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ESTABLISHING A LOWER BOUND FOR CARDINALITIES OF SETS OF GOLDBACH AND PRIME PAIRS

TOM MILNER-GULLAND

This paper proposes, and demonstrates the efficacy of, a method for establishing a lower bound for cardinalities of sets of Goldbach pairs, and shows that the proofs employed may be modified for sets of twin primes. We employ, as one of two variables upon which our proofs turn, the sum, for all i in an interval $[x, y]$ of integers, of $\binom{k(J,i)}{2}$, where $k(J, i)$ is the number of distinct divisors, in a set J of primes, of i. We use our results in conjunction with a perspective on mirror symmetry as found in divisibility distributions, and, implicitly, the Chinese Remainder Theorem. We take the midpoint in $[1, m]$, where m is the square of an odd prime, and tacitly consider the sieve of Eratosthenes in such a way as to find a set of primes whose distribution exhibits a mirror-symmetrical pattern about that midpoint.

Tom Milner-Gulland

Introduction

Let $\mathbb{N}$ be the set of non-negative integers. Throughout this paper, $[x, y]$ will be taken to mean $[x, y] = \{n \in \mathbb{N} : 0 \leq x \leq n \leq y\}$. All intervals are closed bounded unless otherwise stated. Also, p_n for $n = 1, 2, \ldots$ will be the sequence of primes; for a finite subset K of $\mathbb{N}$, $\prod K$ will denote $\prod_{n \in K} n$, and ϕ will be the Euler totient; for sets K and K', $K \oplus K'$ will be the symmetric difference of K and K'.

Additionally, for subsets M and N of $\mathbb{N}$ and $i \in [1, \infty)$, we define

$$\mathrm{Div}(M, i) = \{m \in M : m \mid i\}$$

and

$$\mathrm{Mult}(M, N) = \{m \in M : n \mid m \text{ for some } n \in N\}.$$

Also, $P(n)$ will be $\{p_1, p_2, \ldots, p_n\}$ and J will be any set of primes.

Our essential concept, which is one of a folding of the number scale, will be illustrated diagrammatically and also expressed algebraically. We consider that our inductive step is established by way of the employment of a set of bijections concerning the arithmetic mean (henceforth, A.M) of $|\mathrm{Mult}([i, p_n^2 + i], P(n))|$ for i ranging over $[1, \prod P(n)]$. This is coupled with a general study of maxima and minima, among intervals I all of equal length, found by functions concerning sets of sets $\mathrm{Mult}(I, \{p, q\})$ and $\mathrm{Mult}(I, \{p, q, r\})$ such that p, q and r are distinct elements of $P(n)$. From Theorem 2 onwards, key to our proofs is the tacit use of the Chinese Remainder Theorem.

1. ***Folding the Number Scale***

Assume until Theorem 1 that $y - x$ is even. Our forthcoming argument employs mirror symmetry in the context of a and b in $[x, y]$ for which $a + b = x + y$. Indeed, our employment of $(x + y)/2$ will imply a rephrasing of the Goldbach Conjecture as *every integer greater than one is the A.M. of two primes.*

For any integer k, let

$$A(k, r, i) = \{m \in [0, r) : i \mid (k + m)\},$$

and let

$$A'(k, r, i) = \{m \in [0, r) : i \mid (k - m)\}.$$

For $n > 1$, and $q \in P(n) \setminus \{2\}$ for which $q \mid (x + y)/2$ and k for which $q \mid k$, $(x + y)/2 - k$ is an integer multiple of q, so q also divides both $(x + y)/2 + k$ and $(x + y)/2 - k$. Thus for any r, $A((x + y)/2, r, q) = A'((x + y)/2, r, q)$, while for i for which $q \nmid ((x + y)/2 + i)$,

$$|A\left(\frac{x+y}{2} + i, r, q\right) \cup A'\left(\frac{x+y}{2} + i, r, q\right)| = 2|A\left(\frac{x+y}{2} + i, r, q\right)| + u,$$

where $u \in \{-1, 0, 1\}$. Where, in the forthcoming exposition, we employ a set of primes that excludes the integer two, we are respecting the fact that two divides either $(x+y)/2$ or $(x+y)/2 - 1$ and $(x+y)/2 + 1$, and in both cases

$$\{m \in [0, r) : 2 \mid \left(\frac{x+y}{2} - m\right)\} = \{m \in [0, r) : 2 \mid \left(\frac{x+y}{2} + m\right)\}.$$

For any set M of integers, let

$$L(k, r, M) = \{A(k, r, i) : i \in M\} \cup \{A'(k, r, i) : i \in M\}.$$

Suppose that $0 \leq m < k$. If $k + m$ and $k - m$ are both prime, and e is even, with respect to the Goldbach equation $p + q = e$, the substitution of $2k$ for e yields a solution. By employing $\bigcup L(k, r, P(n))$ we may devise a method to study divisibility distributions in the context of what may be conceived as a folded number scale (see Figure 1). For any integers i and r and any set K of primes, let

$$D(r, i, K) = \{p \in K : i \in \bigcup L(r, i, \{p\})\}.$$

We note that, since there is no $0 < m < p_n^2$ such that $\mathrm{Div}(P(n), m) = \{p_{n+k}\}$, where $k > 0$, there is no composite in $[1, p_n^2] \setminus \mathrm{Mult}([1, p_n^2], P(n))$ (hence our respecting the sieve of Eratosthenes, by employing p_n^2 or $p_n^2 + 1$ as the cardinality of intervals with which we shall ultimately be working). Therefore, for any even $e < p_n^2$,

$$\{m \in \left[2, \frac{e}{2}\right] : \ D\left(\frac{e}{2}, \frac{e}{2} - 1, P(n)\right) = \emptyset\}$$

is a subset, of cardinality $e/2 - 1 - |\bigcup L(e/2, e/2 - 1, P(n))|$, of the set of primes p such that $p + q = e$ is the Goldbach equation. (To prove that, for every integer i and every n such that $p_n^2 < 2i \leq p_{n+1}^2$, either $[0, i-2] \setminus \bigcup L(i, i, P(n)) \neq \emptyset$ or for every i there is a known solution $p + q = 2i$ to the Goldbach equation, is to prove the Goldbach Conjecture.)

