The critical line and the roots of the Riemann Zeta-function

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Starting with the functional equation of the Riemann ζ-function, it is shown, that all of its components may be written as polynoms. The roots of these polynoms in the central form are on the imaginary axes and on the real axes. If the roots on the imaginary axes are shifted to the critical line, than the resulting form is identical with the functional equation: Thus, the roots of all components of the components of the functional equation have roots on the critical line and/or on the real axes.

Key words: Riemann Zeta- Function

1. The functional equation and infinite product series

The critical line of the the Riemann $\zeta(\sigma)$-function is defined as follows:

$$s = \sigma + i\cdot\tau = \frac{1}{2} + i\cdot\tau \;;\; \sigma = \frac{1}{2} \;;\; \sigma = 1 - \sigma \qquad (1.1)$$

The functional equation of the Riemann $\zeta(s)$-function /2/ results from the extension of the validity of the function by algebraic continuation. It may be written as follows:

$$\xi(s) = \Gamma\left(\frac{s}{2}\right)\cdot\pi^{-\frac{s}{2}}\cdot\zeta(s) = \Gamma\left(\frac{1-s}{2}\right)\cdot\pi^{-\frac{1-s}{2}}\cdot\zeta(1-s) = \xi(1-s)$$

$$\frac{\xi(s)}{\xi(1-s)} = \frac{\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{1-s}{2}\right)}\cdot\frac{\pi^{-\frac{s}{2}}}{\pi^{-\frac{1-s}{2}}}\cdot\frac{\zeta(s)}{\zeta(1-s)} = 1 \qquad (1.2)$$

This equation states, that ($\xi(s)$) is central symmetrical over ($\sigma = 1/2$):

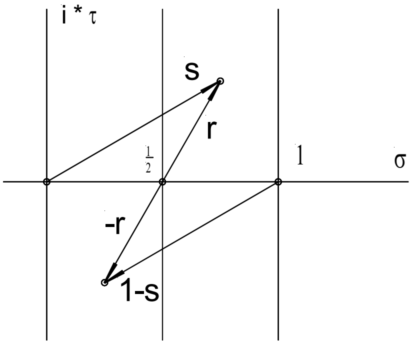

Figure 1.1: The symmetry conditions of the ξ(s)-function

It is shown in Annex 2, that all components of the above functional equation of the Riemann Zeta function may be written as polynoms with all roots on the real and on the imaginary axes. Further, it is shown, that any polynoms having roots on the imaginary and on the real axes have - after the coordinate transformation shifting all the roots from the imaginary axes to the critical line - the resulting form is identical with the functional equation.
First it is shown in Annex 2, that in case a polynom has roots on the positive imaginary axes and its conjugate complex pair has roots on the negative imaginary axes, compose with lemma A2.1 polynom quotient equations of the form:

$$\frac{\prod_{j=1}^{\infty}\left[-1+\frac{r}{\cdot i\, a_{(j)}}\right]}{\prod_{j=1}^{\infty}\left[-1+\frac{r}{-i\cdot a_{(j)}}\right]} = -1 \quad \text{or} \quad = 1 \qquad (1.3)$$

Than it is shown, that after shifting all roots of the polynoms in the polynom quotient equations to the critical line ($r_{tr_{(j)}} = \frac{1}{2} + i \cdot a_{(j)}$) respectively ($r_{tr_{(j)}} = \frac{1}{2} - i \cdot a_{(j)}$) with lemma A2.4 these equations have the form:

$$\frac{\prod_{j=1}^{\infty}\left[1-\frac{s}{r_{tr_{(j)}}}\right]}{\prod_{j=1}^{\infty}\left[1+\frac{1-s}{r_{tr_{(j)}}}\right]} = \frac{\prod_{j=1}^{\infty}\left[1-\frac{s}{\frac{1}{2}+i \cdot a_{(j)}}\right]}{\prod_{j=1}^{\infty}\left[1+\frac{1-s}{\frac{1}{2}-i \cdot a_{(j)}}\right]} = 1 \quad \text{or} \quad = -1 \tag{1.4}$$

Thus, if the function ($\xi(s)$) may be written as a polynom, with all roots on the imaginary and/or on the real axes, and the shift of all roots parallel to the real axes by ($\sigma = 1/2$),to the critical line, results the functional equation, than the functional equation defines all roots on the critical line and/or on the real axes, but nowhere else.

With (A2.4.7) from annex 2 the shifting of all roots of the gamma function to the critical line results for the first component in (1.2) as following form :

$$\frac{\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{1-s}{2}\right)} = \lim_{q \to \infty}\left[\prod_{k=0}^{n}\frac{1-\frac{s}{\frac{1}{2}+i \cdot \frac{2 \cdot b(k,q)}{\ln(\sqrt{q})}}}{1-\frac{1-s}{\frac{1}{2}-i \cdot \frac{2 \cdot b(k,q)}{\ln(\sqrt{q})}}} \cdot \prod_{j=0}^{\sqrt{n}}\left[\frac{1-\frac{1-s}{\frac{1}{2}-i \cdot (2 \cdot j+1)}}{1-\frac{s}{\frac{1}{2}+i \cdot (2 \cdot j+1)}}\right]\right] \tag{1.5}$$

with the constants ($b(k,q)$) being with (A2.3.1)) positive real numbers:

$$b(k,q) = \frac{\left(2 \cdot k+\sqrt{q}\right)^2}{2 \cdot \sqrt{q}} \tag{1.6}$$

With (A2.3.18) from annex 2 the roots of the exponential function - in the central form being on the imaginary axes - shifted to the critical line results for the second component in (1.2) the following form:

$$\frac{\pi^{-\frac{s}{2}}}{\pi^{-\frac{1-s}{2}}} = \frac{e^{-\frac{s \cdot \ln(\pi)}{2}}}{e^{-\frac{(1-s) \cdot \ln(\pi)}{2}}} = \lim_{q \to \infty}\prod_{k=0}^{q}\frac{1-\frac{s}{\frac{1}{2}+i \cdot \frac{2 \cdot b(k,q)}{\ln(\pi)}}}{1+\frac{1-s}{\frac{1}{2}-i \cdot \frac{2 \cdot b(k,q)}{\ln(\pi)}}} \tag{1.7}$$

With (A2.5.4) from annex 2 the third component in (1.2) - the Riemann Zeta function ($\zeta(s)$), in the central form having all roots on the imaginary and on the real axes - have, after the shifting of the roots from the imaginary axes to the critical line, the form of the following polynom coefficient equation:

$$\frac{\zeta(s)}{\zeta(1-s)} = \frac{\prod_{n=1}^{\infty}\lim_{q \to \infty}\prod_{k=0}^{q}\left[1-\frac{s}{\frac{1}{2}+i \cdot \frac{b(k,q)}{\ln\left[P_{(n)}\right]}}\right]}{\prod_{n=1}^{\infty}\lim_{q \to \infty}\prod_{k=0}^{q}\left[1-\frac{(1-s)}{\frac{1}{2}-i \cdot \frac{b(k,q)}{\ln\left[P_{(n)}\right]}}\right]} \cdot \frac{\prod_{n=1}^{\infty}\left[\prod_{j=1}^{\infty}\left[1-\left(\frac{1-s}{\frac{1}{2}-i \cdot \frac{j \cdot \pi}{\ln(P_n)}}\right)^2\right]\right]}{\prod_{n=1}^{\infty}\left[\prod_{j=1}^{\infty}\left[1-\left(\frac{s}{\frac{1}{2}+i \cdot \frac{j \cdot \pi}{\ln(P_n)}}\right)^2\right]\right]} = -1 \quad \text{or} \quad = 1 \tag{1.8}$$

With (1.5), (1.7) and with (1.8) the functional equation of the ζ-function defines roots exclusively on the critical line and on the real axes.

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Annexes

Annex 1: The binomial coefficients and the normal distribution

Annex 2: Splitting of polynoms

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Annex 1: The binomial coefficients and the normal distribution

The present annex gives the proof, that the binomial coefficients normed with their maximum value have the normal distribution as their limit:

$$\lim_{n\to\infty}\frac{B(j,n)}{B_{max}(n)} = \lim_{n\to\infty}\frac{\left(\frac{n}{2}!\right)^2}{j!\cdot n-j!} = e^{-\frac{(2\cdot j-n)^2}{2\cdot n}} = \prod_{k=0}^{n^2}\left[1-\left(\frac{2\cdot j-n}{2\cdot k+n}\right)^2\right] = e^{-x} = \prod_{k=0}^{n^2}\left[1-\frac{2\cdot n\cdot x}{(2\cdot k+n)^2}\right]$$

$$x = \frac{(2\cdot j-n)^2}{2\cdot n} \qquad \text{(A1.0.1)}$$

Using this infinite product representation for the exponential function similar representations for the trigonometric functions sine and cosine, for the gamma function and for the the Rieman ($\zeta(s)$) function are given. As an interesting byproduct infinite product series for (π) are given as well.

A1.1. The normed binomial coefficients

The binomial coefficients are for any positive integer ($n = 1, 2..\infty$) defined as:

$$B(j,n) = \frac{n!}{j!\cdot(n-j)!} = \frac{\Gamma(n+1)}{\Gamma(j+1)\cdot\Gamma(n-j+1)} \qquad j = 1, 2..n \qquad \text{(A1.1.1)}$$

The maximum is reached at ($j = n/2$):

$$B_{max}(n) = \frac{n!}{\left(\frac{n}{2}!\right)^2} = \frac{\Gamma(n+1)}{\Gamma\left(\frac{n}{2}+1\right)^2} = \prod_{j=1}^{\frac{n}{2}}\frac{j+\frac{n}{2}}{j} = \prod_{j=1}^{\frac{n}{2}}\left(\frac{n}{2\cdot j}+1\right) \qquad \text{(A1.1.2)}$$

Normed with this maximum the binomial coefficients may be written as:

$$B_{norm}(j,n) = \frac{B(j,n)}{B_{max}(n)} = \frac{\left(\frac{n}{2}!\right)^2}{j!\cdot n-j!} = \frac{\Gamma\left(\frac{n}{2}+1\right)^2}{\Gamma(j+1)\cdot\Gamma(n-j+1)} \qquad \text{(A1.1.3)}$$

The normal distribution centered at ($j = n/2$) and having ($\sigma = \frac{\sqrt{n}}{2}$) as standard deviation is:

$$F(j,n) = \frac{1}{\sqrt{2\cdot\pi}\cdot\frac{\sqrt{n}}{2}}\cdot e^{-\frac{1}{2}\cdot\left[\frac{j-\left(\frac{n}{2}\right)}{\frac{\sqrt{n}}{2}}\right]^2} \qquad \text{(A1.1.4)}$$

Multiplied by the denominator gives the function, which best approaches the binomial coefficients as shown in the figure below. For simplicity this resulting function will be called in the following the normal distribution:

$$E_{norm}(j,n) = F(j,n)\cdot\left(\sqrt{2\cdot\pi}\cdot\frac{\sqrt{n}}{2}\right) \qquad \text{(A1.1.5)}$$

Using the following transformation of the abscise:

$$x(j,n) := \left(j-\frac{n}{2}\right)\cdot\sqrt{\frac{2}{n}} \qquad j = x\cdot\sqrt{\frac{n}{2}}+\frac{n}{2} \qquad \text{(A1.1.6)}$$

both functions, the normed binomial coefficients and the normal distribution will be centered around the origin, as shown in the figure below for the range ($j := 1..100$) and for ($n := 100$):

$$B_{norm\_c}(j,n) := \frac{\Gamma\left(\frac{n}{2}+1\right)}{\Gamma\left(\frac{n}{2}+1+x(j,n)\cdot\sqrt{\frac{n}{2}}\right)}\cdot\frac{\Gamma\left(\frac{n}{2}+1\right)}{\Gamma\left(\frac{n}{2}+1-x(j,n)\cdot\sqrt{\frac{n}{2}}\right)} \qquad \text{(A1.1.7)}$$

$$E_{norm\_c}(j,n) := e^{-x(j,n)^2}$$

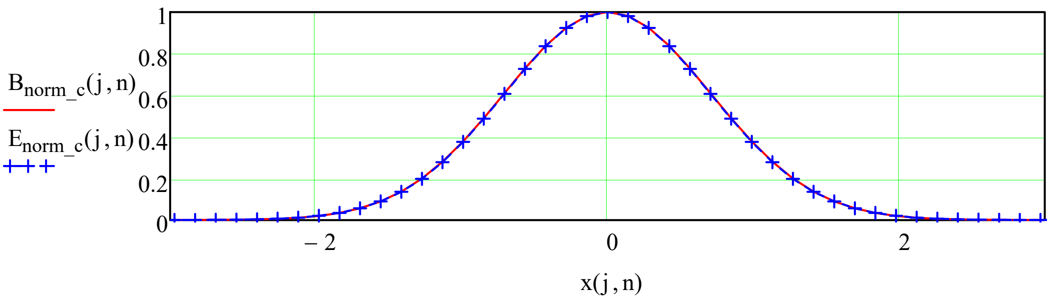


Figure A1.1.1: Comparison of the normal distribution with the normed binomial coefficients

A1.2. The difference between normed binomial coefficients and the normal distribution

Regarding the last figure it seems to be obvious that the two functions are identical. In order to prove this the difference at an arbitrary value will be evaluated. As the arbitrary value the standard deviation is chosen, because at about this value the difference reaches its maximum.

With (A1.1.6) the transformed standard deviation and the corresponding value of the abscise are:

$$\sigma = \frac{\sqrt{n}}{2}\cdot\sqrt{\frac{2}{n}} = \frac{1}{\sqrt{2}} \qquad j_\sigma(n) = \sigma\cdot\sqrt{\frac{n}{2}} + \frac{n}{2} = \frac{\sqrt{n}}{2} + \frac{n}{2} \qquad \text{(A1.2.1)}$$

The values of the normal distribution and of the normed binomial coefficients at this point are:

$$E_{norm\_\sigma} = e^{-\sigma^2} = \frac{1}{\sqrt{e}} \qquad B_{norm\_\sigma}(n) := \frac{\Gamma\left(\frac{n}{2}+1\right)}{\Gamma\left(\frac{n}{2}+1+\frac{\sqrt{n}}{2}\right)}\cdot\frac{\Gamma\left(\frac{n}{2}+1\right)}{\Gamma\left(\frac{n}{2}+1-\frac{\sqrt{n}}{2}\right)} \qquad \text{(A1.2.2)}$$

The difference between the two functions (A1.1.3) and (A1.1.5) has its maximum around the standard deviation, as shown in the figure below:

$$Error(j,n) := B_{norm\_c}(j,n) - E_{norm\_c}(j,n) \qquad \text{(A1.2.3)}$$

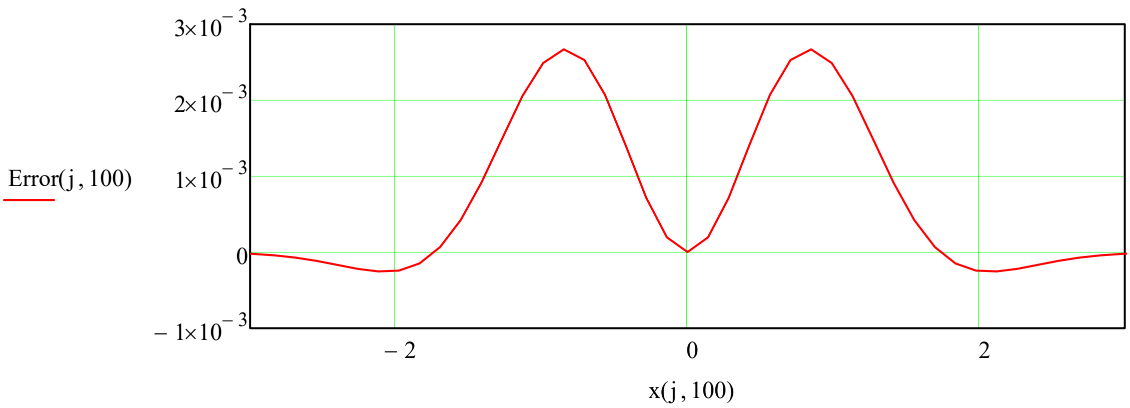


Figure A1.2.1: Difference between normed binomial coefficients and the normal distribution

The difference between the normed binomial coefficients and the normal distribution at the standard deviation (A1.2.1) is with (A1.2.2) a function of the number of the binomial coefficients and it is decreasing for rising number of binomial coefficients as shown in the figure below for the range ($n := 10, 12 .. 100$):

$$\Delta_{B\_E}(n) := B_{norm\_\sigma}(n) - \frac{1}{\sqrt{e}} \qquad \text{(A1.2.4)}$$

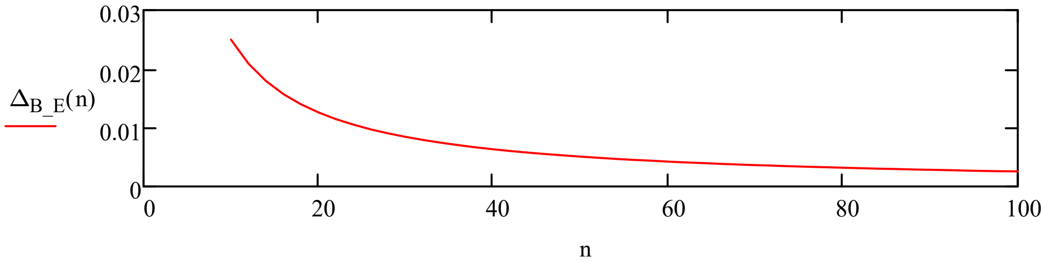


Figure A1.2.2: Evolution of the maximum of the difference between normed binomial coefficients and the normal distribution as function of the number of coefficients considered

Using the variable substitution:

$$p = \frac{\sqrt{n}}{2} \qquad n = 4 \cdot p^2 \qquad \text{(A1.2.5)}$$

the value of the normed binomial coefficients (A1.2.2) at the standard deviation will be:

$$B_{norm\_\sigma}(p) = \frac{\Gamma\left(2 \cdot p^2 + 1\right)}{\Gamma\left(2 \cdot p^2 + p + 1\right)} \cdot \frac{\Gamma\left(2 \cdot p^2 + 1\right)}{\Gamma\left(2 \cdot p^2 - p + 1\right)} \qquad \text{(A1.2.6)}$$

Since all arguments of the Gamma function here are integer values, the function may be written as follows:

$$B_{norm\_\sigma}(p) = \frac{\prod_{j=1}^{2 \cdot p^2} j}{\prod_{j=1}^{2 \cdot p^2 + p} j} \cdot \frac{\prod_{j=1}^{2 \cdot p^2} j}{\prod_{j=1}^{2 \cdot p^2 - p} j} = \frac{1}{\prod_{j=2 \cdot p^2+1}^{2 \cdot p^2 + p} j} \cdot \frac{\prod_{j=2 \cdot p^2 - p + 1}^{2 \cdot p^2} j}{1} \qquad \text{(A1.2.7)}$$

Both components have exactly (p) multiplicands, therefore they may be shortened as follows:

$$B_{norm\_\sigma}(p) = \prod_{j=1}^{p} \frac{2 \cdot p^2 - j + 1}{2 \cdot p^2 + j} \qquad \text{(A1.2.8)}$$

The terms of this product for any ($j = 1 .. p$) are smaller then unity:

$$\frac{2 \cdot p^2 - p + 1}{2 \cdot p^2 + p} < 1 \qquad \text{(A1.2.9)}$$

Therefore the infinite product (A1.2.8) is monotonously decreasing, as shown in the figure above.

