n-1}{4}$, $\frac{n+3}{4}$, and $\frac{n-1}{2}$, we can see that any positive odd integer n will indeed lead to 1, as it must always shrink with every iteration, and it cannot shrink beyond 1.

Now suppose we choose a node n and move the lower endpoint of its lower connection (or edge) to *any other node* on another branch. (Obviously we cannot reattach a node to one of its own children, as that would detach the node and its children from the rest of the tree and create a loop.) After reattaching this node, we know that it and all of its infinite children will *still* ultimately lead to 1, as whatever node we reattach it to must itself lead to 1.

And so it is probably obvious what our final task is. We must shift the connections in this tree so that it is equivalent to our odd-only Collatz variant. If we can show that the odd-only Collatz variant is merely a *shuffle* of this tree, then we will know that all odd integers lead to 1 and that the conjecture is true.

2.4 The Shuffle

You have perhaps already noticed that one set of connections need not be moved: the connections defined by the $\frac{n-1}{4}$ formula. This formula appears in both trees connecting such numbers as $13 \Rightarrow 3$, $85 \Rightarrow 21$, and $21 \Rightarrow 5$. These connections are safe and need not be touched.

Let us consider the connections defined by the second formula, $\frac{n+3}{4}$. These connections include $9 \rightarrow 3$ and $25 \rightarrow 7$. We can disconnect these and reattach them to the node given by the corresponding formula found in 2.2.1, $\frac{3n+1}{4}$. For example, 9 will be redirected to 7, 25 will be redirected to 19. Because $n > \frac{3n+1}{4}$ for any positive odd integer n (disregarding 1), we know the node will never be reattached to one of its own children.

This shuffle gives us the tree presented in **Figure 5**, where we know all nodes are ultimately connected to 1.

This may look a bit unbalanced, as I've only included the numbers from the simple tree in Figure 4. Note that even after this shuffle, the children of each node are still always greater than their parent. That is, for any node n, $n < 2n + 1$, $n < 4n + 1$, and $n < \frac{4n-1}{3}$. (Being the inverses of our tree-generating formulas, these formulas define the values of a given node's children.)

Finally, we disconnect and reconnect the connections defined by the bottom formula in 2.3.1, $\frac{n-1}{2}$. These connections include $11 \to 5$ and $15 \to 7$. They must be reattached according to the corresponding formula from 2.2.1, $\frac{3n+1}{2}$. Thus 11 will be redirected to 17 and 15 will be redirected to 23.

We just have to be sure that a node will not be reconnected to one of its own children. With this shuffle, simply comparing node values won't work so nicely as nodes may now have children with smaller values as we shuffle. For example, when we reconnect 11 to 17, we must consider that 11 now has the 7 node as a child along with its infinite children.

But the task is still not difficult. Recall our inverse formulas: $2n + 1$, $4n + 1$, $\frac{4n-1}{3}$ (from our first shuffle), and now $\frac{2n-1}{3}$ (the inverse of $\frac{3n+1}{2}$). Even if we apply these formulas to one another to create composite formulas corresponding to a node's grandchildren, it will be impossible for any child node to equal $\frac{3n+1}{2}$; the coefficients of n will never be equal, nor will the denominators. (Note that we are not trying to solve for n; we already know n. We are only concerned with whether or not any child of n will equal $\frac{3n+1}{2}$.) So we are guaranteed that a node n will never contain $\frac{3n+1}{2}$ as a child or grandchild at any depth. Reconnecting node n to $\frac{3n+1}{2}$ will not separate n from the tree.

Notice that with this shuffle, we are rearranging *half* of all the odd numbers, attaching each of them to an odd number greater than itself. It's no wonder the Collatz tree seems such a chaotic mess!

After reshuffling all nodes accordingly, our tree has become the Collatz Odd-Only Variant presented in **Figure 3**. Since we know that neither of our two shuffles could result in any node being detached from the tree or creating a loop, we know that all odd numbers in our Collatz Odd-Only Variant are also ultimately connected to 1. This means all odd numbers in the original Collatz tree are connected to 1.

Therefore, the Collatz Conjecture is true.

3 Contact

I'd be interested in any thoughts or feedback readers may have; my email is seanthebest@gmail.com.