Since for some $g \in [0, i)$ and some m, $A(k, r, i) = \{g + ni : n \in [0, m)\}$ and $||A'(k, r, i)| - |A(k, r, i)|| = u$, where $u \in \{0, 1\}$, while

$$|\mathrm{Mult}([1, r], \{i\})| = |A(k, r, i)| - v,$$

where $v \in \{0, 1\}$, in our method $\bigcup L(k, \lfloor (y-x)/2 \rfloor + 1, P(n))$ is given a treatment similar to that which may be afforded to $\mathrm{Mult}([x, y], P(n))$ in regard to whether or not $[x, y] \setminus \mathrm{Mult}([x, y], P(n)) = \emptyset$.

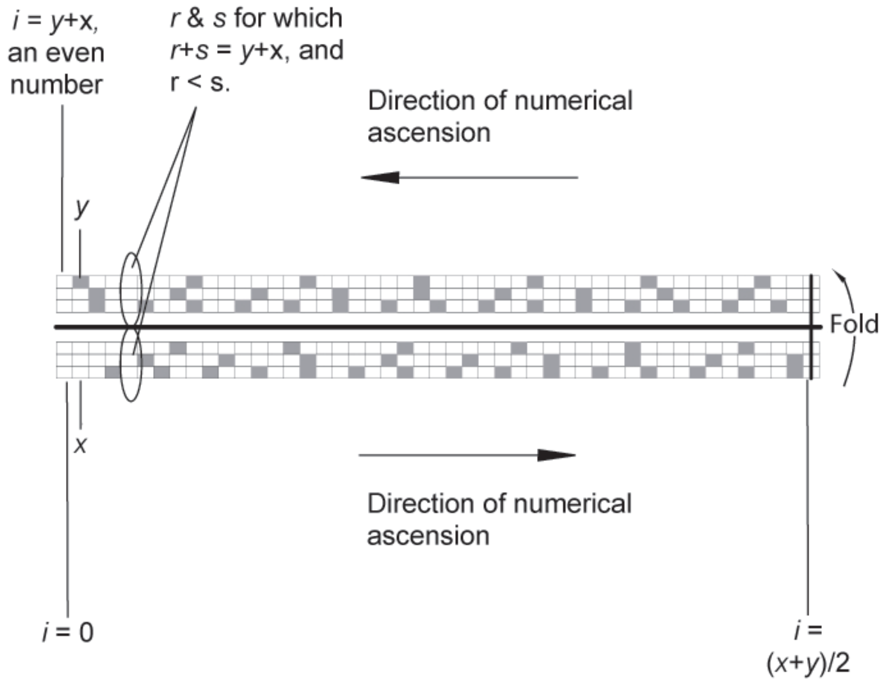
Let $H(J)$ be the set of two-element subsets of J.

Theorem 1. Let I and I' be intervals for which $|I| = |I'|$ and

$$|\mathrm{Mult}(I, J)| < |\mathrm{Mult}(I', J)|.$$

Then

$$|\mathrm{Mult}(I', J)| - |\mathrm{Mult}(I, J)| < 2|H(J)| + 1.$$



NOTES:

1) For each set of rows on either side (top or bottom) of the fold, a column on that side signifies a distinct integer (except, if odd, for (x+y)/2, which extends over both sides). A shaded box signifies a prime factor of an integer thus represented.

2) The direction of numerical ascension is reversed at (*x*+*y*)/2, giving the folded number scale.

3) When all the members of *P*(*n*) are incorporated (in further rows), each vertically coupled pair of columns (comprising, in sum, 2n grid squares) for which no shaded boxes appear in either side of the fold (see reference to r & s) will represent a solution to the Goldbach equation.

Figure 1. The Folded Number Scale for Divisors $3, 5$ and 7 and an Interval $[x, y]$

Proof

The following proof is a consequence of the fact that two is the least of all elements of the set of $|S|$ such that $S \subseteq \mathrm{Div}(J, m)$ with $|S| > 1$, in view of the fact that for some $|S|$-element subset M of $\mathbb{N}$ there is a bijection $b : S \to \mathrm{Mult}(M, S)$ defined, for each $p \in S$, by $p \in S = \mathrm{Div}(S, b(p))$. It is also a consequence of the fact that, for any integer i for which $i \nmid |I|$, $|\mathrm{Mult}(I, \{i\})| = \lfloor |I|/i \rfloor + r$, where $r \in \{0, 1\}$. Let K be a three-element set of primes. For any sets M and M' of integers for which

I. $|M| = |M'|$,

II. for each proper subset R of K,

$$\sum_{N \in R} |\mathrm{Mult}(M, \{\prod R\})| = \sum_{N \in R} |\mathrm{Mult}(M', \{\prod R\})|$$

III. $|\mathrm{Mult}(M', \{\prod K\})| = |\mathrm{Mult}(M, \{\prod K\})| + 1$,

let $g_{M,M',K} : M \to M'$ be a bijection that minimizes

$$\sum_{k \in M} |\mathrm{Div}(K, k) \oplus \mathrm{Div}(K, g_{M,M',K}(k))|.$$

For any sets N and N' of integers for which $|N| = |N'|$ and, for each $p \in K$, $|\mathrm{Mult}(N', \{p\})| - |\mathrm{Mult}(N, \{p\})| = r$, where $r \in \{0, 1\}$, let $X(N, N', K)$ be the set of pairs (C, C') of sets of integers such that

I. $|C| = |C'| = 3$,

II. there exists $g_{N \setminus C, N' \setminus C', K}$,

III. for each $m \in C \cup C'$, $|\mathrm{Div}(K, m)| \leq 1$.

Let $S(I, I', J)$ be the set of all three-element subsets G of J such that, for some $(C, C') \in X(I, I', G)$ there exists $g_{I \setminus C, I' \setminus C', G}$. Let $K \in S(I, I', J)$. Suppose that $\mathrm{Div}(K, g_{I \setminus C, [0,|I|) \setminus C', K}(i)) = K$, implying $|\mathrm{Div}(K, i)| = 2$. Recall that $H(J)$ is the set of two-element subsets of J. Let $Y(I, I', K)$ be the set of all $E \in H(K)$ for which, for some $(C, C') \in X(I, I', K)$,

$$|\mathrm{Mult}(I \setminus \{i\}), \{\prod E\})| = |\mathrm{Mult}(I' \setminus \{g_{I \setminus C, I' \setminus C', K}(i)\}, \{\prod E\})| + 1$$

and $Y'(I, I', K)$ be the set of all $p \in K$ for which

$$|\mathrm{Mult}(I \setminus (C \cup \{i\}), \{p\})| = |\mathrm{Mult}(I' \setminus (C' \cup \{g_{I \setminus C, I' \setminus C', K}(i)\}), \{p\})| + 1.$$

Then $|\mathrm{Mult}(I', J)| - |\mathrm{Mult}(I, J)| = |Y(I, I', K)| - |Y'(I, I', K)| = 2 - 1 = 1$.