On the other hand each of the components except the last one will be rendered smaller, if (j) is replaced by (p). Therefore the infinite product series (A1.2.8) has a lower bound as:

$$\prod_{j=1}^{p} \frac{2 \cdot p^2 - j + 1}{2 \cdot p^2 + j} > \prod_{j=1}^{p} \frac{2 \cdot p^2 - p}{2 \cdot p^2 + p} = \frac{1}{\prod_{j=1}^{p} \frac{2 \cdot p + 1}{2 \cdot p - 1}} = \frac{1}{\prod_{j=1}^{p} \frac{2 \cdot p - 1 + 2}{2 \cdot p - 1}} = \frac{1}{\left(1 + \frac{1}{p - \frac{1}{2}}\right)^{p - \frac{1}{2}} \cdot \sqrt{1 + \frac{1}{p - \frac{1}{2}}}}$$

or:

$$\lim_{p \to \infty} \frac{1}{\left(1+\frac{1}{p-\frac{1}{2}}\right)^{p-\frac{1}{2}} \cdot \sqrt{1+\frac{1}{p-\frac{1}{2}}}} = \frac{1}{e \cdot 1} \qquad \text{(A1.2.10)}$$

Since the infinite series (A1.2.8) is monotonously decreasing and has a lower bound, it converges to a limit value. The limit value may be evaluated, if the central elements of the numerator and of the denominator of each of the (p) components are considered:

$$2 \cdot p^2 + \frac{p}{2} \qquad \text{and} \qquad 2 \cdot p^2 - \frac{p}{2} \qquad \text{(A1.2.11)}$$

All other elements of the products in the numerator and in the denominator may be written as difference to the central elements:

$$B_{norm\_\sigma}(p) = \prod_{j=1}^{\frac{p}{2}} \frac{\left[\left(2 \cdot p^2 - \frac{p}{2}\right) + j\right] \cdot \left[\left(2 \cdot p^2 - \frac{p}{2}\right) - j\right]}{\left[\left(2 \cdot p^2 + \frac{p}{2}\right) + j\right] \cdot \left[\left(2 \cdot p^2 + \frac{p}{2}\right) - j\right]} = \prod_{j=1}^{\frac{p}{2}} \frac{\left(2 \cdot p^2 - \frac{p}{2}\right)^2 - j^2}{\left(2 \cdot p^2 + \frac{p}{2}\right)^2 - j^2} \qquad \text{(A1.2.12)}$$

Since the number of binomial coefficients is rising without limit, certainly the following relation holds:

$$\left(2 \cdot p^2 + \frac{p}{2}\right)^2 >> \left(\frac{p}{2}\right)^2 > j^2 \qquad \text{resp.} \qquad \left(2 \cdot p^2 - \frac{p}{2}\right)^2 >> \left(\frac{p}{2}\right)^2 > j^2 \qquad \text{(A1.2.13)}$$

Therefore the products may be written as powers of the central elements. After some elementary operations we obtain:

$$\lim_{p \to \infty} B_{norm\_\sigma}(p) = \lim_{p \to \infty} \prod_{j=1}^{\frac{p}{2}} \frac{\left(2 \cdot p^2 - \frac{p}{2}\right)^2 - j^2}{\left(2 \cdot p^2 + \frac{p}{2}\right)^2 - j^2} = \lim_{p \to \infty} \frac{\left(2 \cdot p^2 - \frac{p}{2}\right)^p}{\left(2 \cdot p^2 + \frac{p}{2}\right)^p} =$$

$$= \lim_{p \to \infty} \frac{1}{\left[\frac{\left(2 \cdot p^2 - \frac{p}{2}\right) + p}{2 \cdot p^2 - \frac{p}{2}}\right]^p} = \lim_{p \to \infty} \frac{1}{\left(1+\frac{1}{2 \cdot p - \frac{1}{2}}\right)^p} \qquad \text{(A1.2.14)}$$

Using again the definition of (e) for the number of elements rising without limit the limit of (A1.2.14) results exactly the value of the normal distribution (A.2.2.4):

$$\lim_{p \to \infty} B_{norm\_\sigma}(p) = \lim_{p \to \infty} \frac{1}{\sqrt{\left(1+\frac{1}{2 \cdot p - \frac{1}{2}}\right)^{2 \cdot p - \frac{1}{2}} \cdot \left(1+\frac{1}{2 \cdot p - \frac{1}{2}}\right)^{\frac{1}{2}}}} = \frac{1}{\sqrt{e \cdot 1}} \qquad \text{(A1.2.15)}$$

Since the difference between the two functions, the normal distribution and the binomial coefficients disappears at the arbitrary chosen abscise value of the standard deviation for (p) growing without limit, it disappears for all values. For any other value a similar procedure may be repeated resulting an other power value of (e). Therefore it may be stated, that for the number of binomial coefficients rising without limit the binomial coefficients are approaching without limit the function of the normal distribution.

A1.3. The product representation of the exponential function

As proven in the last section the normal distribution and the limit value of the normed binomial coefficients (A1.1.7) are equals:

$$\lim_{n \to \infty} e^{-x(j,n)^2} = \lim_{n \to \infty} \left(\frac{\Gamma\left(\frac{n}{2}+1\right)}{\Gamma\left(\frac{n}{2}+1+x(j,n)\cdot\sqrt{\frac{n}{2}}\right)} \cdot \frac{\Gamma\left(\frac{n}{2}+1\right)}{\Gamma\left(\frac{n}{2}+1-x(j,n)\cdot\sqrt{\frac{n}{2}}\right)} \right) \quad \text{(A1.3.1)}$$

Using the following coordinate transformation: (A1.3.2)

$$z(j,n) := x(j,n)^2$$

and neglecting unity versus $(n/2)$, this may be written as:

$$\lim_{n \to \infty} e^{-z(j,n)} = \lim_{n \to \infty} \left(\frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n}{2}+\frac{\sqrt{2\cdot n\cdot z(j,n)}}{2}\right)} \cdot \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n}{2}-\frac{\sqrt{2\cdot n\cdot z(j,n)}}{2}\right)} \right) \quad \text{(A1.3.3)}$$

Using the definition of the gamma function from Gauss:

$$\Gamma(x) = \lim_{n \to \infty} \left[n^x \cdot \frac{(n-1)!}{x\cdot(x+1)\cdot(x+2)..(x+n-1)} \right] = \lim_{n \to \infty} \left(\frac{n^x}{n} \cdot \prod_{k=0}^{n-1} \frac{k+1}{k+x} \right) \quad \text{(A1.3.4)}$$

the gamma functions in equation (A1.3.3) may be written as infinite products:

$$\frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n+\sqrt{2\cdot z\cdot n}}{2}\right)} = \lim_{m \to \infty} \frac{\frac{m^{\frac{n}{2}}}{m} \cdot \prod_{k=0}^{m-1} \frac{k+1}{k+\frac{n}{2}}}{\frac{m^{\frac{n}{2}+\frac{\sqrt{2\cdot z\cdot n}}{2}}}{m} \cdot \prod_{k=0}^{m-1} \frac{k+1}{k+\frac{n}{2}+\frac{\sqrt{2\cdot z\cdot n}}{2}}} = \lim_{m \to \infty} \left[m^{-\frac{\sqrt{2\cdot z\cdot n}}{2}} \cdot \prod_{k=0}^{m-1} \left(1+\frac{\sqrt{2\cdot z\cdot n}}{2\cdot k+n}\right) \right]$$

$$\frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-\sqrt{2\cdot z\cdot n}}{2}\right)} = \lim_{m \to \infty} \left[m^{\frac{\sqrt{2\cdot z\cdot n}}{2}} \cdot \prod_{k=0}^{m-1} \left(1-\frac{\sqrt{2\cdot z\cdot n}}{2\cdot k+n}\right) \right] \quad \text{(A1.3.5)}$$

Inserted into (A1.3.3) gives: (A1.3.6)

$$e^{-z} = \lim_{n \to \infty} \lim_{m \to \infty} \left[\prod_{k=0}^{m} \left(1+\frac{\sqrt{2\cdot z\cdot n}}{2\cdot k+n}\right) \cdot \prod_{k=0}^{m} \left(1-\frac{\sqrt{2\cdot z\cdot n}}{2\cdot k+n}\right) \right] = \lim_{n \to \infty} \lim_{m \to \infty} \prod_{k=0}^{m} \left[1-\frac{2\cdot z\cdot n}{(2\cdot k+n)^2} \right]$$

Since ($k = 0$) gives the smallest multiplicand in the above product, the lower limit of the product is:

$$\text{Limit}_{low} = \lim_{n \to \infty} \lim_{m \to \infty} \prod_{k=0}^{m} \left(1-\frac{2\cdot z\cdot n}{n^2}\right) = \lim_{n \to \infty} \lim_{m \to \infty} \left[\left(1-\frac{1}{\frac{n^2}{2\cdot z\cdot n}}\right)^{\frac{n^2}{2\cdot z\cdot n}} \right]^{\frac{2\cdot z\cdot n\cdot m}{n^2}} \quad \text{(A1.3.7)}$$

The limit of the above function within the bracelets and the limit of the power are for $(n = m)$:

$$\lim_{n \to \infty} \left(1 - \frac{1}{\frac{n^2}{2 \cdot z \cdot n}}\right)^{\frac{n^2}{2 \cdot z \cdot n}} = \frac{1}{e} \qquad \lim_{n \to \infty} \lim_{m \to \infty} \frac{2 \cdot z \cdot n \cdot m}{n^2} = 2 \cdot z \tag{A1.3.8}$$

Therefore the lower limit is:

$$\text{Limit}_{low} = \frac{1}{e^{2 \cdot z}} \tag{A1.3.9}$$

The value of the product (A1.3.6) is monotonously decreasing, since each of the multiplicands is smaller then unity and smaller then the previous one:

$$\prod_{k=0}^{m} \left[1 - \frac{2 \cdot z \cdot n}{(2 \cdot k + n)^2}\right] > \prod_{k=0}^{m+1} \left[1 - \frac{2 \cdot z \cdot n}{(2 \cdot k + n)^2}\right] = \prod_{k=0}^{m} \left[1 - \frac{2 \cdot z \cdot n}{(2 \cdot k + n)^2}\right] \cdot \left[1 - \frac{2 \cdot z \cdot n}{(2 \cdot m + 2 + n)^2}\right] \tag{A1.3.10}$$

Since the product on the right hand side of (A1.3.6) is bound from below and monotonously decreasing, a limit value exists for $(n = m)$.

The limit for $(n \cdot m)$ in (A1.3.6) is certainly equal to the limit for (m):

$$\lim_{n \to \infty} \lim_{m \to \infty} \prod_{k=0}^{n \cdot m} \left[1 - \frac{2 \cdot z \cdot n}{(2 \cdot k + n)^2}\right] = e^{-z} \tag{A1.3.11}$$

But the limit for $(\ n\,(m = 1,\ 2\ \ldots \infty)\ )$ remains unchanged for any (m). Therefore it is the same for (n^2) as well as shown in the figure below:

$$\lim_{n \to \infty} \lim_{m \to \infty} \prod_{k=0}^{n \cdot m} \left[1 - \frac{2 \cdot z \cdot n}{(2 \cdot k + n)^2}\right] = \lim_{n \to \infty} \prod_{k=0}^{n^2} \left[1 - \frac{2 \cdot z \cdot n}{(2 \cdot k + n)^2}\right] = e^{-z} \tag{A1.3.12}$$

For the graphical representation in the range $(j := 10..1000)$ and with $(n := 10^5)$ the following function is defined for $(z = 1)$. Since the evaluation is quite time consuming, the results are written to a file. They are read in case of the evaluation of the present paper:

$$F(j, n) := \prod_{k=1}^{j} \left[1 - \frac{2 \cdot n}{(2 \cdot k + n)^2}\right] \qquad F_{exp\_z_j} := F(j \cdot n, n) \tag{A1.3.13}$$

$$\text{WRITEPRN}(\text{"EXP\_MINUS\_ZED.PRN"}) := F_{exp\_z} \qquad F_{exp\_z} := \text{READPRN}(\text{"EXP\_MINUS\_ZED.PRN"})$$

$$last\_ := \text{length}(F_{exp\_z}) - 1$$

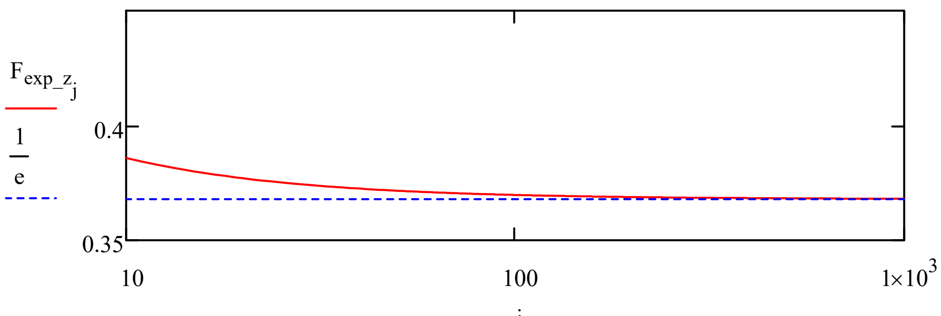


Figure A1.3.1: Convergence of the product representation of the exponential function:

The gamma functions on the right hand side of (A1.3.3) may be regarded as the split components of the exponential function (see annex 2 for variable splitting). Consequently the following functions as split components are defined for $(z > 0,\ s = \sqrt{z})$ and real:

$$\theta_{n\_n\_e}(s) := \lim_{n \to \infty} \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n + i \cdot s \cdot \sqrt{2 \cdot n}}{2}\right)} \qquad \theta_{n\_p\_e}(s) := \lim_{n \to \infty} \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n - i \cdot s \cdot \sqrt{2 \cdot n}}{2}\right)}$$

$$\theta_{p\_n\_e}(s) := \lim_{n \to \infty} \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n + s \cdot \sqrt{2 \cdot n}}{2}\right)} \qquad \theta_{p\_p\_e}(s) := \lim_{n \to \infty} \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n - s \cdot \sqrt{2 \cdot n}}{2}\right)} \qquad \text{(A1.3.14)}$$

For large but limited (n) this may be written as:

$$\theta_{n\_n\_e\_}(s,n) := \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n + i \cdot s \cdot \sqrt{2 \cdot n}}{2}\right)} \qquad \theta_{n\_p\_e\_}(s,n) := \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n - i \cdot s \cdot \sqrt{2 \cdot n}}{2}\right)}$$

$$\theta_{p\_n\_e\_}(s,n) := \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n + s \cdot \sqrt{2 \cdot n}}{2}\right)} \qquad \theta_{p\_p\_e\_}(s,n) := \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n - s \cdot \sqrt{2 \cdot n}}{2}\right)} \qquad \text{(A1.3.15)}$$

Using the product representation analog to (A1.3.6) these functions will be:

$$\theta_{n\_n\_e}(z,n) = n^{-i \cdot \frac{\sqrt{2 \cdot z \cdot n}}{2}} \cdot \prod_{k=0}^{n^2} \left(1 + i \cdot \frac{\sqrt{2 \cdot z \cdot n}}{2 \cdot k + n}\right) \qquad \theta_{n\_p\_e}(z,n) = n^{i \cdot \frac{\sqrt{2 \cdot z \cdot n}}{2}} \cdot \prod_{k=0}^{n^2} \left(1 - i \cdot \frac{\sqrt{2 \cdot z \cdot n}}{2 \cdot k + n}\right)$$

$$\text{(A1.3.16)}$$

$$\theta_{p\_n\_e}(z,n) = n^{-\frac{\sqrt{2 \cdot z \cdot n}}{2}} \cdot \prod_{k=0}^{n^2} \left(1 + \frac{\sqrt{2 \cdot z \cdot n}}{2 \cdot k + n}\right) \qquad \theta_{p\_p\_e}(z,n) = n^{\frac{\sqrt{2 \cdot z \cdot n}}{2}} \cdot \prod_{k=0}^{n^2} \left(1 - \frac{\sqrt{2 \cdot z \cdot n}}{2 \cdot k + n}\right)$$

The products on the right side of the two first expressions here are complex functions. Replacing (n) by ($\sqrt{n}$) the product components in the above formula may be defined as new functions:

$$a(k,n) := \frac{(2 \cdot k + \sqrt{n})^2}{2 \cdot \sqrt{n}} \qquad \text{(A1.3.17)}$$

$$\eta_{n\_n\_e\_}(z,n) := \prod_{k=0}^{n} \left(1 + \frac{i \cdot \sqrt{z}}{\sqrt{a(k,n)}}\right) \qquad \eta_{n\_n\_e}(z) = \lim_{n \to \infty} \eta_{n\_n\_e\_}(z,n) = \lim_{n \to \infty} \left(\theta_{n\_n\_e}(z,n) \cdot n^{i \cdot \frac{\sqrt{2 \cdot z \cdot n}}{2}}\right)$$

$$\eta_{n\_p\_e\_}(z,n) := \prod_{k=0}^{n} \left(1 - \frac{i \cdot \sqrt{z}}{\sqrt{a(k,n)}}\right) \qquad \eta_{n\_p\_e}(z) = \lim_{n \to \infty} \eta_{n\_p\_e\_}(z,n) = \lim_{n \to \infty} \left(\theta_{n\_p\_e}(z,n) \cdot n^{-i \cdot \frac{\sqrt{2 \cdot z \cdot n}}{2}}\right)$$

$$\eta_{p\_n\_e\_}(z,n) := \prod_{k=0}^{n} \left(1 + \frac{\sqrt{z}}{\sqrt{a(k,n)}}\right) \qquad \eta_{p\_n\_e}(z) = \lim_{n \to \infty} \eta_{p\_n\_e\_}(z,n) = \lim_{n \to \infty} \left(\theta_{p\_n\_e}(z,n) \cdot n^{\frac{\sqrt{2 \cdot z \cdot n}}{2}}\right)$$

$$\eta_{p\_p\_e\_}(z,n) := \prod_{k=0}^{n} \left(1 - \frac{\sqrt{z}}{\sqrt{a(k,n)}}\right) \qquad \eta_{p\_p\_e}(z) = \lim_{n \to \infty} \eta_{p\_p\_e\_}(z,n) = \lim_{n \to \infty} \left(\theta_{p\_p\_e}(z,n) \cdot n^{-\frac{\sqrt{2 \cdot z \cdot n}}{2}}\right)$$

The exponential function with positive exponents, expressed by the infinite products of the gamma function will be with (A1.3.12) and with (A1.3.17) as follows:

$$e^z = e_p(z) = \lim_{n \to \infty} e_p(z,n) = \lim_{n \to \infty} \left(\eta_{n\_n\_e}(z)\cdot\eta_{n\_p\_e}(z)\right) = \lim_{n \to \infty} \frac{1}{\eta_{p\_n\_e}(z)\cdot\eta_{p\_p\_e}(z)}$$

$$e_{p\_}(z,n) := \prod_{k=0}^{n}\left(1+\frac{z}{a(k,n)}\right) = \eta_{n\_n\_e\_}(z,n)\cdot\eta_{n\_p\_e\_}(z,n) = \frac{1}{\eta_{p\_n\_e\_}(z,n)\cdot\eta_{p\_p\_e\_}(z,n)} \qquad \text{(A1.3.18)}$$

The conjugates of each of these functions are equal to their inverse:

$$e^{-z} = e_p(-z) = \lim_{n \to \infty} e_n(z,n) = \lim_{n \to \infty} \frac{1}{\eta_{n\_n\_e}(z)\cdot\eta_{n\_p\_e}(z)} = \lim_{n \to \infty} \left(\eta_{p\_n\_e}(z)\cdot\eta_{p\_p\_e}(z)\right)$$

$$e_{n\_}(z,n) := \prod_{k=0}^{n}\left(1-\frac{z}{a(k,n)}\right) = \frac{1}{\eta_{n\_n\_e\_}(z,n)\cdot\eta_{n\_p\_e\_}(z,n)} = \eta_{p\_n\_e\_}(z,n)\cdot\eta_{p\_p\_e\_}(z,n) \qquad \text{(A1.3.19)}$$

Writing the complex function in polar form the radius and the angle of the first of these functions will be:

$$r(z,n) := \sqrt{\prod_{k=0}^{n}\left(1+\frac{z}{a(k,n)}\right)} \qquad \phi_{im}(z,n) := \mathrm{Im}\left(\eta_{n\_n\_e\_}(z,n)\right) \qquad \phi_{re}(z,n) := \mathrm{Re}\left(\eta_{n\_n\_e\_}(z,n)\right)$$

$$\phi_1(z,n) := \mathrm{if}\left(\phi_{re}(z,n) > 0, \mathrm{atan}\left(\frac{\phi_{im}(z,n)}{\phi_{re}(z,n)}\right), \pi - \mathrm{atan}\left(\frac{\phi_{im}(z,n)}{-\phi_{re}(z,n)}\right)\right) \qquad \text{(A1.3.20)}$$

$$\phi_2(z,n) := \mathrm{if}\left(\phi_{re}(z,n) < 0, \pi - \mathrm{atan}\left(\frac{\phi_{im}(z,n)}{-\phi_{re}(z,n)}\right), 2\cdot\pi + \mathrm{atan}\left(\frac{\phi_{im}(z,n)}{\phi_{re}(z,n)}\right)\right)$$

$$\phi(z,n) := \mathrm{if}\left(\phi_{re}(z,n) > 0, \mathrm{if}\left(z < 0.4, \phi_1(z,n), \phi_2(z,n)\right), \pi - \mathrm{atan}\left(\frac{\phi_{im}(z,n)}{-\phi_{re}(z,n)}\right)\right)$$

whereas the angle is evaluated using the imaginary and the real parts of the split exponential function for the graphical representation below. Note that the square of the radius above is equal to the exponential function. The radius compared to the exponential function and the angle are shown in the figure below for the ranges ($x := 0, 0.1..2$) resp. ($y := 0, 0.002..1.25$). This figure on the left shows at the same time the quality of the representation of the exponential function as infinite product:

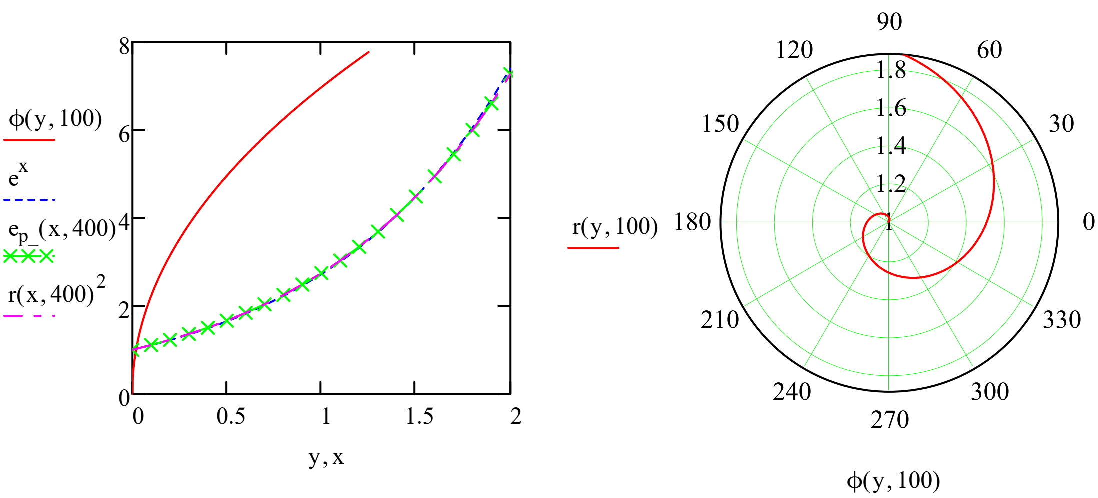


Figure A1.3.2: The exponential function expressed as infinite product and the polar representation of the split exponential function

The same result is obtained if the transformation (A1.1.6) is inserted into (A.2.3.19):

$$e^{-\left[\left(j-\frac{\sqrt{n}}{2}\right)\cdot\sqrt{\frac{2}{\sqrt{n}}}\right]^2} = \prod_{k=0}^{n}\left[1-\frac{\left(2\cdot j-\sqrt{n}\right)^2}{\left(2\cdot k+\sqrt{n}\right)^2}\right] = \prod_{k=0}^{n}\left(1-\frac{z}{a(k,n)}\right) = e^{-\frac{\left(2\cdot j-\sqrt{n}\right)^2}{2\cdot\sqrt{n}}} = e^{-z} \qquad \text{(A1.3.21)}$$

$$e^{\left[\left(j-\frac{\sqrt{n}}{2}\right)\cdot\sqrt{\frac{2}{\sqrt{n}}}\right]^2} = \prod_{k=0}^{n}\left[1+\frac{\left(2\cdot j-\sqrt{n}\right)^2}{\left(2\cdot k+\sqrt{n}\right)^2}\right] = \prod_{k=0}^{n}\left(1+\frac{z}{a(k,n)}\right) = e^{\frac{\left(2\cdot j-\sqrt{n}\right)^2}{2\cdot\sqrt{n}}} = e^{z}$$

Since the first two functions (A1.3.17) are conjugate complex for real (x) values, they are the split components (see annex 2) of the product representation of the exponential function (A1.3.19).

A1.4. The product representation of the trigonometric functions

The exponential function is defined for any complex number as exponent, therefore the infinite products (A1.3.17) may be regarded as functions of the complex variable (z). The hyperbolic functions expressed as infinite products will be than:

$$\cosh_{p\_n}(z,n) := \frac{1}{2}\cdot\left(e_{p\_}(z,n) + e_{n\_}(z,n)\right) \qquad \sinh_{p\_n}(z,n) := \frac{1}{2}\cdot\left(e_{p\_}(z,n) - e_{n\_}(z,n)\right) \qquad \text{(A1.4.1)}$$

Following Euler's relations the hyperbolic functions taken for imaginary values result the trigonometric functions:

$$\cos_{p\_n}(t,n) := \cosh_{p\_n}(i\cdot t,n) \qquad \sin_{p\_n}(t,n) := -i\cdot\sinh_{p\_n}(i\cdot t,n) \qquad \text{(A1.4.2)}$$

These functions approximated as infinite products are compared with the corresponding functions in the figure below for the range ($j := 0..15$, $t_{(j)} := 0.1\cdot\pi\cdot j$). In the infinite products the number of binomial coefficients considered is limited ($n := 1000000$). This because of the limited computational capabilities for very large numbers. Since the evaluation is quite time consuming, the results are written to a file. They are read in case of the evaluation of the present paper:

$F_{cos\_p\_n_j} := \cos_{p\_n}\left(t_j,n\right)$ $\qquad$ $F_{sin\_p\_n_j} := \sin_{p\_n}\left(t_j,n\right)$

WRITEPRN("COS_P_N.PRN") := $F_{cos\_p\_n}$ $\qquad$ WRITEPRN("SIN_P_N.PRN") := $F_{sin\_p\_n}$

$F_{cos\_p\_n}$:= READPRN("COS_P_N.PRN") $\qquad$ $F_{sin\_p\_n}$:= READPRN("SIN_P_N.PRN")

$k := 0 .. \text{length}\left(F_{cos\_p\_n}\right) - 1$

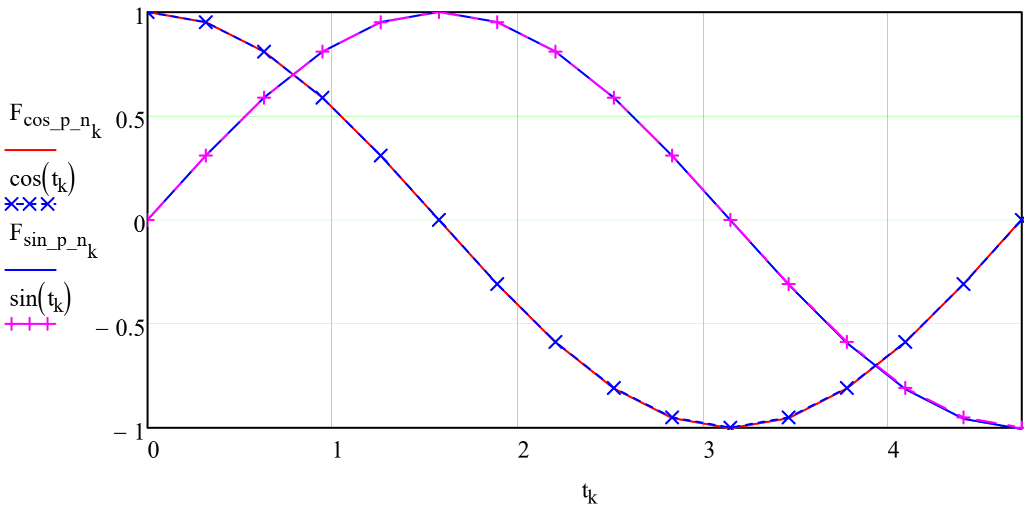


Figure A1.4.1: The trigonometric functions expressed as infinite products

It is well known and obvious from Euler's relations (A1.4.1), that all the roots of the trigonometric functions for the complex variable (z) correspond to the roots of the hyperbolic functions on the imaginary axes resp. at the origin:

$$\cosh(z) = \frac{1}{2}\cdot\left(e^{z} + e^{-z}\right) = 0 \qquad \sinh(z) = \frac{1}{2}\cdot\left(e^{z} - e^{-z}\right) = 0 \tag{A1.4.3}$$

With (1.4), in accordance with (A1.4.9) follows, that writing with (A1.3.19) and (A1.3.20) the exponential functions in infinite product form results the following quotient equations:

$$\frac{e^{z}}{e^{-z}} = \lim_{n\to\infty}\prod_{k=0}^{n}\frac{1+\dfrac{z}{a(k,n)}}{1-\dfrac{z}{a(k,n)}} = \lim_{n\to\infty}\prod_{k=0}^{n}\frac{a(k,n)+z}{a(k,n)-z} = -1 \quad \text{or} \quad = 1 \tag{A1.4.4}$$

In case of (-1) the cosine function is defined as adjoint function of the exponential function on the adjoint, on the real axes, with real roots. In case of (1), the sine function is defined as adjoint function of the exponential function on the adjoint, on the imaginary axes, with all imaginary roots.

With the product definition of the exponential function (A1.3.19) and (A.2.3.20) the relations from Euler may be written as follows:

$$\cos(x) = \lim_{n\to\infty}\left[\left[\prod_{k=0}^{n}\left(1+i\cdot\frac{x}{a(k,n)}\right) + \prod_{k=0}^{n}\left(1-i\cdot\frac{x}{a(k,n)}\right)\right]\cdot\frac{1}{2}\right] = \cosh(i\cdot x)$$

$$\sin(x) = \lim_{n\to\infty}\left[\left[\prod_{k=0}^{n}\left(1+i\cdot\frac{x}{a(k,n)}\right) - \prod_{k=0}^{n}\left(1-i\cdot\frac{x}{a(k,n)}\right)\right]\cdot\frac{-i}{2}\right] = -i\cdot\sinh(i\cdot x) \tag{A1.4.5}$$

Here the sum of two complex functions result a periodic function. From where does the periodicity originate? This will be clear, if the products of the sum are evaluated as infinite series:

$$e^{i\cdot x} = \prod_{k=0}^{n}\left(1+i\cdot\frac{x}{a(k,n)}\right) = 1 + i\cdot h_1\cdot x - h_2\cdot x^2 - i\cdot h_3\cdot x^3 + h_4\cdot x^4 + i\cdot h_5\cdot x^5 - h_6\cdot x^6 \ldots$$

$$e^{-i\cdot x} = \prod_{k=0}^{n}\left(1-i\cdot\frac{x}{a(k,n)}\right) = 1 - i\cdot h_1\cdot x - h_2\cdot x^2 + i\cdot h_3\cdot x^3 + h_4\cdot x^4 - i\cdot h_5\cdot x^5 - h_6\cdot x^6 \ldots \tag{A1.4.6}$$

The sum and the difference of these series result the known series for the trigonometric functions:

$$\cos(x) = \cosh(i\cdot x) = 1 - h_2\cdot x^2 + h_4\cdot x^4 - h_6\cdot x^6 \ldots, \text{ with } h_2 = \frac{1}{2!},\ h_4 = \frac{1}{4!},\ h_6 = \frac{1}{6!} \ldots$$

$$\sin(x) = -i\cdot\sinh(i\cdot x) = h_1\cdot x - h_3\cdot x^3 + h_5\cdot x^5 \ldots, \text{ with } h_1 = \frac{1}{1!},\ h_3 = \frac{1}{3!},\ h_5 = \frac{1}{5!} \ldots \tag{A1.4.7}$$

In these series the value of the last term always overtakes the sum of all previous terms. Since the sign of the last term is varying, the function becomes periodic, the root values converging to odd resp. even multiples of $(\frac{\pi}{2})$. In fact the value $(\frac{\pi}{2})$ is defined by this convergence.