We note that

a) for each $M \in S(I, [0, |I|), J)$, for some $(C, C') \in X(I, [0, |I|), M)$ we have $|\mathrm{Mult}([0, |I|) \setminus C', M)| = |\mathrm{Mult}(I \setminus C, M)| + 1$;

b) for any $\{p, q\} \in H(J)$ for which

$$\sum_{r \in \{p,q\}} |\mathrm{Mult}(I, \{r\})| = \sum_{r \in \{p,q\}} |\mathrm{Mult}([0, |I|), \{r\})|$$

and $|\mathrm{Mult}(I, \{pq\})| = |\mathrm{Mult}([0, |I|), \{pq\})| + 1$, we have

$$|\mathrm{Mult}(I, \{p, q\})| = |\mathrm{Mult}([0, |I|), \{p, q\})| - 1. \tag{1}$$

Let $H'(J) = \{M \subseteq J : |M| = 3\}$ and $G(J) = \{\prod N : N \in H'(J)\}$. The following is intended to invoke the fact that it is possible that $|\mathrm{Mult}(I, G(J))\}|$ is not equal to $|\mathrm{Mult}([0, |I|), G(J))\}|$ notwithstanding that, subject to further scrutiny, it is possible that also

$$\sum_{M \in H'(J)} |\mathrm{Mult}(I, \{\prod M\})| = \sum_{M \in H'(J)} |\mathrm{Mult}([0, |I|), \{\prod M\})|.$$

For distinct elements W and W' of $S(I, [0, |I|), J)$, let N and N' be proper subsets of W and W' respectively, such that $N \cup N' \in S(I, [0, |I|), J)$. Then since a), above, is true, it follows that for some $(C, C') \in X(I, [0, |I|), N \cup N')$,

$$|\mathrm{Mult}(I \setminus C, N \cup N')| = |\mathrm{Mult}([0, |I|) \setminus C', N \cup N')| - 1.$$

If each three-element subset of J is in $S(I, [0, |I|), J)$ then, noting that for each $p \in J$ we have $|\mathrm{Mult}([0, |I|), \{p\})| = \lfloor |I|/p \rfloor + 1$, it follows that

$$\max\{|\mathrm{Mult}([i, |I| + i), J)| : i \in \mathbb{N}\} = |\mathrm{Mult}([0, |I|), J)|. \tag{2}$$

Conversely, if there is a nonempty subset N of $H'(J)$ for which each element of N is not in $S(I, [0, |I|), J)$, then for each for $M \in N$ either there exists $E \in H(M)$ for which

$$|\mathrm{Mult}([0, |I|), \{\prod E\})| > |\mathrm{Mult}(I, \{\prod E\})|$$

or $|\mathrm{Mult}([0, |I|), \{\prod M\})| = |\mathrm{Mult}(I, \{\prod M\})|$. We note that for any two sets R and R' of integers for which $\sum_{p \in J} |\mathrm{Mult}(R', \{p\})| = \sum_{p \in J} |\mathrm{Mult}(R, \{p\})|$,

$$\sum_{\substack{m \in R \\ |\mathrm{Div}(J,m)|>1}} (|\mathrm{Div}(J, m)| - 1) - \sum_{\substack{n \in R' \\ |\mathrm{Div}(J,n)|>1}} (|\mathrm{Div}(J, n)| - 1) = |\mathrm{Mult}(R', J)| - |\mathrm{Mult}(R, J)|. \tag{3}$$

For any sets V and V' of integers, let

$$t(V, V', J) = \sum_{n \in V} |H(\mathrm{Div}(J, n))| - \sum_{m \in V'} |H(\mathrm{Div}(J, m))|.$$

Assume from now until the completion of this paper, that $|J| > 2$. For any two sets R and R' of integers for which $t(R, R', J) = 0$ let

$$b_{R,R',J} : \{(H(\mathrm{Div}(J, k)), k) : k \in R\} \to \{(H(\mathrm{Div}(J, m)), m) : m \in R'\}$$

be a bijection. Then if $t(I, [0, |I|), J) = 0$, it follows that there is a $|H(J)|$-element subset, U, of $\{(H(\mathrm{Div}(J, k)), k) : k \in I\}$ for which, for each $(u, v) \in U$, for some $T \in H(J)$, we have $b_{I,[0,|I|),J}(u, v) = (T, 0)$. Also, for any nonempty subset K of J,

$$|\mathrm{Mult}([1, |I|), \{\prod K\})| = \left\lfloor \frac{|I|}{\prod K} \right\rfloor,$$

so

$$|\{(H(\mathrm{Div}(J, k)), k) : k \in I\}| - |\{(H(\mathrm{Div}(J, m)), m) : 0 < m < |I|\}| \leq |H(J)|. \tag{4}$$

In view of (4), let $G(J)$ be the set of all pairs (E, E') of sets of integers for which, for each $M \in \{E, E'\}$,

$$\sum_{m \in M} |H(\mathrm{Div}(J, m))| = |H(J)|$$

and for which $\sum_{m \in E} |\mathrm{Div}(J, m)| = \sum_{n \in E'} |\mathrm{Div}(J, n)|$. Let (R, R') be an element of $G(J)$. Suppose that, in the set of all $m \in \mathrm{Mult}(R, J)$ for which $|\mathrm{Div}(J, m)| > 1$ there exists no n for which $|\mathrm{Div}(J, n)| > 2$ and there exists $k \in \mathrm{Mult}(R', J)$ for which $\mathrm{Div}(J, k) = J$. Then, assuming $|\mathrm{Mult}(R', J)| > |\mathrm{Mult}(R, J)|$, it follows by (3) that

$$\begin{aligned} &|\mathrm{Mult}(R', J)| - |\mathrm{Mult}(R, J)| \\ &= \sum_{\substack{m \in R', \\ |\mathrm{Div}(J,m)|>1}} (|\mathrm{Div}(J, m)| - 1) - \sum_{\substack{n \in R, \\ |\mathrm{Div}(J,n)|>1}} (|\mathrm{Div}(J, n)| - 1) \\ &= |H(J)| - |J| + 1. \end{aligned}$$

Since two is the least, and $|J|$ the highest of all elements of the set of all $|S|$ such that $S \subseteq J$ and $H(S) \neq \emptyset$, this implies that, for any $(K, K') \in G(J)$,

$$|\mathrm{Mult}(K', J)| - |\mathrm{Mult}(K, J)| \leq |H(J)| - |J| + 1. \tag{5}$$

Let $U(J)$ be the set of all $|J|$-element sets of primes. Recall that for any sets V and V' of integers,

$$t(V, V', J) = \sum_{n \in V} |H(\mathrm{Div}(J, n))| - \sum_{m \in V'} |H(\mathrm{Div}(J, m))|.$$

Let $h(J)$ be the highest element in the set of all $|\mathrm{Mult}(M, N)| - |\mathrm{Mult}(M', N)|$ such that $N \in U(J)$ and M and M' are intervals for which $t(M, M', N) = 0$. It follows by (1) and (2) that, for some $0 \leq m \leq t([0, |I|), I, J)$,

$$|\mathrm{Mult}([0, |I|), J)| \leq |\mathrm{Mult}(I, J)| - m + h(J). \tag{6}$$

We note that

$$|\{p \in J : |\mathrm{Mult}(M, \{p\})| \neq |\mathrm{Mult}(M', \{p\})|\}| \leq |J|.$$

It follows by (2) coupled with (4), and (5), that $h(J) \leq |H(J)| - |J| + 1 + |J|$. Hence

$$h(J) \leq |H(J)| + 1. \tag{7}$$

Since $m \leq |H(J)|$, (6) gives us

$$|\mathrm{Mult}(I', J)| - |\mathrm{Mult}(I, J)| \leq 2|H(J)| + 1.$$

□

Recall that

$$A(k, r, i) = \{m \in [0, r) : i \mid (k + m)\},$$

and

$$A'(k, r, i) = \{m \in [0, r) : i \mid (k - m)\},$$

and

$$L(k,r,M) = \{A(k,r,i) : i \in M\} \cup \{A'(k,r,i) : i \in M\}.$$

1.1. Remark.

For any k, i and r,

$$\begin{aligned}&|\{p \in J : |\mathrm{Mult}((k-r,k+r),\{p\})| > |\mathrm{Mult}((i-r,i+r),\{p\})|\}|\\ &\quad = |\{p \in J : |L(k,r,\{p\})| > |L(i,r,\{p\})|\}|\\ &\quad \leq |J|.\end{aligned}$$

Theorem 2. Let k, r and i be integers for which

$$|\bigcup L(k,r,J)| > |\bigcup L(i,r,J)|.$$

Then

$$|\bigcup L(k,r,J)| - |\bigcup L(i,r,J)| \leq 4|H(J)| - |J| + 2.$$

Proof

For any distinct primes p and q, let

$$V(k,r,\{p,q\}) = (A'(k,r,p) \cap A(k,r,q)) \cup (A'(k,r,q) \cap A(k,r,p)).$$

Choose k and r so that for each $K \in H(J)$ there is a distinct $V(k,r,K)$ in $\{V(k,r,M) : M \in H(J)\}$ and likewise a distinct $\mathrm{Mult}((k-r,k+r),K)$ in the set of all $\mathrm{Mult}((k-r,k+r),M)$ such that $M \in H(J)$. Then

$$|\{V(k,r,M) : M \in J\}| = |H(J)|. \tag{8}$$

Also, since in any interval I of positive integers for which $|I| = pq$, there is precisely one element of $A'(k,\max I,p) \cap A(k,\max I,q) \cap I$, and since

$$\min(A'(k,r,p) \cap A(k,r,q)) + \min(A'(k,r,q) \cap A(k,r,p)) = pq, \tag{9}$$

we have $||V(k,r,\{p,q\})| - |V(i,r,\{p,q\})|| \in \{0,1\}$. Let

$$Q(k,r,J) = \{\mathrm{Mult}((k-r,k+r),M) : M \in H(J)\}.$$

Then it follows by (8) that

$$|Q(k,r,J)| + |\{V(k,r,M) : M \in H(J)\}| = 2|H(J)|. \tag{10}$$

Let $j(k, i, r, J) = \sum_{M \in H(J)}(|V(i, r, M)| - |V(k, r, M)|)$. Then $j(k, i, r, J) \leq |H(J)|$. Also, there exist u and v for which

$$\sum_{M \in H(J)} (|\mathrm{Mult}([u, u+2r), M)| - |\mathrm{Mult}([v, y+2r), M)|) = j(k, i, r, J). \tag{11}$$

The forthcoming exposition is a consequence of the fact that, for any interval M and for any proper subsets G and G' of J for which $|G'| > |G|$,

$$|\mathrm{Mult}(M, G') \setminus \mathrm{Mult}(M, G)| = |\mathrm{Mult}(M, G')| - |\mathrm{Mult}(M, G)|.$$

Thus, for any M' for which $|\mathrm{Mult}(M', J)| > |\mathrm{Mult}(M, J)|$, we may choose G and $q \in J \setminus G$ so that

$$|\mathrm{Mult}(M, G \cup \{q\}) \setminus \mathrm{Mult}(M, G)| < |\mathrm{Mult}(M', G \cup \{q\}) \setminus \mathrm{Mult}(M', G)|.$$

Let K be any subset of J for which $|K| > 1$. For any t and each $q \in K$, any $m \in [0, r)$ is in $\bigcup L(t, r, \{q\}) \setminus \bigcup L(t, r, K \setminus \{q\})$ if and only if, for each subset S of $L(t, r, K \setminus \{q\})$, we have $m \in \bigcup L(t, r, \{q\}) \setminus \bigcup S$.