Equations (A1.4.6) may be written for even and for odd powers of (x) separated as follows

$$e^{i\cdot x} = \prod_{k=0}^{n}\left(1+i\cdot\frac{x}{a(2\cdot k,n)}\right) + \prod_{k=0}^{n}\left(1+i\cdot\frac{x}{a(2\cdot k+1,n)}\right) =$$

$$= \sum_{k=0}^{n}\frac{(-1)^{k}\cdot x^{2\cdot k}}{(2\cdot k)!} - i\cdot\sum_{k=0}^{n}\frac{(-1)^{k}\cdot x^{2\cdot k+1}}{(2\cdot k+1)!} = \cos(x) + i\cdot\sin(x) \tag{A1.4.8}$$

$$e^{-i\cdot x} = \prod_{k=0}^{n}\left(1 - i\cdot\frac{x}{a(2\cdot k,n)}\right) + \prod_{k=0}^{n}\left(1 - i\cdot\frac{x}{a(2\cdot k+1,n)}\right) = \tag{A1.4.8}$$

$$= \sum_{k=0}^{n}\frac{(-1)^k\cdot x^{2\cdot k}}{(2\cdot k)!} + i\cdot\sum_{k=0}^{n}\frac{(-1)^k\cdot x^{2\cdot k+1}}{(2\cdot k+1)!} = \cos(x) - i\cdot\sin(x)$$

These give the well known relations:

$$\sinh(x) = \frac{e^x - e^{-x}}{2} = -i\cdot\sin(i\cdot x) \quad ; \quad \cosh(x) = \frac{e^x + e^{-x}}{2} = \cos(i\cdot x) \tag{A1.4.9}$$

A1.5. The product representation of the gamma function

With the polynom product representation of the gamma function from Gauss (A1.3.4) written for ($\frac{1}{2}$), ($\frac{1+x}{2}$) and for ($\frac{1-x}{2}$) and shortened gives the following quotients, both composed of an exponential part and of a periodic part:

$$\frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1+x}{2}\right)} = \lim_{n\to\infty}\left[n^{-\frac{x}{2}}\cdot\prod_{j=0}^{n-1}\left(1 + \frac{x}{2\cdot j+1}\right)\right]; \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1-x}{2}\right)} = \lim_{n\to\infty}\left[n^{\frac{x}{2}}\cdot\prod_{j=0}^{n-1}\left(1 - \frac{x}{2\cdot j+1}\right)\right] \tag{A1.5.1}$$

The product of these above quotients eliminate the exponential components, leaving the periodic components, which give the known relation for the cosines function:

$$\frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1+t}{2}\right)}\cdot\frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1-t}{2}\right)} = \lim_{n\to\infty}\left(n^{-\frac{t}{2}}\cdot\prod_{j=0}^{n-1}\frac{2\cdot j+1+t}{2\cdot j+1}\cdot n^{\frac{t}{2}}\cdot\prod_{j=0}^{n-1}\frac{2\cdot j+1-t}{2\cdot j+1}\right) = \tag{A1.5.2}$$

$$= \lim_{n\to\infty}\prod_{j=0}^{n-1}\frac{(2\cdot j+1+t)\cdot(2\cdot j+1-t)}{(2\cdot j+1)^2} = \prod_{j=1}^{\infty}\left[1 - \left(\frac{t}{2\cdot j-1}\right)^2\right] = \cos\left(t\cdot\frac{\pi}{2}\right)$$

The sines function with the use of the approximation (A1.7.8) for ($\pi/2$) may be written as product of two infinite products, whereas for negative arguments both will be imaginary:

$$\sin\left(t\cdot\frac{\pi}{2}\right) = \cos\left[(t-1)\cdot\frac{\pi}{2}\right] = \prod_{j=1}^{\infty}\left[1 - \left(\frac{t-1}{2\cdot j-1}\right)^2\right] = \prod_{j=1}^{\infty}\frac{2\cdot j-1-(t-1)}{2\cdot j-1}\cdot\prod_{j=0}^{\infty}\frac{2\cdot j+1+(t-1)}{2\cdot j+1} =$$

$$= \prod_{j=1}^{\infty}\frac{2\cdot j-t}{2\cdot j-1}\cdot\prod_{j=0}^{\infty}\frac{2\cdot j+t}{2\cdot j+1} = \prod_{j=1}^{\infty}\frac{(2\cdot j)^2}{(2\cdot j-1)\cdot(2\cdot j+1)}\cdot t\cdot\left[\prod_{j=1}^{\infty}\left[\left(1 - \frac{t}{2\cdot j}\right)\cdot\left(1 + \frac{t}{2\cdot j}\right)\right]\right] =$$

$$= \frac{\pi}{2}\cdot t\cdot\prod_{j=1}^{\infty}\left[1 - \left(\frac{t}{2\cdot j}\right)^2\right] = \sqrt{\frac{\pi}{2}\cdot t}\cdot\prod_{j=1}^{\infty}\left(1 - \frac{t}{2\cdot j}\right)\cdot\sqrt{\frac{\pi}{2}\cdot t}\cdot\prod_{j=1}^{\infty}\left(1 + \frac{t}{2\cdot j}\right) \tag{A1.5.3}$$

With these last two expressions (A1.5.2) and (A1.5.3) the law of Pythagoras yields:

$$\cos(t)^2 + \sin(t)^2 = \prod_{j=0}^{\infty}\left[1 - \left(\frac{t}{2\cdot j+1}\right)^2\cdot\left(\frac{2}{\pi}\right)^2\right]^2 + t^2\cdot\prod_{j=1}^{\infty}\left[1 - \left(\frac{t}{2\cdot j}\right)^2\cdot\left(\frac{2}{\pi}\right)^2\right]^2 = 1 \tag{A1.5.4}$$

The relations for the exponential function (see (A1.5.2) and (A1.5.3) for the representation of the cosines and sines functions as infinite products) gives:

$$e^{t\cdot\frac{\pi}{2}} = \cos\left(i\cdot t\cdot\frac{\pi}{2}\right) - i\cdot\sin\left(i\cdot t\cdot\frac{\pi}{2}\right) = \prod_{j=1}^{\infty}\left[1+\left(\frac{t}{2\cdot j-1}\right)^2\right] + t\cdot\frac{\pi}{2}\cdot\prod_{j=1}^{\infty}\left[1+\left(\frac{t}{2\cdot j}\right)^2\right]$$

$$e^{-t\cdot\frac{\pi}{2}} = \cos\left(i\cdot t\cdot\frac{\pi}{2}\right) + i\cdot\sin\left(i\cdot t\cdot\frac{\pi}{2}\right) = \prod_{j=1}^{\infty}\left[1+\left(\frac{t}{2\cdot j-1}\right)^2\right] - t\cdot\frac{\pi}{2}\cdot\prod_{j=1}^{\infty}\left[1+\left(\frac{t}{2\cdot j}\right)^2\right] \qquad \text{(A1.5.5)}$$

From equation (A1.4.8) follows for even and for odd powers of (x), with (A1.5.2) and with (A1.5.3):

$$\frac{1}{2}\cdot\left(e^{i\cdot x}+e^{-i\cdot x}\right) = \sum_{k=0}^{n}\frac{(-1)^k\cdot x^{2\cdot k}}{(2\cdot k)!} = \cos(x) = \prod_{j=1}^{\infty}\left[1-\left(\frac{x}{2\cdot j-1}\cdot\frac{2}{\pi}\right)^2\right]$$

$$\frac{1}{2}\cdot\left(e^{i\cdot x}-e^{-i\cdot x}\right) = -i\cdot\sum_{k=0}^{n}\frac{(-1)^k\cdot x^{2\cdot k+1}}{(2\cdot k+1)!} = i\cdot\sin(x) = i\cdot x\cdot\prod_{j=1}^{\infty}\left[1-\left(\frac{x}{2\cdot j}\cdot\frac{2}{\pi}\right)^2\right] \qquad \text{(A1.5.6)}$$

From the identity of the constants within equation (A1.4.8) follows for even powers of (x):

$$h_j = \lim_{n\to\infty}\sum_{B(2\cdot j,n)}\prod_{i=1}^{2\cdot j}\frac{1}{a_2\left[1_{(i)},n\right]} = \frac{1}{j!} = \lim_{n\to\infty}\sum_{B(j,n)}\prod_{i=1}^{j}\left[\frac{1}{2\cdot 1_{(i)}-1}\cdot\frac{2}{\pi}\right]^{2\cdot j} \qquad \text{(A1.5.7)}$$

whereas the components of the products $(a_2\left[1_{(i)},n\right])$ respectively $(2\cdot 1_{(i)}-1)$ has to be taken for all possible combinations of $(2\cdot j)$ respectively (j) elements out of all integers $(1,2..n)$ yielding total of $\left(B(2\cdot j,n)=\frac{n!}{(2\cdot j)!\cdot(n-2\cdot j)!}\right)$ respectively $\left(B(j,n)=\frac{n!}{j!\cdot(n-j)!}\right)$ combinations without repetition.

Similarly for odd powers of (x):

$$h_j = \lim_{n\to\infty}\sum_{B[(2\cdot j-1),n]}\prod_{i=1}^{2\cdot j-1}\frac{1}{a_2\left[1_{(i)},n\right]} = \frac{1}{j!} = \lim_{n\to\infty}\left[\sum_{B(j,n)}\prod_{i=1}^{j}\left[\frac{1}{2\cdot 1_{(i)}}\cdot\frac{2}{\pi}\right]^{2\cdot j}\right] \qquad \text{(A1.5.8)}$$

With the split cosines resp. with the split sines function (A2.3.6) defined as:

$$Q_{p\_n\_e}\left(s\cdot\frac{\pi}{2}\right) = \lim_{n\to\infty}\prod_{j=1}^{n}\left(1+\frac{s}{2\cdot j-1}\right) \quad \text{and} \quad Q_{p\_p\_e}\left(s\cdot\frac{\pi}{2}\right) = \lim_{n\to\infty}\prod_{j=1}^{n}\left(1-\frac{s}{2\cdot j-1}\right)$$

$$\text{(A1.5.9)}$$

$$R_{p\_n\_e}\left(s\cdot\frac{\pi}{2}\right) = \lim_{n\to\infty}\left[\sqrt{\frac{\pi}{2}}\cdot s\cdot\prod_{j=1}^{n}\left(1+\frac{s}{2\cdot j}\right)\right] \quad \text{and} \quad R_{p\_p\_e}\left(s\cdot\frac{\pi}{2}\right) = \lim_{n\to\infty}\left[\sqrt{\frac{\pi}{2}}\cdot s\cdot\prod_{j=1}^{n}\left(1-\frac{s}{2\cdot j}\right)\right]$$

results the original cosines, resp. sines functions (A1.5.2) resp. (A1.5.3):

$$Q_{p\_e}\left(s\cdot\frac{\pi}{2}\right) = Q_{p\_n\_e}\left(s\cdot\frac{\pi}{2}\right)\cdot Q_{p\_p\_e}\left(s\cdot\frac{\pi}{2}\right) \quad \text{and} \quad R_{p\_e}\left(s\cdot\frac{\pi}{2}\right) = R_{p\_n\_e}\left(s\cdot\frac{\pi}{2}\right)\cdot R_{p\_p\_e}\left(s\cdot\frac{\pi}{2}\right) \qquad \text{(A1.5.10)}$$

The gamma functions (A1.5.1) may be written with the exponential function (A1.3.20) and with the split cosines function from (A1.5.9) as follows:

$$\frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1+s}{2}\right)} = \lim_{m \to \infty}\left(e^{-\frac{s\cdot\ln(m)}{2}}\cdot Q_{p\_n\_e}\left(s\cdot\frac{\pi}{2}, m\right)\right) =$$

$$= \lim_{m \to \infty}\left[\lim_{n \to \infty}\prod_{k=0}^{n}\left(1-\frac{s\cdot\ln(m)}{2\cdot a(k,n)}\right)\cdot\prod_{j=1}^{m}\left(1+\frac{s}{2\cdot j-1}\right)\right]$$

$$\frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1-s}{2}\right)} = \lim_{m \to \infty}\left(e^{\frac{s\cdot\ln(m)}{2}}\cdot Q_{p\_p\_e}\left(s\cdot\frac{\pi}{2}, m\right)\right) = \qquad \text{(A1.5.11)}$$

$$= \lim_{m \to \infty}\ \lim_{n \to \infty}\left[\prod_{k=0}^{n}\left(1+\frac{s\cdot\ln(m)}{2\cdot a(k,n)}\right)\cdot\prod_{j=1}^{m}\left(1-\frac{s}{2\cdot j-1}\right)\right]$$

With the same reasoning as in (A1.3.12) and replacing ($m=\sqrt{n}$) lets write for the approximation of the gamma function by infinite products, as a polynom:

$$\Gamma_{rel\_appr}(x,n) := \prod_{k=0}^{n}\left(1-\frac{x\cdot\ln(\sqrt{n})}{2\cdot a(k,n)}\right)\cdot\prod_{j=0}^{\sqrt{n}}\frac{2\cdot j+1+x}{2\cdot j+1}$$

$$\frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1+\zeta}{2}\right)} = \lim_{n \to \infty}\left(e^{-\frac{\zeta\cdot\ln(\sqrt{n})}{2}}\cdot\prod_{j=0}^{\sqrt{n}}\frac{2\cdot j+1+\zeta}{2\cdot j+1}\right) = \prod_{k=0}^{n}\left(1-\frac{\zeta\cdot\ln(\sqrt{n})}{2\cdot a(k,n)}\right)\cdot\prod_{j=0}^{\sqrt{n}}\frac{2\cdot j+1+\zeta}{2\cdot j+1}$$

$$\frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1-\zeta}{2}\right)} = \lim_{n \to \infty}\left[\prod_{k=0}^{n}\left[1+\frac{\zeta\cdot\ln(\sqrt{n})\cdot\sqrt{n}}{(2\cdot k+\sqrt{n})^2}\right]\cdot\prod_{j=0}^{\sqrt{n}}\frac{2\cdot j+1-\zeta}{2\cdot j+1}\right] \qquad \text{(A1.5.12)}$$

Because of symmetry reasons the corresponding (Δ) functions for roots at even integers for the split trigonometric function (A2.2.5) - but replacing the split cosine by the shifted sine function, as in (A1.5.3) - may be written as follows:

$$\Delta_{rel\_appr}(x,n) := \prod_{k=0}^{n}\left(1-\frac{x\cdot\ln(\sqrt{n})}{2\cdot a(k,n)}\right)\cdot\prod_{j=0}^{\sqrt{n}}\frac{2\cdot j+x}{2\cdot j+1}\ ;\ \Delta_{rel\_appr\_}(x,n) := e^{-\frac{x\cdot\ln(\sqrt{n})}{2}}\cdot\prod_{j=0}^{\sqrt{n}}\frac{2\cdot j+x}{2\cdot j+1}$$

$$\frac{\Delta\left(\frac{1}{2}\right)}{\Delta\left(\frac{1+\zeta}{2}\right)} = \prod_{k=0}^{n}\left(1-\frac{x\cdot\ln(\sqrt{n})}{2\cdot a(k,n)}\right)\cdot\prod_{j=0}^{\sqrt{n}}\frac{2\cdot j+\zeta}{2\cdot j+1}$$

$$\frac{\Delta\left(\frac{1}{2}\right)}{\Delta\left(\frac{1-\zeta}{2}\right)} = \lim_{n \to \infty}\left[\prod_{k=0}^{n}\left(1+\frac{x\cdot\ln(\sqrt{n})}{2\cdot a(k,n)}\right)\cdot\prod_{j=0}^{\sqrt{n}}\frac{2\cdot j-\zeta}{2\cdot j+1}\right] \qquad \text{(A1.5.13)}$$

In the range ($j := 1..150$) for the variable ($x_{(j)} := \frac{j-50}{10}$) and for the number of quotients considered ($n := 1000000$) the quotient of the gamma function and its approximation as infinite product as well as the approximations of the delta function are compared in the figure below. Since the evaluation is quite time consuming, the results are written to a file. They are read in case of the evaluation of the present paper:

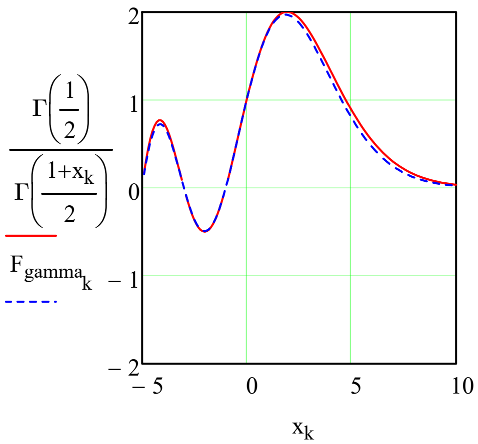
$F_{gamma_j} := \Gamma_{rel\_appr}(x_j, n)$ $\quad F_{delta_j} := \Delta_{rel\_appr}(x_j, n)$ $\quad F_{delta\_j} := \Delta_{rel\_appr\_}(x_j, n)$

WRITEPRN("GAMMA_appr.PRN") := F_{gamma} $\quad$ WRITEPRN("DELTA_appr.PRN") := F_{delta}

WRITEPRN("DELTA_appr_.PRN") := $F_{delta\_}$ $\quad$ F_{delta} := READPRN("DELTA_appr.PRN")

F_{gamma} := READPRN("GAMMA_appr.PRN") $\quad$ $F_{delta\_}$:= READPRN("DELTA_appr_.PRN")