Therefore, we may couple (11) with (1), (2), (7), (10) and Remark 1.1, and we have

$$\begin{aligned} &|\mathrm{Mult}((k+r, k-r), J)| - |\mathrm{Mult}((i-r, i+r), J)| + 2|H(J)| - |J| + 1 \\ &\qquad\qquad\qquad \geq |\bigcup L(k, r, J)| - |\bigcup L(i, r, J)|. \end{aligned}$$

It follows by Theorem 1 that

$$\begin{aligned} |\bigcup L(k, r, J)| - |\bigcup L(i, r, J)| &\leq 4|H(J)| - 2(|J| - 1) + |J| \\ &= 4|H(J)| - |J| + 2. \end{aligned}$$

□

1.1.2. Remark.

Let us substitute $P(n)$ for J in the statement of Theorem 2. From the facts that $A(k, r, 2) = A'(i, r, 2) = \bigcup L(i, r, \{2\})$ and that $P(n)$ contains the prime two, and that $|\{M \in H(P(n)) : 2 \in M\}| = n - 1$, and since $|H(P(n))| = n(n-1)/2$, we may subtract $n - 1$ from the right-hand side of the above equality. Thus

$$\begin{aligned} |\bigcup L(k, r, &P(n)) - |\bigcup L(i, r, P(n))| \\ &\leq 2n(n-1) - n + 2 - (n-1) \\ &= 2n(n-2) + 3. \end{aligned}$$

1.2. *A Lower Bound for Cardinalities of Sets of Goldbach Pairs*

Here we reach the final part of the first of our two chief proofs. Our forthcoming Theorem employs the A.M. of $|\text{Mult}([i, p_n^2 + i], P(n))|$ for i ranging over $[1, \prod P(n)]$, in conjunction with Theorems 1 and 2, which in turn have utilised, by way of the expression $\bigcup L(k, r, J)$, mirror symmetry about the midpoint k of an interval $(k - r, k + r)$.

For any integer k for which no element of J divides k, let

$$f(n) = \frac{\prod P(n) \setminus \{2\} - |\bigcup L(k, \prod P(n) \setminus \{2\}, P(n) \setminus \{2\})|}{\prod P(n)}.$$

Lemma. The value of $f(n)$ is equal to $\prod_{p \in P(n) \setminus \{2\}} (p - 2) / \prod P(n)$.

Proof

For any prime $q \notin J$, we have $\text{Mult}([1, q \prod J], \{q\}) = \{iq : i \in [1, \prod J]\}$. Suppose that $i \in [1, \prod J]$. Then $iq \in \text{Mult}([1, q \prod J], J)$ if and only if $i \in \text{Mult}([1, \prod J], J)$, so

$$\text{Mult}([1, q \prod J], J) \cap \text{Mult}([1, q \prod J], \{q\}) = \{iq : i \in \text{Mult}([1, \prod J], J)\}.$$

In particular, the map

$$g : \text{Mult}([1, \prod J], J) \to \text{Mult}([1, q \prod J], J) \cap \text{Mult}([1, q \prod J], \{q\})$$

defined by $g(i) = iq$ is a bijection. Thus, since there are precisely $\prod J$ integer multiples of q in $[1, q \prod J]$,

$$\begin{aligned} \phi(\prod J) = \prod_{p \in J} (p - 1) &= |\{r \in [1, q \prod J] : \text{Div}(J \cup \{q\}, r) = \{q\}| \\ &= \frac{|\{r \in [1, q \prod J] : \text{Div}(J, r) = \emptyset\}|}{q}. \end{aligned} \tag{12}$$

Assume until the completion of this proof, that $2 \notin J$. Let k be an integer for which no $p \in J \cup \{q\}$ divides k. Recall that, for any integers i and r and any set K of primes,

$$D(r, i, K) = \{p \in K : i \in \bigcup L(r, i, \{p\})\}.$$

We note that for any integers i and j, $\text{Div}(J, i) = \text{Div}(J, j \prod J \pm i)$, and for each $m \in \bigcup L(k, \prod J, J)$, some element of J divides either $k + m$ or $k - m$. Then for a ranging over $[1, \prod J]$,

$$\begin{aligned} &|\{a \prod J + u : D(k, u, J \cup \{q\}) = \{q\},\ 0 < u \leq \prod J\}| \\ &\qquad = |\bigcup L(k, \prod J, \{q\}) \setminus \bigcup L(k, \prod J, J)| \end{aligned}$$

while for each $m \in [a\prod J+1, (a+1)\prod J]$, we have $D(k,m,J) = D(k, m+\prod J, J)$. Therefore, for any interval I of cardinality $q\prod J$,

$$\begin{aligned}&|\{m \in \bigcup L(k, \max I, \{q\}) \cap I : D(k,m,J) = \{q\}\}| \\ &= 2|\{i \in \mathrm{Mult}([1, q\prod J], \{q\}) : \mathrm{Div}(J, i) = \{q\}\}|. \qquad (13)\end{aligned}$$

Let

$$c(p,J) = \frac{|[0,\prod J) \setminus \mathrm{Mult}([0,\prod J), J\setminus\{p\})|}{|\{m \in \mathrm{Mult}([0,\prod J), J\setminus\{p\}) : \mathrm{Div}(J,m) = \{p\}\}|}$$