$k := 1 .. \mathrm{length}(F_{gamma}) - 1 \qquad x_k := \frac{k-50}{10}$



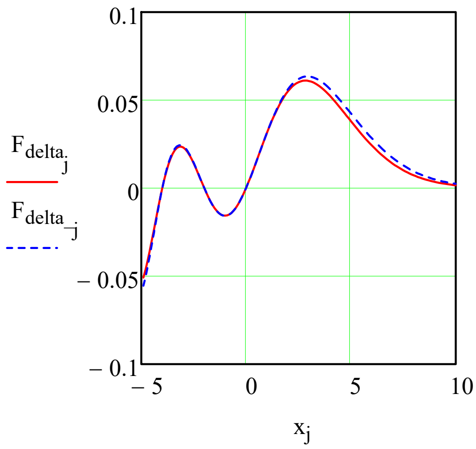


Figure A1.5.1: The gamma and the delta functions expressed as infinite products

The gamma function with the parameters (m) and (p) and using the definition by Gauss (see (A1.3.4) may be written as follows:

$$\frac{\Gamma\left(\frac{p}{2\cdot m}\right)}{\Gamma\left(\frac{p+s}{2\cdot m}\right)} = \lim_{n\to\infty} \frac{\frac{n^{\frac{p}{2\cdot m}}}{n}\cdot\prod_{j=0}^{n}\frac{j+1}{j+\frac{p}{2\cdot m}}}{\frac{n^{\frac{p+s}{2\cdot m}}}{n}\cdot\prod_{j=0}^{n}\frac{j+1}{j+\frac{p+s}{2\cdot m}}} = \lim_{n\to\infty}\left[e^{-\frac{s\cdot\ln(n)}{2\cdot m}}\cdot\prod_{j=0}^{n}\left(1+\frac{s}{2\cdot j\cdot m+p}\right)\right] \qquad \text{(A1.5.14)}$$

As this formula shows, the parameter (p) influences only the periodic component of the gamma function, while the parameter (m) influences both, the exponential and the periodic components. The periodic component is analog to the split cosines function defined in (A1.5.9):

$$Q_{p\_n\_mp}(s, m, p, n) = \lim_{n\to\infty}\prod_{j=0}^{\sqrt{n}}\left(1+\frac{s}{2\cdot j\cdot m+p}\cdot\frac{2}{\pi}\right)$$
$$Q_{p\_p\_mp}(s, m, p, n) = \lim_{n\to\infty}\prod_{j=0}^{\sqrt{n}}\left(1-\frac{s}{2\cdot j\cdot m+p}\cdot\frac{2}{\pi}\right) \qquad \text{(A1.5.15)}$$

For ($p = 1$) and for ($m = 2$) - therefore with ($j = 2\cdot k$) for all even numbers - the periodic components of the gamma function are:

$$Q_{p\_n\_even}(s, n) = Q_{p\_n\_mp}(s, 2, 1, n) = \prod_{j=0}^{\sqrt{n}}\left[1+\frac{s}{2\cdot(2\cdot j)+1}\cdot\frac{2}{\pi}\right]$$
$$Q_{p\_p\_even}(s, n) = Q_{p\_p\_mp}(s, 2, 1, n) = \prod_{j=0}^{\sqrt{n}}\left[1-\frac{s}{2\cdot(2\cdot j)+1}\cdot\frac{2}{\pi}\right] \qquad \text{(A1.5.16)}$$

The corresponding first degree split function (which corresponds for ($m = 1$) to the cosines function) is shown in the figure below for the range ($t := 0, 0.5..20$), whereas the same function evaluated by the gamma functions is shown for comparison as well:

$$Q_{p\_even}(s,n) := \prod_{j=0}^{\sqrt{n}} \left[1 - \left[\frac{s}{2\cdot(2\cdot j)+1}\cdot\frac{2}{\pi}\right]^2\right] = Q_{p\_n\_even}(s,2,1,n)\cdot Q_{p\_p\_even}(s,2,1,n)$$

$$Q_{p\_even\_}(s) := \frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{1+s}{4}\right)}\cdot\frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{1-s}{4}\right)} \qquad \text{(A1.5.17)}$$

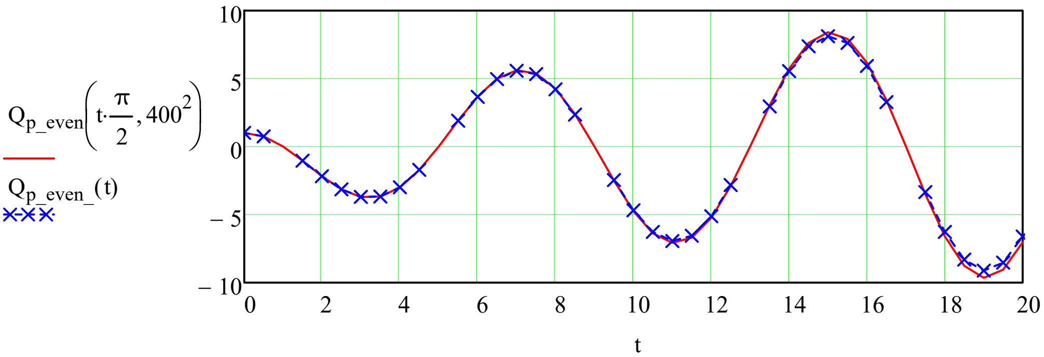


Figure A1.5.2: The periodic component of the gamma polynom with all real roots

With this identity and with the periodic components defined as:

$$Q_{p\_p\_even}(s,n) := \prod_{j=0}^{\sqrt{n}} \left[1 - \frac{s}{2\cdot(2\cdot j)+1}\cdot\frac{2}{\pi}\right] \;;\; Q_{p\_p\_odd}(s,n) := \prod_{j=0}^{\sqrt{n}} \left[1 - \frac{s}{2\cdot(2\cdot j+1)+1}\cdot\frac{2}{\pi}\right] \qquad \text{(A1.5.18)}$$

The first degree split function (corresponding to the split cosines function (A1.5.9) may be written as product of the subset functions for even and for odd integers:

$$Q_{p\_p}(s,n) := Q_{p\_p\_even}(s,n)\cdot Q_{p\_p\_odd}(s,n) \;;\; Q_{p\_p}(s) = \lim_{n\to\infty}\left(Q_{p\_p\_even}(s,n)\cdot Q_{p\_p\_odd}(s,n)\right) \qquad \text{(A1.5.19)}$$

Similarly the complementing split even and odd functions are:

$$Q_{p\_n\_even}(s,n) := \prod_{j=0}^{\sqrt{n}} \left[1 + \frac{s}{2\cdot(2\cdot j)+1}\cdot\frac{2}{\pi}\right] \;;\; Q_{p\_n\_odd}(s,n) := \prod_{j=0}^{\sqrt{n}} \left[1 + \frac{s}{2\cdot(2\cdot j+1)+1}\cdot\frac{2}{\pi}\right]$$

Herewith and with the corresponding second degree split functions:

$$Q_{p\_n}(s,n) := Q_{p\_n\_even}(s,n)\cdot Q_{p\_n\_odd}(s,n) \;;\; Q_{p\_n}(s) = \lim_{n\to\infty}\left(Q_{p\_n\_even}(s,n)\cdot Q_{p\_n\_odd}(s,n)\right) \qquad \text{(A1.5.20)}$$

the first degree split function may be written as product of the even and of the odd first degree split functions:

$$Q_p\left(s\cdot\frac{\pi}{2}\right) = Q_{p\_p}\left(s\cdot\frac{\pi}{2}\right) Q_{p\_n}\left(s\cdot\frac{\pi}{2}\right) = Q_{p\_p\_even}\left(s\cdot\frac{\pi}{2}\right)\cdot Q_{p\_p\_odd}\left(s\cdot\frac{\pi}{2}\right)\cdot Q_{p\_n\_even}\left(s\cdot\frac{\pi}{2}\right)\cdot Q_{p\_n\_odd}\left(s\cdot\frac{\pi}{2}\right) =$$

$$= Q_{p\_even}\left(s\cdot\frac{\pi}{2}\right)\cdot Q_{p\_odd}\left(s\cdot\frac{\pi}{2}\right) = \cos\left(s\cdot\frac{\pi}{2}\right) \qquad \text{(A1.5.21)}$$

The first degree even and odd functions are shown in the figure below for the range ($t := 0, 0.5..20$):

$$Q_{p\_even\_\_}(s,n) := Q_{p\_n\_even}(s,n)\cdot Q_{p\_p\_even}(s,n) \;;\; Q_{p\_odd}(s,n) := Q_{p\_n\_odd}(s,n)\cdot Q_{p\_p\_odd}(s,n)$$

$$Q_{p\_res}(s,n) := Q_{p\_even}(s,n)\cdot Q_{p\_odd}(s,n) \qquad \text{(A1.5.22)}$$

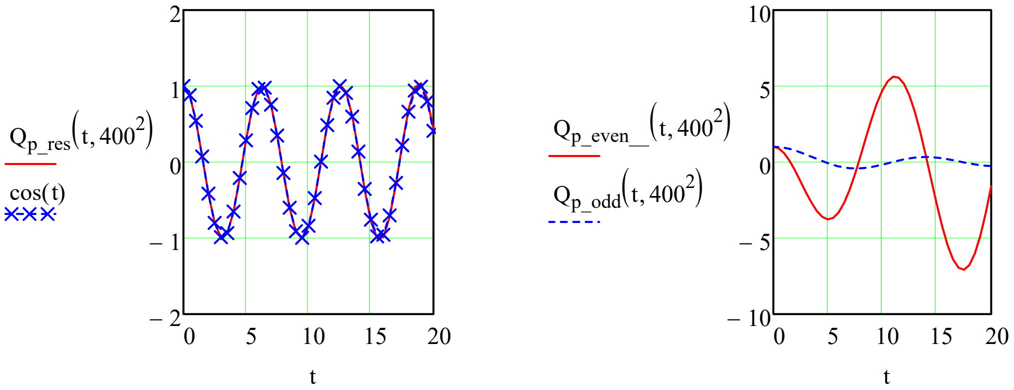


Figure A1.5.3: The odd and even components of the cosines function

The complete gamma functions will be than with (A1.5.11), ($p = 1$) and ($m = 2$):

$$\frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{1+s}{4}\right)} = Q_{p\_n\_mp}\left(s\cdot\frac{\pi}{2},2,1,n\right)\cdot \lim_{n\to\infty} e^{-\frac{s\cdot\ln\left(\sqrt{n}\right)}{4}} = \lim_{n\to\infty}\left[\prod_{k=0}^{n}\left(1-\frac{\ln\left(\sqrt{n}\right)\cdot s}{2\cdot a(k,n)}\right)\cdot Q_{p\_n\_even}\left(s\cdot\frac{\pi}{2},n\right)\right]$$

and for ($m = 2$), ($p = 3$) it is: (A1.5.23)

$$\frac{\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{3+s}{4}\right)} = Q_{p\_n\_mp}\left(s\cdot\frac{\pi}{2},2,3,n\right)\cdot \lim_{n\to\infty} e^{-\frac{s\cdot\ln\left(\sqrt{n}\right)}{4}} = \lim_{n\to\infty}\left[\prod_{k=0}^{n}\left(1-\frac{\ln\left(\sqrt{n}\right)\cdot s}{2\cdot a(k,n)}\right)\cdot Q_{p\_n\_odd}\left(s\cdot\frac{\pi}{2},n\right)\right]$$

On the other hand it is known generally, that for ($2\cdot m = 2^n$) between the gamma functions the following identities holds:

$$\frac{\Gamma\left(\frac{1}{2^n}\right)}{\Gamma\left(\frac{1+x}{2^n}\right)} = \frac{\Gamma\left(\frac{1}{2^{n+1}}\right)}{\Gamma\left(\frac{1+x}{2^{n+1}}\right)}\cdot\prod_{k=1}^{n}\frac{\Gamma\left[\frac{\left(1+2^k\right)}{2^{k+1}}\right]}{\Gamma\left[\frac{\left(1+2^k\right)+x}{2^{k+1}}\right]} \qquad \frac{\Gamma\left(\frac{1}{2^n}\right)}{\Gamma\left(\frac{1-x}{2^n}\right)} = \frac{\Gamma\left(\frac{1}{2^{n+1}}\right)}{\Gamma\left(\frac{1-x}{2^{n+1}}\right)}\cdot\prod_{k=1}^{n}\frac{\Gamma\left[\frac{\left(1+2^k\right)}{2^{k+1}}\right]}{\Gamma\left[\frac{\left(1+2^k\right)-x}{2^{k+1}}\right]}$$

(A1.5.24)

This identity allows for checking the above relations (A1.5.23). For ($n = 2$) these identities (A1.5.24) gives:

$$\frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1+x}{2}\right)} = \frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{1+x}{4}\right)}\cdot\frac{\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{3+x}{4}\right)} \qquad \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1-x}{2}\right)} = \frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{1-x}{4}\right)}\cdot\frac{\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{3-x}{4}\right)} \qquad \text{(A1.5.25)}$$

With (A1.5.11) and with (A1.5.14) these last identities may be written for ($m \to \infty$) as:

$$e^{-\frac{x\cdot\ln\left(\sqrt{m}\right)}{2}}\cdot Q_{p\_n}\left(x\cdot\frac{\pi}{2},m\right) = e^{-\frac{x\cdot\ln\left(\sqrt{n}\right)}{4}}\cdot Q_{p\_n\_mp}\left(x\cdot\frac{\pi}{2},2,1,m\right)\cdot e^{-\frac{x\cdot\ln\left(\sqrt{n}\right)}{4}}\cdot Q_{p\_n\_mp}\left(x\cdot\frac{\pi}{2},2,3,m\right)$$

$$e^{\frac{x\cdot\ln\left(\sqrt{m}\right)}{2}}\cdot Q_{p\_p}\left(x\cdot\frac{\pi}{2},m\right) = e^{\frac{x\cdot\ln\left(\sqrt{n}\right)}{4}}\cdot Q_{p\_p\_mp}\left(x\cdot\frac{\pi}{2},2,1,m\right)\cdot e^{\frac{x\cdot\ln\left(\sqrt{n}\right)}{4}}\cdot Q_{p\_p\_mp}\left(x\cdot\frac{\pi}{2},2,3,m\right) \qquad \text{(A1.5.26)}$$

These equations certainly hold for the exponential parts:

$$\lim_{m \to \infty} e^{-\frac{x \cdot \ln(\sqrt{m})}{2}} = \lim_{n \to \infty} e^{-\frac{x \cdot \ln(\sqrt{n})}{4}} \cdot \lim_{n \to \infty} e^{-\frac{x \cdot \ln(\sqrt{n})}{4}}$$

$$\lim_{m \to \infty} e^{\frac{x \cdot \ln(\sqrt{m})}{2}} = \lim_{n \to \infty} e^{\frac{x \cdot \ln(\sqrt{m})}{4}} \cdot \lim_{n \to \infty} e^{\frac{x \cdot \ln(\sqrt{m})}{4}} \qquad \text{(A1.5.27)}$$

The infinite product parts may be written with (A1.5.5) and with (A1.5.14) as follows:

$$\prod_{j=0}^{\infty} \left(1 - \frac{s}{2 \cdot j + 1}\right) = \prod_{j=0}^{\infty} \left(1 - \frac{s}{4 \cdot j + 1}\right) \cdot \prod_{j=0}^{\infty} \left(1 - \frac{s}{4 \cdot j + 3}\right) \qquad \text{(A1.5.28)}$$

$$\prod_{j=0}^{\infty} \left(1 + \frac{s}{2 \cdot j + 1}\right) = \prod_{j=0}^{\infty} \left(1 + \frac{s}{4 \cdot j + 1}\right) \cdot \prod_{j=0}^{\infty} \left(1 + \frac{s}{4 \cdot j + 3}\right)$$

what confirms (A1.5.25), since the set of integers in the denominator on the left is the union of the sets of integers in the denominator on the right:

$$N_{left} = \{\, 1 + 2 \cdot n : n \in Z \,\} \qquad N_{right\_1} = \{\, 1 + 4 \cdot n : n \in Z \,\} \qquad N_{right\_2} = \{\, 3 + 4 \cdot n : n \in Z \,\}$$

$$N_{left} = N_{right\_1} \cup N_{right\_2} \qquad Z = 0,\ 1,\ 2,\ \ldots \infty \qquad \text{(A1.5.29)}$$

A1.6. Product series for the Riemann zeta function

The Zeta function ($\zeta(s)$) may be written for the series of primes ($P_{(n)}$) as infinite product:

$$\zeta(s) = \prod_{n=1}^{\infty}\left[1-\frac{1}{\left[P_{(n)}\right]^{s}}\right]^{-1} \;;\; \zeta(s)^{-1} = \prod_{n=1}^{\infty}\left[1-\frac{1}{\left[P_{(n)}\right]^{s}}\right] = \prod_{n=1}^{\infty}\left[1-e^{-s\cdot\ln\left[P_{(n)}\right]}\right] \qquad \text{(A1.6.1)}$$

$$= \prod_{n=1}^{\infty}\frac{e^{\frac{s}{2}\cdot\ln\left[P_{(n)}\right]}}{e^{\frac{s}{2}\cdot\ln\left[P_{(n)}\right]}-e^{-\frac{s}{2}\cdot\ln\left[P_{(n)}\right]}} = \zeta_1\left(\frac{1+r}{2}\right)\cdot\zeta_2\left(\frac{1+r}{2}\right)$$

The Zeta function ($\zeta(s)$) has two components:

$$\zeta_1(s) = e^{\frac{s}{2}\cdot\ln\left[P_{(n)}\right]}$$

$$\zeta_2(\sigma) = \frac{1}{e^{\frac{\sigma}{2}\cdot\ln\left[P_{(n)}\right]}-e^{-\frac{\sigma}{2}\cdot\ln\left[P_{(n)}\right]}} = \frac{1}{2\cdot i\cdot\sin\left[\frac{i\cdot\tau}{2}\cdot\ln\left[P_{(n)}\right]\right]} \qquad \text{(A1.6.2)}$$

$$\zeta_2(i\cdot\tau) = \frac{1}{e^{\frac{i\cdot\tau}{2}\cdot\ln\left[P_{(n)}\right]}-e^{-\frac{i\cdot\tau}{2}\cdot\ln\left[P_{(n)}\right]}} = \frac{1}{2\cdot i\cdot\sin\left[\frac{\sigma}{2}\cdot\ln\left[P_{(n)}\right]\right]}$$

With (A1.4.8) and (A1.4.9) the exponential function may be written as infinite product:

$$e^{\frac{\sigma}{2}\cdot\ln\left[P_{(n)}\right]} = \lim_{q\to\infty}\prod_{k=0}^{q}\left[1-i\cdot\frac{\sigma\cdot\ln\left[P_{(n)}\right]}{b(k,q)}\right]$$

$$e^{\frac{i\cdot\tau}{2}\cdot\ln\left[P_{(n)}\right]} = \lim_{q\to\infty}\prod_{k=0}^{q}\left[1+i\cdot\frac{\tau\cdot\ln\left[P_{(n)}\right]}{b(k,q)}\right] \qquad \text{(A1.6.3)}$$

where the constants are all positive real numbers:

$$b(k,q) := \frac{\left(2\cdot k+\sqrt{q}\right)^{2}}{2\cdot\sqrt{q}} \;;\quad q = 1,2,3..\infty\;;\quad k = 0,1,2..\infty \qquad \text{(A1.6.4)}$$

Annex 2: Splitting of polynoms

The present annex shows that any infinite polynom product with all positive real roots may be split in two stages into the product of four infinite polynom product functions: two with all imaginary and two with all real roots. Between the real roots both sides next to zero of the second degree split functions with all real roots there are only roots on the imaginary axes. Equations with quotients of such infinite products are defined. Examples defining roots exclusively on the imaginary axes for such quotient functions are given for the exponential, the trigonometric, the gamma and for the Riemann zeta functions.

A2.1. Splitting the polynom with all positive roots

Is (ζ) a complex number, so any function ($f(\zeta)$) is a complex number again. The sum of such a complex number and of its conjugate, as well as the difference of the complex number and its transpose result a real number ($f(\zeta) + \overline{f(\zeta)} = f(\zeta) - f(\zeta)^T = (\text{real})$). The sum of such a complex number and of its transpose, as well as the difference of the complex number and its conjugate result an imaginary number ($f(\zeta) + f(\zeta)^T = f(\zeta) - \overline{f(\zeta)} = (\text{imaginary})$). These obvious relations will be applied to polynom functions.

Definition of the complete polynom function:

Taking the sets of the reciprocal of the positive real numbers ($a_{0_{(i)}} \neq a_{0_{(j)}}$) with ($i,\ j = 1,2..N,\ 1 < N < \infty$) composing the vector ($A_0$), the sets of the reciprocal of the square roots ($a_{1_{(j)}} = \sqrt{a_{0_{(j)}}}$) of these numbers composing the vector (A_1) and the sets of the reciprocal of the power of one fourth ($a_{2_{(j)}} = \sqrt{a_{1_{(j)}}}$) of these same numbers composing the vector (A_2), than the infinite product ($P(\zeta_0, A_0)$) - standing for the **complete polynom function** with the complex variable (ζ_0) and with all roots positive real at ($a_{0_{(j)}}$) - in the normed form is defined as follows:

$$P(\zeta_0, A_0) = \prod_{j=1}^{\infty} \left[1 - \frac{\zeta_0}{a_{0_{(j)}}} \right] \qquad a_{0_{(j)}} > 0 \qquad \text{(A2.1.1)}$$

Definition of the first degree split polynoms:

Splitting the complex variable (ζ_0) by the substitution ($\zeta_1 = \sqrt{\zeta_0}$) the polynom will be split:

$$P(\zeta_0, A_0) = \prod_{j=1}^{\infty} \left[1 - \frac{\zeta_1}{a_{1_{(j)}}} \right] \cdot \prod_{j=1}^{\infty} \left[1 + \frac{\zeta_1}{a_{1_{(j)}}} \right] = P_p(\zeta_1, A_1) \cdot P_n(\zeta_1, A_1) \qquad a_{1_{(j)}} = \sqrt{a_{0_{(j)}}} > 0 \qquad \text{(A2.1.2)}$$

The first of these **first degree split polynoms**, ($P_p(\zeta_1, A_1)$) has all positive real roots at ($a_{1_{(j)}}$) and is similar to the complete polynom ($P(a_0, A_0)$), the other ($P_n(\zeta_1, A_1)$) has all negative real roots at ($-a_{1_{(j)}}$):

$$P_p(\zeta_1, A_1) = \prod_{j=1}^{\infty} \left[1 - \frac{\zeta_1}{a_{1_{(j)}}} \right] \qquad P_n(\zeta_1, A_1) = \prod_{j=1}^{\infty} \left[1 + \frac{\zeta_1}{a_{1_{(j)}}} \right] \qquad \text{(A2.1.3)}$$

Definition of the second degree split polynoms:

Continuing the splitting by the variable substitution ($\zeta_2 = \sqrt{\zeta_1}$) of the first degree split polynoms ($P_n(\zeta_1, A_1)$) and ($P_p(\zeta_1, A_1)$), they will be split into two components resulting the **second degree split polynoms**:

$$P(\zeta_0, A_0) = P_p(\zeta_1, A_1) \cdot P_n(\zeta_1, A_1) = P_{p\_p}(\zeta_2, A_2) \cdot P_{p\_n}(\zeta_2, A_2) \cdot P_{n\_p}(\zeta_2, A_2) \cdot P_{n\_n}(\zeta_2, A_2) \qquad \text{(A2.1.4)}$$

The splitting of the first degree split polynom ($P_p(\zeta_1, A_1)$) is similar to the splitting of the original polynom ($P(\zeta_0, A_0)$), since it has exclusively positive real roots ($a_{1_{(j)}} > 0$):

$$P_p(\zeta_1, A_1) = \prod_{j=1}^{\infty} \left[1 - \frac{\zeta_2}{a_{2_{(j)}}} \right] \cdot \prod_{j=1}^{\infty} \left[1 + \frac{\zeta_2}{a_{2_{(j)}}} \right] = P_{p\_p}(\zeta_2, A_2) \cdot P_{p\_n}(\zeta_2, A_2) \qquad \text{(A2.1.5)}$$

The first of the second degree split polynoms ($P_{p\_p}(\zeta_2, A_2)$) has all positive real roots at ($a_{2_{(j)}}$) and is similar again to the original polynom ($P(a_0, A_0)$), the other ($P_{p\_n}(\zeta_2, A_2)$) has all negative real roots at ($-a_{2_{(j)}}$):

$$P_{p\_p}(\zeta_2, A_2) = \prod_{j=1}^{\infty} \left[1 - \frac{\zeta_2}{a_{2_{(j)}}}\right] \qquad P_{p\_n}(\zeta_2, A_2) = \prod_{j=1}^{\infty} \left[1 + \frac{\zeta_2}{a_{2_{(j)}}}\right] = P_{p\_p}(\zeta_2, -A_2) \tag{A2.1.6}$$

The further splitting of the first degree split polynom ($P_n(\zeta_1, A_1)$) with all negative real roots is different: it splits into two polynoms each of them with all roots on the imaginary axes:

$$P_n(\zeta_1, A_1) = \prod_{j=1}^{\infty} \left[1 - \frac{\zeta_2}{i \cdot a_{2_{(j)}}}\right] \cdot \prod_{j=1}^{\infty} \left[1 + \frac{\zeta_2}{i \cdot a_{2_{(j)}}}\right] = P_{n\_p}(\zeta_2, A_2) \cdot P_{n\_n}(\zeta_2, A_2) \tag{A2.1.7}$$

Definition of the adjoint axes:

This splitting of the negative numbers renders the introduction of the imaginary and of the complex numbers necessary, in order to increase the degree of liberty. The two independent axes - the real and the imaginary - may be regarded as **adjoint axes.**

The third of the second degree split polynoms ($P_{n\_p}(\zeta_2, A_2)$) has all, positive imaginary roots at ($i \cdot a_{2_{(j)}}$), the other ($P_{n\_n}(\zeta_2, A_2)$) has al negative imaginary roots at ($-i \cdot a_{2_{(j)}}$):

$$P_{n\_p}(\zeta_2, A_2) = \prod_{j=1}^{\infty} \left[1 - \frac{\zeta_2}{i \cdot a_{2_{(j)}}}\right] \qquad P_{n\_n}(\zeta_2, A_2) = \prod_{j=1}^{\infty} \left[1 + \frac{\zeta_2}{i \cdot a_{2_{(j)}}}\right] = P_{n\_p}(\zeta_2, -A_2) \tag{A2.1.8}$$

The product of all second degree split polynoms as components is symmetrical over both, over the real and over the imaginary axes:

$$P(\zeta_0, A_0) = \prod_{j=1}^{\infty} \left[\left[1 - \frac{\zeta_2}{a_{2_{(j)}}}\right] \cdot \left[1 + \frac{\zeta_2}{a_{2_{(j)}}}\right] \cdot \left[1 - \frac{\zeta_2}{i \cdot a_{2_{(j)}}}\right] \cdot \left[1 + \frac{\zeta_2}{i \cdot a_{2_{(j)}}}\right]\right] \tag{A2.1.9}$$

Between the second degree split polynoms the following identity relations exist per definition:

$$\begin{aligned}
P_{p_p}(\zeta_2, A_2) &= \prod_{j=1}^{\infty} \left[1 - \frac{\zeta_2}{a_{2_{(j)}}}\right] = P_{p_n}[\zeta_2, (-A_2)] = P_{n_p}[\zeta_2, (i \cdot A_2)] = P_{n_n}[\zeta_2, (-i \cdot A_2)] = \\
&= P_{p_n}[(-\zeta_2), A_2] = P_{n_p}[(-i \cdot \zeta_2), A_2] = P_{n_n}[(i \cdot \zeta_2), A_2] \\
P_{p_n}(\zeta_2, A_2) &= \prod_{j=1}^{\infty} \left[1 + \frac{\zeta_2}{a_{2_{(j)}}}\right] = P_{p_p}[\zeta_2, (-A_2)] = P_{n_p}[\zeta_2, (-i \cdot A_2)] = P_{n_n}[\zeta_2, (i \cdot A_2)] = \\
&= P_{p_p}[(-\zeta_2), A_2] = P_{n_p}[(i \cdot \zeta_2), A_2] = P_{n_n}[(-i \cdot \zeta_2), A_2] \\
P_{n_p}(\zeta_2, A_2) &= \prod_{j=1}^{\infty} \left[1 - \frac{\zeta_2}{i \cdot a_{2_{(j)}}}\right] = P_{p_p}[\zeta_2, (i \cdot A_2)] = P_{p_n}[\zeta_2, (-i \cdot A_2)] = P_{n_n}[\zeta_2, (-A_2)] = \\
&= P_{p_p}[(i \cdot \zeta_2), A_2] = P_{p_n}[(-i \cdot \zeta_2), A_2] = P_{n_n}[(-\zeta_2), A_2] \\
P_{n_n}(\zeta_2, A_2) &= \prod_{j=1}^{\infty} \left[1 + \frac{\zeta_2}{i \cdot a_{2_{(j)}}}\right] = P_{p_p}[\zeta_2, (-i \cdot A_2)] = P_{p_n}[\zeta_2, (i \cdot A_2)] = P_{n_p}[\zeta_2, (-A_2)] = \\
&= P_{p_p}[(-i \cdot \zeta_2), A_2] = P_{p_n}[(i \cdot \zeta_2), A_2] = P_{n_p}[(-\zeta_2), A_2]
\end{aligned} \tag{A2.1.10}$$

From these relations result per definition the following relations between the first degree split polynoms:

$$P_p(\zeta_1, A_1) = \prod_{j=1}^{\infty}\left[1 - \frac{\zeta_1}{a_{1_{(j)}}}\right] = P_{p\_p}(\zeta_2, A_2)\cdot P_{p\_n}(\zeta_2, A_2) = P_{n\_p}\left[\zeta_2, (i\cdot A_2)\right]\cdot P_{n\_n}\left[\zeta_2, (-i\cdot A_2)\right] =$$

$$= P_n\left[\zeta_1, (i\cdot A_1)\right] = P_n\left[(i\cdot\zeta_1), A_1\right] \qquad \text{(A2.1.11)}$$

$$P_n(\zeta_1, A_1) = \prod_{j=1}^{\infty}\left[1 + \frac{\zeta_1}{a_{1_{(j)}}}\right] = P_{n\_p}(\zeta_2, A_2)\cdot P_{n\_n}(\zeta_2, A_2) = P_{p\_n}\left[\zeta_2, (i\cdot A_2)\right]\cdot P_{p\_p}\left[\zeta_2, (i\cdot A_2)\right] =$$

$$= P_p\left[\zeta_1, (-A_1)\right] = P_p\left[(-\zeta_1), A_1\right]$$

The complete polynom ($P(\zeta_0, A_0)$) after the second degree split (A2.1.9) has symmetrically placed infinite number of roots on the real and on the imaginary axes. Between the two real roots next to zero there are only imaginary roots:

$$\max\left[-a_{2_{(j)}}\right] < \mathrm{Re}(\zeta_2) < \min\left[a_{2_{(j)}}\right] \quad \text{at:} \qquad \zeta_2 = i\cdot a_{2_{(j)}} \qquad \zeta_2 = -i\cdot a_{2_{(j)}} \qquad j = 1, 2..\infty \qquad \text{(A2.1.12)}$$

From the definitions follow the symmetry relations: If a polynom ($P(\zeta_0, A_0)$) may be written in the form (A2.1.1) with all roots positive and real, than it is symmetrical for ($\zeta_0 = \sigma_0$) real and it may be split after the substitution ($\zeta_1 = \sqrt{\zeta_0}$) into two first degree split polynoms, one ($P_p(\zeta_1, A_1)$) having only positive, the other ($P_n(\zeta_1, A_1)$) having only negative real roots and each of them is the transpose function of the other for ($\zeta_1 = \sigma + i\cdot\tau$) complex:

$$P_p(\zeta_1, A_1) = P_n(\zeta_1, A_1)^T \qquad \text{(A2.1.13)}$$

Definition of the polynoms bound by their roots:

Similarly the polynom ($P_p(\zeta_1, A_1)$) having only positive real roots may be split after the substitution ($\zeta_2 = \sqrt{\zeta_1}$) into two second degree split polynoms one ($P_{p\_p}(\zeta_2, A_2)$) having only positive the other ($P_{p\_n}(\zeta_2, A_2)$) having only negative real roots of the same set of real numbers and each of them is the transpose function of the other for ($\zeta_2 = \sigma + i\cdot\tau$) complex:

$$P_{p\_p}(\zeta_2, A_2) = P_{p\_n}(\zeta_2, A_2)^T \qquad \text{(A2.1.14)}$$

This pair of second degree split polynoms have roots on the same axes: they are **bound by their roots** being on the same axes, on the real axes and having the same absolute values.

Taking the polynom ($P_n(\zeta_2, A_2)$) with all negative real roots, after the substitution ($\zeta_2 = \sqrt{\zeta_1}$) it may be split once more into two second degree split polynoms like (A2.1.8) both having all roots on the imaginary axes. The two split polynoms are symmetrical over the real axes, meaning they are conjugate complex:

$$P_{n\_p}(\zeta_2, A_2) = \overline{P_{n\_n}(\zeta_2, A_2)} \qquad \text{(A2.1.15)}$$

This pair of second degree split polynoms have roots on the same axes: they are **bound by their roots** being on the same axes, on the imaginary axes and having the same absolute values.

Taking the roots of the polynoms (A2.1.11) on the imaginary, instead of on the real axes results polynoms formally identical to the polynoms (A2.1.15). Therefore they are conjugate complex as well:

$$P_p(\zeta_1, i\cdot A_1) = \overline{P_n(\zeta_1, i\cdot A_1)} \qquad \text{(A2.1.16)}$$

What is the representation of a polynom having all its roots on one of the adjoint axes, if the coordinate system is modified that way, that the same axes is shifted parallel by a constant value? Taking the polynoms ($P_{n\_p}(\zeta_2, A_2)$) and ($P_{n\_n}(\zeta_2, A_2)$) defined in (A2.1.8), with all roots on the imaginary axis ($r_{(j)} = i \cdot a_{(j)}$) and ($r_{(j)} = -\left[i \cdot a_{(j)}\right]$) and shifting the roots to the line ($s = \zeta + \frac{b}{2}$), than they will be:

$$\left(r_{tr_{(j)}} = \frac{b}{2} + i \cdot a_{(j)}\right) \text{ and } \left(r_{tr_{(j)}} = \frac{b}{2} - i \cdot a_{(j)}\right) \quad \text{(A2.1.17)}$$

The variable (ζ) will be replaced by the variable ($s = \zeta + \frac{b}{2}$), and the imaginary axes will be replaced by a line starting from ($\sigma = \frac{b}{2}$), instead from the origin. For ($b = 1$) this corresponds to the critical line of the Riemann Zeta function. For each of the components of the infinite product it may be written:

$$1 - \frac{\zeta}{i \cdot a_{(j)}} = 1 - \frac{s - \frac{b}{2}}{r_{tr_{(j)}} - \frac{b}{2}} = \frac{r_{tr_{(j)}}}{r_{tr_{(j)}} - \frac{b}{2}} - \frac{s}{r_{tr_{(j)}} - \frac{b}{2}} = \frac{r_{tr_{(j)}}}{r_{tr_{(j)}} - \frac{b}{2}}\left[1 - \frac{s}{r_{tr_{(j)}}}\right] \quad \text{and}$$

$$1 + \frac{\zeta}{i \cdot a_{(j)}} = \frac{r_{tr_{(j)}}}{\left[r_{tr_{(j)}} - \frac{b}{2}\right]}\left[1 + \frac{s - b}{r_{tr_{(j)}}}\right] \quad \text{(A2.1.18)}$$

The polynoms (A2.1.8) will have the form:

$$\left[\prod_{j=1}^{\infty} \frac{r_{tr_{(j)}}}{r_{tr_{(j)}} - \frac{b}{2}}\right] \cdot \left[\prod_{j=1}^{\infty} \left[1 - \frac{s}{r_{tr_{(j)}}}\right]\right] \quad \text{and} \quad \left[\prod_{j=1}^{\infty} \frac{r_{tr_{(j)}}}{r_{tr_{(j)}} - \frac{b}{2}}\right] \cdot \left[\prod_{j=1}^{\infty} \left[1 + \frac{s - b}{r_{tr_{(j)}}}\right]\right] \quad \text{(A2.1.19)}$$

In this representation the point of central symmetry is changed from the origin to the point ($\sigma = \frac{b}{2}$). Correspondingly the roots of the second degree split functions with all roots on the real axes will be shifted by the same value ($s = \zeta + \frac{b}{2}$), yielding:

$$\left(r_{tr_{(j)}} = \frac{b}{2} + a_{(j)}\right) \text{ and } \left(r_{tr_{(j)}} = \frac{b}{2} - a_{(j)}\right) \quad \text{(A2.1.20)}$$

Again, for each of the components of the infinite product it may be written:

$$1 - \frac{\zeta}{a_{(j)}} = \frac{r_{tr_{(j)}}}{r_{tr_{(j)}} - \frac{b}{2}}\left[1 - \frac{s}{r_{tr_{(j)}}}\right] \quad \text{and} \quad 1 + \frac{\zeta}{a_{(j)}} = \frac{r_{tr_{(j)}}}{r_{tr_{(j)}} - \frac{b}{2}}\left[1 + \frac{s - b}{r_{tr_{(j)}}}\right] \quad \text{(A2.1.21)}$$

The corresponding polynoms (A2.1.8) will have the same form as (A2.1.19).