Then

$$\phi(\prod J) = \prod_{p\in J}\left(p - \frac{c(p,J)}{\frac{|[0,\prod J)|}{|[0,\prod J\setminus\{p\})|}}\right)$$

and $c(p,J) = p$. With reference to the definition of $c(p,J)$, if we substitute $\bigcup L(k, \prod J, J\setminus\{p\})$ for $\mathrm{Mult}([0,\prod J), J\setminus\{p\})$, and $D(k,m,J)$ for $\mathrm{Div}(J,m)$, it follows by (12) coupled with (13) that, for any interval I of cardinality $\prod J$,

$$\prod J - |\bigcup L(k, \max I, J) \cap I| = \prod_{p\in J}(p-2).$$

Then, noting that

$$\prod p \in P(n)\setminus\{2\}(p-2)(2-1) = \prod p \in P(n)\setminus\{2\}(p-2),$$

it follows that

$$f(n) = \frac{\prod_{p\in P(n)\setminus\{2\}}(p-2)}{\prod P(n)}.$$

□

Theorem 3. For any k and any $n > 4$

$$\frac{p_n^2-1}{2} - |\bigcup L(k, \frac{p_n^2-1}{2} - 1, P(n))| \geq 1.$$

Proof

For any sets M and M' of integers for which there is a bijection $b : M \to M'$, let $B(b)$ be the set of all $b(m)$ such that $m \in M$ and $\mathrm{Div}(P(n), m) = \emptyset$ while $\mathrm{Div}(P(n), b(m)) \neq \emptyset$. For any intervals N and M, each of cardinality p_n^2+1, let $h_{N,M,P(n)} : N \to M$ be a bijection for which

$$|\mathrm{Mult}(M, P(n))| - |\mathrm{Mult}(N, P(n))| = |B(h_{N,M,P(n)})|.$$

Let $I(n,a)$ be $[a, a+p_n^2]$ for which

$$|\mathrm{Mult}([a, a+p_n^2], P(n)))| = \min\{|\mathrm{Mult}([x, x+p_n^2], P(n))| : 0 < x \leq \prod P(n)\}.$$

Then it follows by Theorem 1 that, for any interval I for which $|I| = p_n^2 + 1$,

$$|B(h_{I(n,a),I,P(n)})| \leq 2|H(P(n))| + 1.$$

Let i be an element of $B(h_{I(n,a),[0,p_n^2],P(n)})$. Then

$$\begin{aligned}&\{i + x : \mathrm{Div}(P(n), i + x) = \emptyset,\ 0 < x \leq \prod P(n)\}\\ &= [i, \prod P(n) + i] \setminus \mathrm{Mult}([i, \prod P(n) + i], P(n)).\end{aligned} \tag{14}$$

Let

$$b(n) = \frac{|\mathrm{Mult}([1, \prod P(n)], P(n))|}{\prod P(n)}.$$

Then it follows by (14) that $b(n)(p_n^2 + 1)$ is the A.M. of $|\mathrm{Mult}([x, x + p_n^2], P(n))|$ for x ranging from one to $\prod P(n)$.

For any interval I of cardinality $p_n^2 + 1$, let $j_{I,n} : [0, |I|) \to I$ be a bijection defined, for each $r \in [0, |I|)$, by $j_{I,n}(r) = \min I + r$ and let $j'_{I,n} : I \to [0, |I|)$ be a bijection defined, for each $s \in I$, by $j'_{I,n}(s) = s - \min I$. Then, for any x,

$$\begin{aligned}&|\mathrm{Mult}([0, p_n^2], P(n))| - |\mathrm{Mult}([x, x + p_n^2], P(n))|\\ &\qquad = |B(j'_{[x,x+p_n^2],n})| - |B(j_{[x,x+p_n^2],n})|.\end{aligned}$$

The following is a consequence of the fact that, for any $k \in B(j'_{I(n,a),n})$ and $n > 1$, it is for a proper fraction, only, of all elements m of

$$\{j_{M,n}(k) : M \in \{[x, x + p_n^2] : 0 < x \leq \prod P(n)\},$$

that $\mathrm{Div}(P(n), m)$ is empty. Thus, our forthcoming method is tantamount to choosing any $|B(h_{I(n,a),[0,p_n^2],P(n)})|$-element subset of $B(j'_{I(n,a),n})$ in order to determine an upper bound for $m - b(n)(p_n^2 + 1)$, for m ranging over

$$\{|\mathrm{Mult}([x, x + p_n^2], P(n))| : 0 < x \leq \prod P(n)\}.$$

It follows by (14) that, for any $m \in [0, p_n^2]$, the A.M. of $|\mathrm{Mult}(\{j_{[x,x+p_n^2],n}(m)\}, P(n))|$ for x ranging from one to $\prod P(n)$, is equal to $b(n)$. Since for any x we have

$$\begin{aligned}&|\mathrm{Mult}(I(n,a), P(n))| + |B(h_{I(n,a),[x,x+p_n^2],P(n)})|\\ &\qquad = |\mathrm{Mult}([x, x + p_n^2], P(n))|,\end{aligned}$$

the A.M. of $|B(h_{I(n,a),[x,x+p_n^2],P(n)})|$ for x ranging from one to $\prod P(n)$ is equal to $b(n)(p_n^2 + 1) - |\mathrm{Mult}(I(n,a), P(n))|$. Further, it follows by Theorem 1 that

$$\begin{aligned}|\mathrm{Mult}([0, p_n^2], P(n))| - &|\mathrm{Mult}(I(n,a), P(n))|\\ &= |B(h_{I(n,a),[0,p_n^2],P(n)})|\\ &\leq 2|H(P(n))| + 1,\end{aligned}$$

while also

$$\begin{aligned}&(p_n^2 + 1 - |B(h_{I(n,a),[0,p_n^2],P(n)})|)b(n) + |B(h_{I(n,a),[0,p_n^2],P(n)})|\\ &\qquad = |\mathrm{Mult}([0, p_n^2], P(n))|.\end{aligned} \tag{15}$$

Since the integer one is not in $P(n)$, we shall subtract one from the value, given in 1.1.2, of $2n(n-2)+3$. Let

$$g(n) = \min\{|\bigcup L\left(i, \frac{p_n^2-1}{2}, P(n)\right)| : 0 < i \leq \prod P(n)\}.$$

It follows by Theorem 2 and Remark 1.1.2, that

$$0 \leq |\bigcup L\left(\frac{p_n^2+1}{2}, \frac{p_n^2-1}{2}, P(n)\right)| - g(n) \leq \frac{p_n^2-1}{2} - 2n(n-2)+2.$$

Recall that

$$f(n) = \frac{\prod P(n) \setminus \{2\} - |\bigcup L(k, \prod P(n) \setminus \{2\}, P(n) \setminus \{2\})|}{\prod P(n)}.$$

For $n = 5$, we have $((p_n^2-1)/2 - 2n(n-2)+2)f(n) \approx (60-30+2) \times 0.05844 \approx 1.87$. This necessitates, by our method, at least one solution to the Goldbach equation $p+q=e$ when $e = (p_n^2-1)/2$ with $n=5$.