A2.2. Equations for the roots of the quotients of split polynoms

Definition of the adjoint polynoms:

Taking the second degree split polynom with all real roots (A2.1.6) and its transpose resp. conjugate, for any (ζ) the sum of two mutually transpose values is an imaginary value of the polynom ($2\cdot i\cdot R_p(\tau, D_1)$, τ real), their difference a real value of the polynom ($2\cdot Q_p(\sigma, C_1)$, σ real).Taking the second degree split polynom with all imaginary roots (A2.1.8) and its transpose resp. conjugate, for any (ζ) the sum of two mutually transpose values is an imaginary value of the polynom ($2\cdot i\cdot R_n(\tau, D_1)$, τ real), their difference a real value of the polynom ($2\cdot Q_n(\sigma, C_1)$, σ real). Here and in the following - without restricting the generality - for the complex number (ζ_2) may be written just (ζ), since the square root of any complex number is a complex number again. The sque symmetric polynom ($R_p(\tau, D_1)$) and the symmetric polynom ($Q_p(\sigma, C_1)$) are **adjoint polynoms** to the original split polynom ($P_p(\zeta_1, A_1)$):

$$P_{p\_p}(\zeta,A_2) + P_{p\_p}(\zeta,A_2)^T = P_{p\_p}(\zeta,A_2) + P_{p\_n}(\zeta,A_2) = P_{p\_p}(\zeta,A_2) - \overline{P_{p\_p}(\zeta,A_2)} = (\text{imaginary}) =$$
$$= P_{p\_p}(i\cdot\tau,A_2) + P_{p\_n}(i\cdot\tau,A_2) = 2\cdot Q_{n\_p}(\tau,D_2)\cdot Q_{n\_n}(\tau,D_2) = 2\cdot Q_n(\tau,D_1)$$

$$P_{n\_n}(\zeta,A_2) + P_{n\_n}(\zeta,A_2)^T = P_{n\_n}(\zeta,A_2) - P_{n\_p}(\zeta,A_2) = P_{n\_n}(\zeta,A_2) - \overline{P_{n\_n}(\zeta,A_2)} = (\text{imaginary}) =$$
$$= \left(P_{n\_n}(i\cdot\tau,A_2) - P_{n\_p}(i\cdot\tau,A_2)\right) = 2\cdot i\cdot R_{p\_n}(\tau,D_2)\cdot R_{p\_p}(\tau,D_2) = 2\cdot i\cdot R_p(\tau,D_1) \qquad \text{(A2.2.1)}$$

$$P_{p\_p}(\zeta,A_2) + \overline{P_{p\_p}(\zeta,A_2)} = P_{p\_p}(\zeta,A_2) - P_{p\_n}(\zeta,A_2) = P_{p\_p}(\zeta,A_2) - P_{p\_p}(\zeta,A_2)^T = (\text{real}) =$$
$$= P_{p\_p}(\sigma,A_2) + P_{p\_n}(\sigma,A_2) = 2\cdot R_{n\_p}(\sigma,C_2)\cdot R_{n\_n}(\sigma,C_2) = 2\cdot i\cdot R_n(\sigma,C_1)$$

$$P_{n\_n}(\zeta,A_2) + \overline{P_{n\_n}(\zeta,A_2)} = P_{n\_n}(\zeta,A_2) + P_{n\_p}(\zeta,A_2) = P_{n\_n}(\zeta,A_2) - P_{n\_n}(\zeta,A_2)^T = (\text{real}) =$$
$$= P_{n\_p}(i\cdot\sigma,A_2) + P_{n\_n}(i\cdot\sigma,A_2) = 2\cdot Q_{p\_p}(\sigma,C_2)\cdot Q_{p\_n}(\sigma,C_2) = 2\cdot Q_p(\sigma,C_1)$$

Lemma A2.1:

Polynoms composed as the sum and as the difference of second degree split polynoms bound by their roots define their adjoint polynoms with all roots on the adjoint axes.

Proof: Setting the sum or the difference of split polynoms equal to zero result the following equations for the functions ($Q_n(\tau, C_2)$), ($R_n(\sigma, D_2)$), ($Q_p(\tau, C_2)$) and ($R_p(\sigma, D_2)$):

$$P_{n\_n}(\zeta,A_2) + P_{n\_p}(\zeta,A_2) = 2\cdot Q_p(\sigma,C_1) \qquad P_{p\_p}(\tau,A_2) + P_{p\_n}(\tau,A_2) = 2\cdot Q_n(\tau,D_1)$$
$$P_{n\_n}(\zeta,A_2) - P_{n\_p}(\zeta,A_2) = 2\cdot i\cdot R_p(\tau,D_1) \qquad P_{p\_p}(\sigma,A_2) - P_{p\_n}(\sigma,A_2) = 2\cdot i\cdot R_n(\sigma,C_1)$$

These equations may be regarded as the **generalized relations of Euler.** This because later in chapter (A2.4), where as examples for split polynoms the exponential and trigonometric functions are given, they correspond to the relations of Euler. They define roots on the adjoint axes for the adjoint polynoms, and are the consequences of the symmetry conditions (A2.1.10), concluding the proof.

Multiplying formally all components of the infinite product (A2.1.8) results for the second degree split polynom and for its transpose the polynoms:

$$P_{n\_n}(\zeta,A_2) = 1 + i\cdot k_1\cdot\zeta - k_2\cdot\zeta^2 - i\cdot k_3\cdot\zeta^3 + k_4\cdot\zeta^4 + i\cdot k_5\cdot\zeta^5 \ldots$$
$$P_{n\_p}(\zeta,A_2) = 1 - i\cdot k_1\cdot\zeta - k_2\cdot\zeta^2 + i\cdot k_3\cdot\zeta^3 + k_4\cdot\zeta^4 - i\cdot k_5\cdot\zeta^5 \ldots \qquad \text{(A2.2.2)}$$

The sum of the above two series is again a polynom, an even function. The difference of the above two series is again a polynom, an odd function.

$$P_{n\_n}(\zeta,A_2) + P_{n\_p}(\zeta,A_2) = 2\,(\,1 - k_2\cdot\sigma^2 + k_4\cdot\sigma^4 - k_6\cdot\sigma^6 \ldots) = 2\cdot Q_p(\sigma,C_1) = (\text{real}) \qquad \text{(A2.2.3)}$$

$$P_{n\_n}(\zeta,A_2) - P_{n\_p}(\zeta,A_2) = 2\cdot i\cdot\tau\,(k_1 - k_3\cdot\tau^2 + k_5\cdot\tau^4 - k_7\cdot\tau^6 \ldots\,) = 2\cdot i\cdot R_p(\tau,D_1) = (\text{imaginary})$$

Multiplying formally all components of the infinite product (A2.1.6) results for the second degree split polynom and for its transpose the polynoms:

$$P_{p\_p}(\zeta,A_2) = 1 - k_1\cdot\zeta + k_2\cdot\zeta^2 - k_3\cdot\zeta^3 + k_4\cdot\zeta^4 \ldots$$
$$P_{p\_n}(\zeta,A_2) = 1 + k_1\cdot\zeta + k_2\cdot\zeta^2 + k_3\cdot\zeta^3 + k_4\cdot\zeta^4 \ldots \qquad \text{(A2.2.4)}$$

The sum of the above two series is again a polynom an even function. The difference of the above two series is again a polynom, an odd function, both in accordance with (A2.2.1):

$$P_{p\_p}(\zeta,A_2) + P_{p\_n}(\zeta,A_2) = 2\,(\,1 + k_2\cdot\tau^2 + k_4\cdot\tau^4 + k_6\cdot\tau^6\,...) = 2\cdot Q_n(\tau,D_1) = (\text{imaginary})$$

$$P_{p\_p}(\zeta,A_2) - P_{p\_n}(\zeta,A_2) = 2\,(k_1\cdot\sigma + k_3\cdot\sigma^3 + k_5\cdot\sigma^5 + k_7\cdot\sigma^7\,...\,) =$$

$$= 2\cdot\sigma\,(k_1 + k_3\cdot\sigma^2 + k_5\cdot\sigma^4 + k_7\cdot\sigma^6\,...\,) = 2\cdot i\cdot R_n(\sigma,C_1) = (\text{real}) \qquad \text{(A2.2.5)}$$

All polynoms ($Q_n(\sigma,C_1)$), ($Q_p(\sigma,C_1)$), ($R_n(\tau,D_1)$ resp. ($R_p(\tau,D_1)$) have an infinite number of roots (C_1) resp. (D_1), symmetrically distributed around zero on the real axes, since for ($x = \sigma = \tau$) real the functions are defined exclusively on the real axes. The polynom ($R_n(\tau,D_1)$) has additionally a root at zero.

Formally all four functions may be taken at any complex value (ζ): the functions ($Q_p(\zeta,C_1)$) and ($Q_n(\zeta,C_1)$) remain in this case symmetrical over the imaginary axes and the functions ($R_p(\zeta,D_1)$) and ($R_n(\zeta,D_1)$) skew symmetrical over the origin since from (A2.1.10) follows, that:

$$2\cdot i\cdot R_p(\tau,D_1) = P_{n\_n}(\zeta,A_2) - P_{n\_p}(\zeta,A_2) = -P_{n\_p}(-\zeta,A_2) + P_{n\_n}(-\zeta,A_2) = -2\cdot i\cdot R_p(-\tau,D_1)$$

$$2\cdot i\cdot R_n(\tau,D_1) = P_{p\_p}(\zeta,A_2) - P_{p\_n}(\zeta,A_2) = -P_{p\_n}(-\zeta,A_2) + P_{p\_p}(-\zeta,A_2) = -2\cdot i\cdot R_n(-\tau,D_1) \qquad \text{(A2.2.6)}$$

$$2\cdot Q_p(\sigma,C_1) = P_{n\_n}(\zeta,A_2) + P_{n\_p}(\zeta,A_2) = P_{n\_p}(-\zeta,A_2) + P_{n\_n}(-\zeta,A_2) = 2\cdot Q_p(-\sigma,C_1)$$

$$2\cdot Q_n(\sigma,C_1) = P_{p\_p}(\zeta,A_2) + P_{p\_n}(\zeta,A_2) = P_{p\_n}(-\zeta,A_2) + P_{p\_p}(-\zeta,A_2) = 2\cdot Q_n(-\sigma,C_1)$$

Definition of the polynom quotient equations:

The equations defining the roots of the adjoint polynoms on the adjoint axes may be written with (A2.2.1) as quotients, called **polynom quotient equations**:

$$\frac{P_{p\_p}(\zeta,A_2)}{P_{p\_n}(\zeta,A_2)} = \frac{P_{p\_p}(\zeta,A_2)}{P_{p\_p}(\zeta,A_2)^T} = -1 \qquad \frac{P_{p\_p}(\zeta,A_2)}{P_{p\_n}(\zeta,A_2)} = \frac{P_{p\_p}(\zeta,A_2)}{P_{p\_p}(\zeta,A_2)^T} = 1 \qquad \text{(A2.2.7)}$$

$$\frac{P_{p\_p}(\zeta,A_2)}{\overline{P_{p\_p}(\zeta,A_2)}} = \frac{P_{n\_n}(\zeta,A_2)}{\overline{P_{n\_n}(\zeta,A_2)}} = 1 \qquad \frac{P_{p\_p}(\zeta,A_2)}{\overline{P_{p\_p}(\zeta,A_2)}} = \frac{P_{n\_n}(\zeta,A_2)}{\overline{P_{n\_n}(\zeta,A_2)}} = -1$$

Thus, if the denominator and the numerator are mutually transpose complex numbers, than in case their sum is equal to zero, the variable may take only imaginary values ($\zeta = i\cdot\tau$), in case their difference is equal to zero, the variable has to take real values ($\zeta = \sigma$). If the denominator and the numerator are mutually conjugate complex numbers, than in case their difference is equal to zero, the variable may take only imaginary values ($\zeta = i\cdot\tau$), in case their sum is equal to zero, the variable has to take real values ($\zeta = \sigma$).

All resulting polynoms (A2.2.1) may be written in the product form similarly to the first degree split polynoms (A2.1.3) and therefore may be formally split into second degree split polynoms of their own into the polynoms ($Q_{p\_p}(\zeta,C_2)$), ($Q_{p\_n}(\zeta,C_2)$), ($Q_{n\_n}(\zeta,C_2)$), ($Q_{n\_p}(\zeta,C_2)$), ($R_{p\_p}(\zeta,D_2)$), ($R_{p\_n}(\zeta,D_2)$) ($R_{n\_n}(\zeta,D_2)$), ($R_{n\_p}(\zeta,D_2)$:

$$Q_p(\zeta,C_1) = Q_{p\_p}(\zeta,C_2)\cdot Q_{p\_n}(\zeta,C_2) = \prod_{j=1}^{\infty}\left[1 - \frac{\zeta}{c_{2_{(j)}}}\right]\cdot\prod_{j=1}^{\infty}\left[1 + \frac{\zeta}{c_{2_{(j)}}}\right] = \prod_{j=1}^{\infty}\left[1 - \left[\frac{\zeta}{c_{2_{(j)}}}\right]^2\right] \qquad \text{(A2.2.8)}$$

$$Q_n(\zeta,C_1) = Q_{n\_p}(\zeta,C_2)\cdot Q_{n\_n}(\zeta,C_2) = \prod_{j=1}^{\infty}\left[1 - i\cdot\frac{\zeta}{c_{2_{(j)}}}\right]\cdot\prod_{j=1}^{\infty}\left[1 + i\cdot\frac{\zeta}{c_{2_{(j)}}}\right] = \prod_{j=1}^{\infty}\left[1 + \left[\frac{\zeta}{c_{2_{(j)}}}\right]^2\right]$$

$$R_p(\zeta,D_1) = R_{p\_p}(\zeta,D_2)\cdot R_{p\_n}(\zeta,D_2) = \zeta\cdot\prod_{j=1}^{\infty}\left[1 - \frac{\zeta}{d_{2_{(j)}}}\right]\cdot\prod_{j=1}^{\infty}\left[1 + \frac{\zeta}{d_{2_{(j)}}}\right] = \zeta\cdot\prod_{j=1}^{\infty}\left[1 - \left[\frac{\zeta}{d_{2_{(j)}}}\right]^2\right]$$

$$R_n(\zeta,D_1) = R_{n\_p}(\zeta,D_2)\cdot R_{n\_n}(\zeta,D_2) = \zeta\cdot\prod_{j=1}^{\infty}\left[1 - i\cdot\frac{\zeta}{d_{2_{(j)}}}\right]\cdot\prod_{j=1}^{\infty}\left[1 + i\cdot\frac{\zeta}{d_{2_{(j)}}}\right] = \zeta\cdot\prod_{j=1}^{\infty}\left[1 + \left[\frac{\zeta}{d_{2_{(j)}}}\right]^2\right]$$

With the definitions (A2.1.10) and (A2.2.8) it may be written for the resulting functions listed in lemma 2.1:

$$P_{p\_n}(\zeta,A_2) + P_{p\_p}(\zeta,A_2) = \prod_{k=0}^{\infty}\left[1+\frac{\zeta}{a_{2_{(j)}}}\right] + \prod_{k=0}^{\infty}\left[1-\frac{\zeta}{a_{2_{(j)}}}\right] = 2\cdot\prod_{k=0}^{\infty}\left[1+\left[\frac{\sigma}{c_{2_{(j)}}}\right]^2\right] = 2\cdot Q_n(\sigma,C_2)$$