Among all (a,b) in $\{(p_{k+1}-p_k, k) : 0 < k \leq 6\}$, there is precisely one, the highest, for which $a=4$, and none for which $a > 4$, but for any $i > 4$, there is, on account of the distribution of integer multiples of three, at least one (a', b') in $\{(p_{k+1}-p_k, k) : i < k \leq i+2\}$ for which $a' \geq 4$. Therefore, for any $k > 6$ and $n = 6$, we have $k/p_k < n/p_n < 1/2$. It follows by our Lemma, that $f(n)/f(n+1) = p_{n+1}/(p_{n+1}-2)$. For any r we have $2r^2/2r(r-2) = r/(r-2)$ and if $r = p_n/2$ we have $2r^2 = p_n^2/2$. Therefore for $n > 5$ we have $(p_n^2/2)/(2n(n-2)) > p_n/(p_n-2)$ giving

$$\frac{f(n)}{f(n+1)} < \frac{p_n^2 - 2n(n-2)}{p_{n+1}^2 - 2(n+1)((n+1)-2)}.$$

Thus our method implies a lower bound for the number of $\{p,q\} \subset P(k)$ for which $p+q=e$, for any even e such that $p_{k-1}^2 < e \leq p_k^2+1$, that is an increasing function of k. Coupled with the fact that, for any s for which there exists $p \in P(n) \setminus \{2\}$ such that $p \mid s$,

$$f(k) < \frac{\prod P(k) \setminus \{2\} - |\bigcup L(s, \prod P(k) \setminus \{2\}, P(k) \setminus \{2\})|}{\prod P(k)}$$

this in turn implies that, for any $n > 4$ and any i,

$$\frac{p_n^2-1}{2} - |\bigcup L(i, \frac{p_n^2-1}{2} - 1, P(n))| \geq 1.$$

$\square$

1.2.1. Remark

Since $n/p_n \to 0$, we have $2n(n-2)/(p_n^2-1) \to 0$, so for any integer j greater than one, the number of (p,q), where p and q are primes whose squares exceed $2j$, for which $p+q=2j$ (which, since $2j$ is an even number, is the Goldbach equation) approaches infinity as j approaches infinity.

2. A Modification of Theorem 3 for Sets of Twin Primes

In the following, the approach to infinity of the cardinality of the set of all twin primes (prime pairs) in $[1, p_n^2]$, as $n \to \infty$, is implied by reference to Remark 1.2.1.

2.1 ***Theorem 4.*** For any positive integer n there is a prime $p > n$ such that $p + 2$ is also prime.

Proof

The method presented in the proof of this theorem is a consequence of the idea of the folded number scale illustrated in Figure 1. This is shown in the illustration of two sets that we shall define, $S(x, y)$ and $S'(x, y)$, together with the folded number scale, in Figure 2. For any x and y, let

$$S(x, y) = \{n + 1 \in [x, y] : 4 \mid n\}.$$

Let

$$S'(x, y) = \{n + 3 \in [x, y] : 4 \mid n\}.$$

Assume $y - x$ is even. Consider $s \in S(x, y)$ for which, for some n, there exists $p \in P(n)$ such that mp divides s, where m is an integer greater than one. Then neither $(s, s + 2)$ nor $(s - 2, s)$ are in the set of all pairs (q, r) of primes in $[x, y]$, such that $q + 2 = r$. Likewise, for any $x \leq j \leq (x + y)/2$ for which p divides j, $(j, y - j + x)$ is not in

$$\{(a, b) : a \in [x, y] \backslash \mathrm{Mult}\left(\left(x, \frac{x+y}{2}\right], P(n)\right), b \in [x, y] \backslash \mathrm{Mult}\left(\left[\frac{x+y}{2}, y\right), P(n)\right)\},$$

implying $j \notin [x, (x + y)/2] \setminus \bigcup L((x + y)/2, (x + y)/2, P(n))$. We note that, for any q for which $q \nmid (x + y)/2$, and for any interval I of positive integers,

$$|\bigcup L\left(\frac{x+y}{2}, \max I, \{q\}\right) \cap I| = \left\lfloor \frac{|I|}{q} \right\rfloor + r,$$

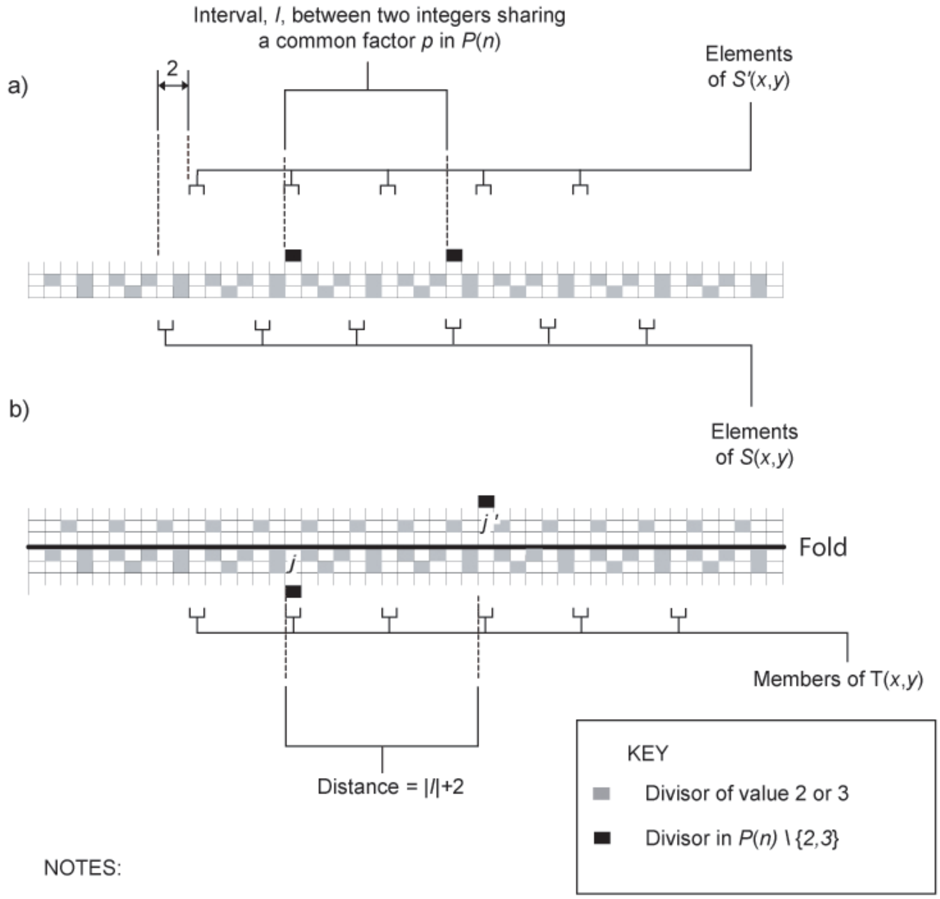
where $r \in \{0, 1\}$.

Consider $n > 3$ and any $K \subset P(n) \setminus \{2\}$ for which, for each $q \in K$, $q \nmid (x + y)/2$. Suppose that $|I| = (y - x)/2$. Then for any set K of integers greater than two, there exists i for which

$$\begin{aligned} &\bigcup \{i + r : r \in \{A(i, \max I, q) \cap I : q \in K\}\} \\ &\qquad = S(x, y) \cap \mathrm{Mult}([x, y], K) \end{aligned}$$

and

$$\begin{aligned} &\bigcup \{i - s : \{s \in A'(i, \max I, q) \cap I : q \in K\}\} \\ &\qquad = S'(x, y) \cap \mathrm{Mult}([x, y], K). \end{aligned}$$



NOTES:

1) In Diagram a), the two black boxes eliminate candidates for being twin prime, each column signifying a distinct integer.

2) In Diagram b), the number scale is folded (as in Figure 1) and the black boxes eliminate candidates for being k such that $k + h = e$ is a solution to the Goldbach equation.

3) Each black box signifies a prime divisor of an integer signified by the column - in b) such is either above (j') or beneath (j) the fold - in which it appears.

4) The set of (i, i+2), where i in $S(x,y)$, and the set of integers m for which $|D(m,J)| = \varnothing$, are candiates for being twin primes (Diagram a)) and k such that $k + h = e$ is a solution to the Goldbach equation (Diagram b)), respectively. Then, the righthand black box being shifted two columns to the right in b), the number of respective candidates eliminated for a) is equal to that for b), although the number of columns (integers) in b) is twice that of a).

Figure 2. The Elimination of Candidates in Subintervals of $[x, y]$

Assume $\min S(x,y) \cup S'(x,y) \in S(x,y)$. Assume also $I = [u,v]$. Let

$$T(i,I,K) = \{s+2 : \{s \in \bigcup\{A'(i,v+2,q) \cap [u,v+2] : q \in K\}.$$

Then, for any such i,

$$\begin{aligned} &|\bigcup\{A(i,\max I,q) \cap I : q \in K\} \cup T(i,I,K)| \\ &= |\{m \in S(x,y) : q \mid m \text{ or } q \mid m+2 \text{ for some } q \in K\}|. \end{aligned}$$

This is illustrated in Figure 2 (see Note 4). We note also that

$$\{(m,m+2) : \mathrm{Div}(P(n),m) = \emptyset,\ \mathrm{Div}(P(n),m+2) = \emptyset,\ p_n < m < p_n^2\}$$

is the set of all pairs $(m,m+2)$, for which m and $m+2$ are each prime, such that m is in the interval (p_n,p_n^2).

Recall that, for any sets M and M' of integers for which there is a bijection $b : M \to M'$, $B(b)$, is the set of all $b(m)$ such that $m \in M$ and $\mathrm{Div}(P(n),m) = \emptyset$ while $\mathrm{Div}(P(n),b(m)) \neq \emptyset$. We note that $|[1,p_n^2]|$ is more than twice the value of the second parameter in the expression $L(k,(p_n^2-1)/2,P(n))$. Let p and q be distinct elements of J. We note that i is not necessarily an element of I or of $[1,p_n^2]$, so (9) is here superfluous. Therefore, with reference to (8), instead of employing, in our method, $V(k,r,\{p,q\}) = (A'(k,r,p) \cap A(k,r,q)) \cup (A'(k,r,q) \cap A(k,r,p))$, we shall take, for any interval I for which $|I| = p_n^2$, the sets

$$U(k,I,\{p,q\}) = A'(k,\max I,p) \cap A(k,\max I,q) \cap I$$

and

$$U'(k,I,\{p,q\}) = A'(k,\max I,q) \cap A(k,\max I,p) \cap I.$$

Let

$$u(k,I,J) = |\{U(k,I,M) : M \in H(J)\}| + |U'(k,I,M) : M \in H(J)\}|.$$

Let $v(k,I,J)$ be

$$|\{A'(k,\max I,\prod M) \cap I : M \in H(J)\}| + |\{A(k,\max I,\prod M) \cap I : M \in H(J)\}|$$

Recall that $Q(k,r,J) = \{\mathrm{Mult}((k-r,k+r),K) : K \in H(J)\}$. Then with reference to (10), we may substitute $v(k,I,J)$ for $|Q(k,r,J)|$ and likewise $u(k,I,J)$ for $|\{V(k,r,M) : M \in H(J)\}|$, giving

$$v(k,I,J) + u(k,I,J) \leq 4|H(J)|. \tag{16}$$

For any r, we have

$$0 \leq |\{p \in J : |\mathrm{Mult}([x,y],\{p\})| < |\mathrm{Mult}([x+r,y+r],\{p\})|\}| \leq |J|.$$

Then, since the right-hand side of (16) is twice the value of the right-hand side of (10), it follows by Theorem 2 that, for any k,i,

$$\begin{aligned} &||\bigcup L(k,\max I,J) \cap I| - |\bigcup L(i,\max I,J) \cap I|| \\ &\qquad \leq 8|H(J)| + 2 - 2|J|. \end{aligned}$$

Then since $n/p_n \to 0$, we have $(8|H(P(n))| + 2)/p_n^2 \to 0$, so coupled with (15) and Theorem 3, it follows that for any positive integer n there is a prime $p > n$ such that $p + 2$ is also prime. $\square$

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