(A2.2.9)

$$P_{p\_n}(\zeta,A_2) - P_{p\_p}(\zeta,A_2) = \prod_{k=0}^{\infty}\left[1+\frac{\zeta}{a_{2_{(j)}}}\right] - \prod_{k=0}^{\infty}\left[1-\frac{\zeta}{a_{2_{(j)}}}\right] = 2\cdot i\cdot\tau\cdot\prod_{j=1}^{\infty}\left[1+\left[\frac{\tau}{d_{2_{(j)}}}\right]^2\right] = 2\cdot i\cdot R_n(\tau,D_2)$$

$$P_{n\_n}(\zeta,A_2) - P_{n\_p}(\zeta,A_2) = \prod_{k=0}^{\infty}\left[1-i\cdot\frac{\zeta}{a_{2_{(j)}}}\right] - \prod_{k=0}^{\infty}\left[1+i\cdot\frac{\zeta}{a_{2_{(j)}}}\right] = 2\cdot i\cdot\tau\cdot\prod_{j=1}^{\infty}\left[1-\left[\frac{\tau}{d_{2_{(j)}}}\right]^2\right] = 2\cdot i\cdot R_p(\tau,D_2)$$

$$P_{n\_n}(\zeta,A_2) + P_{n\_p}(\zeta,A_2) = \prod_{k=0}^{\infty}\left[1-i\cdot\frac{\zeta}{a_{2_{(j)}}}\right] + \prod_{k=0}^{\infty}\left[1+i\cdot\frac{\zeta}{a_{2_{(j)}}}\right] = = 2\cdot\prod_{j=1}^{\infty}\left[1-\left[\frac{\sigma}{c_{2_{(j)}}}\right]^2\right] = 2\cdot Q_p(\sigma,C_2)$$

Addition and subtraction of these above equations with the definitions (A2.1.10) and (A2.2.8) give:

$$P_{p\_p}(\zeta,A_2) = \prod_{k=0}^{\infty}\left[1-\frac{\zeta}{a_{2_{(j)}}}\right] = \prod_{k=0}^{\infty}\left[1+\left[\frac{\sigma}{c_{2_{(j)}}}\right]^2\right] + i\cdot\left[\tau\cdot\prod_{j=1}^{\infty}\left[1+\left[\frac{\tau}{d_{2_{(j)}}}\right]^2\right]\right] = Q_n(\sigma,C_2) + i\cdot R_n(\tau,D_2)$$

$$P_{p\_n}(\zeta,A_2) = \prod_{k=0}^{\infty}\left[1+\frac{\zeta}{a_{2_{(j)}}}\right] = \prod_{k=0}^{\infty}\left[1+\left[\frac{\sigma}{c_{2_{(j)}}}\right]^2\right] - i\cdot\left[\tau\cdot\prod_{j=1}^{\infty}\left[1+\left[\frac{\tau}{d_{2_{(j)}}}\right]^2\right]\right] = Q_n(\sigma,C_2) - i\cdot R_n(\tau,D_2)$$

(A2.2.10)

$$P_{n\_n}(\zeta,A_2) = \prod_{k=0}^{\infty}\left[1-i\cdot\frac{\zeta}{a_{2_{(j)}}}\right] = \prod_{j=1}^{\infty}\left[1-\left[\frac{\sigma}{c_{2_{(j)}}}\right]^2\right] + i\cdot\tau\cdot\left[\prod_{j=1}^{\infty}\left[1-\left[\frac{\tau}{d_{2_{(j)}}}\right]^2\right]\right] = Q_p(\sigma,D_2) + i\cdot R_p(\tau,C_2)$$

$$P_{n\_p}(\zeta,A_2) = \prod_{k=0}^{\infty}\left[1+i\cdot\frac{\zeta}{a_{2_{(j)}}}\right] = \prod_{j=1}^{\infty}\left[1-\left[\frac{\sigma}{c_{2_{(j)}}}\right]^2\right] - i\cdot\tau\cdot\prod_{j=1}^{\infty}\left[1-\left[\frac{\tau}{d_{2_{(j)}}}\right]^2\right] = Q_p(\sigma,C_2) - i\cdot R_p(\tau,D_2)$$

Definition of the generalized relations Pythagoras:

Multiplying these above equations pair wise with each other gives relations, which may be regarded - by the same reasoning as at lemma A2.1 - as the **generalized relations of Pythagoras:**

$$P_n(\sigma,A_1) = P_{n\_p}(\sigma,A_2)\cdot P_{n\_n}(\sigma,A_2) = Q_p(\sigma,C_1)^2 + R_p(\sigma,D_1)^2$$

$$P_p(\tau,A_1) = P_{p\_p}(\tau,A_2)\cdot P_{p\_n}(\tau,A_2) = Q_n(\tau,C_1)^2 + R_n(\tau,D_1)^2$$

(A2.2.11)

Similarly to (A2.2.7) equations (A2.2.10) correspond to setting the original functions ($P_{n\_p}(\zeta,A_2)$), and ($P_{n\_n}(\zeta,A_2)$) resp. ($P_{p\_p}(\zeta,A_2)$), and ($P_{p\_n}(\zeta,A_2)$) equal to zero giving equations defining roots on the adjoint axes: ($P_{n\_p}(\zeta,A_2) = 0$) and ($P_{n\_n}(\zeta,A_2) = 0$) define imaginary, ($P_{p\_p}(\zeta,A_2) = 0$) and ($P_{p\_n}(\zeta,A_2) = 0$) define real roots. Written in the quotient form for the adjoint functions these equations define roots on the adjoint axes. For the real roots of the adjoint functions defining imaginary roots for the original function, resp. the imaginary roots of the adjoint functions defining real roots for the original function the quotient equations are:

$$\frac{Q_p(\zeta,C_1)}{R_p(\zeta,D_1)} = \frac{1}{\zeta}\cdot\prod_{j=1}^{\infty}\frac{1-\left[\frac{\zeta}{c_{2_{(j)}}}\right]^2}{1-\left[\frac{\zeta}{d_{2_{(j)}}}\right]^2} = -i \text{ or } = i \quad\text{and}\quad \frac{Q_n(\zeta,C_1)}{R_n(\zeta,D_1)} = \frac{1}{\zeta}\cdot\prod_{j=1}^{\infty}\frac{1+\left[\frac{\zeta}{c_{2_{(j)}}}\right]^2}{1+\left[\frac{\zeta}{d_{2_{(j)}}}\right]^2} = i \text{ or } = -i$$

(A2.2.12)

Therefore to a polynom with all positive real roots composing the vector (A_2) with equations (A2.2.1) there are adjoint vectors (C_2) and (D_2) defined. Between the original vector and the adjoint vectors there exist relations given in lemma A2.1, which may be regarded as generalized relations from Euler, an inherent relation of split complex polynoms. The addition and subtraction of these equations results equation, which multiplied with each other gives the relations (A2.2.11), which may be regarded as the generalized relationships of Pythagoras, as well an inherent relation of split complex polynoms.

The polynoms defined with the adjoint vectors (C_1) and (D_1) are the adjoint polynoms to the original polynom defined with the vector (A_1). The components of the vector (A_1) on the imaginary axes ($i\cdot A_1$) taken as constants of the original polynoms are the adjoint polynoms to the polynoms defined with the components of the vectors (C_1) and (D_1) on the real axes.

With all this the following lemma may be formulated:

Lemma A2.2:

Polynom quotient equations of the adjoint functions with all roots on one axes define the roots of the original function on the adjoint axes.

Proof: The quotient equations (A2.2.12) define roots of the original function on the adjoint axes, as stated in the lemma, concluding the proof.

All rules deduced from the polynom quotient equations for the placement of the roots on the adjoint axes are valid for the inverse variable ($\xi = 1/s$) as well, since ($b_{(j)} = 1/a_{(j)}$) is as well a sets of positive real numbers:

$$\frac{f(s)}{f(-s)} = \prod_{j=1}^{\infty} \frac{s - a_{(j)}}{s + a_{(j)}} = \prod_{j=1}^{\infty} \frac{b_{(j)} - \xi}{b_{(j)} + \xi} = \frac{F(\xi)}{F(-\xi)} = 1 \qquad \text{(A2.2.13)}$$

Generally, if a polynom quotient equation with the positive real numbers ($a_{(j)}$) defines roots on one of the adjoint axes, say on the imaginary axes, than the same equation with the imaginary numbers ($i\cdot a_{(j)}$) define roots on the adjoint, on the real axes:

$$\frac{f(i\cdot x)}{f(-i\cdot x)} = \prod_{j=1}^{\infty} \frac{i\cdot x - a_{(j)}}{i\cdot x + a_{(j)}} = 1 \qquad \frac{f(x)}{f(-x)} = \prod_{j=1}^{\infty} \frac{x - i\cdot a_{(j)}}{x + i\cdot a_{(j)}} = 1 \qquad \text{(A2.2.14)}$$

Inversely, what can be sad about the roots ($r_{(j)}$) of a function ($f(s) = \Pi(r_{(j)} - s)$) if it is known, that for the function the polynom quotient equations are valid?

$$\frac{f(s)}{f(-s)} = \prod_{j=1}^{\infty} \frac{s - r_{(j)}}{s + r_{(j)}} = 1 \qquad \frac{f(s)}{f(-s)} = \prod_{j=1}^{\infty} \frac{s - r_{(j)}}{s + r_{(j)}} = -1 \qquad \text{(A2.2.15)}$$

If no constants are known, it is not possible to define by this equations the placement of the roots: with equations (A2.2.1) they may be all on the imaginary axes or all on the real axes, or an infinity of roots may be on both axes. But all roots are on one of the adjoint axes in case of ($r_{(j)} = a_{(j)}$) or ($r_{(j)} = i\cdot a_{(j)}$), thus the roots are uniform.

These relations (A2.2.15) requires the following equations for the real and for the imaginary components::

$\mathrm{Re}(f(s)) = \mathrm{Re}(f(-s))$ and $\mathrm{Im}(f(s)) = \mathrm{Im}(f(-s))$ or $\mathrm{Re}(f(s)) = -\mathrm{Re}(f(-s))$ and $= -\mathrm{Im}(f(s))$ (A2.2.16)

With all this the following lemma may be formulated:

Lemma A2.3:

Polynom quotient equations with all imaginary or with all real constant roots define roots exactly on one of the axes: they are all either imaginary or all real.

Proof: The quotients in equation (A2.2.15) may be equal to unity resp. to minus one only in case the numerator and the denominator are conjugate complex resp. transpose, thus the roots defined by the equations has to be exactly on one of the axes, concluding the proof.

With (A2.1.17) and (A2.1.20) the coordinate transformation shifting the roots of the second degree split polynoms from the imaginary axes to the critical line ($\frac{b}{2} + i\cdot\tau$), respectively shifting the roots on the real axes by ($\frac{b}{2}$) by the coordinate transformation ($s = \zeta + \frac{b}{2}$) yields the roots:

$$\left(r_{tr_{(j)}} = \frac{b}{2} + i\cdot a_{(j)}\right),\ \left(r_{tr_{(j)}} = \frac{b}{2} - i\cdot a_{(j)}\right),\ \left(r_{tr_{(j)}} = \frac{b}{2} + a_{(j)}\right) \text{ and } \left(r_{tr_{(j)}} = \frac{b}{2} - a_{(j)}\right) \qquad \text{(A2.2.17)}$$

With this transformation the point of central symmetry is shifted from the origin to the point ($\sigma = \frac{b}{2}$).

Because of the symmetry condition with (A2.2.1) the quotient equations defining the roots of the symmetric and of the sque symmetric adjoint functions on the adjoint axes will be:

$$P_{n\_p\_tr}(r, R_{tr}) - P_{n\_n\_tr}(r, R_{tr}) = 0 \text{ and } P_{n\_p\_tr}(r, R_{tr}) + P_{n\_n\_tr}(r, R_{tr}) = 0$$

respectively: (A2.2.18)

$$P_{n\_p\_tr}(r, R_{tr}) - P_{n\_n\_tr}(r, R_{tr}) = 0 \text{ and } P_{n\_p\_tr}(r, R_{tr}) + P_{n\_n\_tr}(r, R_{tr}) = 0$$

Inserting resulting the infinite polynom polynom products (A1.1.19) yields the polynom quotient equations with the roots either on the critical line or on the real axes:

$$\frac{\prod_{j=1}^{\infty}\left[1 - \frac{s}{r_{tr_{(j)}}}\right]}{\prod_{j=1}^{\infty}\left[1 - \frac{b-s}{r_{tr_{(j)}}}\right]} = 1 \text{ and } \frac{\prod_{j=1}^{\infty}\left[1 - \frac{s}{r_{tr_{(j)}}}\right]}{\prod_{j=1}^{\infty}\left[1 - \frac{b-s}{r_{tr_{(j)}}}\right]} = -1 \qquad \text{(A2.2.19)}$$

Thus, any infinite polynom having polynom quotient equations of similar form, will have all the roots either on the critical line ($\frac{b}{2} + i\cdot\tau$) or on the real axes within the transformed coordinate system, corresponding to roots either on the imaginary axes or on the real axes within the original coordinate system.

Lemma A2.4:

The roots of polynoms in a transformed coordinate system, the transformation being a shift of the imaginary axes parallel by a constant value ($\frac{b}{2}$) are exclusively on the critical line ($\frac{b}{2} + i\cdot\tau$), if the roots of the same polynoms in the original coordinate system are all on the imaginary axes.

Proof: The roots of a polynom being exclusively on the imaginary axes are shifted by the transformation of the coordinate system to the critical line parallel to the imaginary axes at the distance ($\frac{b}{2}$), if the transformation is the shifting of the imaginary axes parallel by the same distance, concluding the proof.

A2.3. The exponential and the trigonometric functions

In this and in the next chapters different infinite sets of positive integers composing the vector (A) and therewith the corresponding polynom will be considered. The polynoms will be identified as known analytical functions with their adjoint vectors (C_2) and (D_2). As a first example the sets of positive integers ($a_2(k,n)$) - defined in (A1.3.17) - composing the vector (A_2) with ($k = 0, 1, 2.. n$) are:

$$a_2(k,n) = \frac{\left(2\cdot k + \sqrt{n}\right)^2}{2\cdot\sqrt{n}} \qquad \lim_{n\to\infty} a_2(k,n) = \infty \tag{A2.3.1}$$

With equations (A1.3.19) and (A1.3.20) the corresponding second degree split polynoms with the constants ($a_2(n)_{(k)}$) are the exponential functions written as infinite products:

$$P_{p\_p\_e}(\zeta, A_2) = e^{-\sigma} = \lim_{n\to\infty} \prod_{k=0}^{n} \left(1 - \frac{\sigma}{a_2(k,n)}\right)$$

$$P_{p\_n\_e}(\zeta, A_2) = e^{\sigma} = \lim_{n\to\infty} \prod_{k=0}^{n} \left(1 - \frac{\sigma}{-a_2(k,n)}\right) = \lim_{n\to\infty} \prod_{k=0}^{n} \left(1 + \frac{\sigma}{a_2(k,n)}\right) \tag{A2.3.2}$$

$$P_{n\_p\_e}(\zeta, A_2) = e^{-i\cdot\tau} = \lim_{n\to\infty} \prod_{k=0}^{n} \left(1 - \frac{\tau}{i\cdot a_2(k,n)}\right)$$

$$P_{n\_n\_e}(\zeta, A_2) = e^{i\cdot\tau} = \lim_{n\to\infty} \prod_{k=0}^{n} \left(1 - \frac{\tau}{-i\cdot a_2(k,n)}\right)$$

The corresponding first degree split polynoms have unity as limit value, specially for this function:

$$P_{p\_e}(\zeta_1, A_1) = P_{p\_p\_e}(\zeta, A_2)\cdot P_{p\_n\_e}(\zeta, A) = e^{-\zeta}\cdot e^{\zeta} = \lim_{n\to\infty} \prod_{k=0}^{n} \left(1 - \frac{\zeta_1}{a_2(k,n)^2}\right) = 1$$

$$P_{n\_e}(\zeta_1, A_1) = P_{n\_p\_e}(\zeta, A_2)\cdot P_{n\_n\_e}(\zeta, A_2) = e^{-i\cdot\zeta}\cdot e^{i\cdot\zeta} = \lim_{n\to\infty} \prod_{k=0}^{n} \left(1 + \frac{\zeta_1}{a_2(k,n)^2}\right) = 1 \tag{A2.3.3}$$

The polynom quotient equations (A2.2.7) written for these polynoms give:

$$\frac{P_{p\_p}(\zeta, A_2)}{P_{p\_n}(\zeta, A_2)} = \lim_{n\to\infty} \prod_{k=0}^{n} \left(\frac{a_2(k,n) - \zeta}{a_2(k,n) + \zeta}\right) = -1 \text{ or } = 1$$

$$\frac{P_{n\_p}(\zeta, A_2)}{P_{n\_n}(\zeta, A_2)} = \lim_{n\to\infty} \prod_{k=0}^{n} \left(\frac{i\cdot a_2(k,n) - \zeta}{i\cdot a_2(k,n) + \zeta}\right) = -1 \text{ or } = 1 \tag{A2.3.4}$$

For ($\zeta_1 = i\cdot\tau$) the zeros of the adjoint functions ($Q_p(\tau, C_1)$) resp. ($R_p(\tau, D_1)$) are at odd resp. at even multiples of ($\frac{\pi}{2}$). In fact, the convergence of the zeros define ($\frac{\pi}{2}$). These odd resp. even multiples of ($\frac{\pi}{2}$) are with lemma A2.1 the roots of the adjoint function. They are the constants ($c_{2_{(j)}}$) and ($d_{2_{(j)}}$) with ($j = 1, 2.. n$) composing the vectors (C_2) and (D_2):

$$c_{2_{(j)}} = (2\cdot j - 1)\cdot\frac{\pi}{2} \qquad d_{2_{(j)}} = (2\cdot j)\cdot\frac{\pi}{2} = c_{2_{(j)}} + \frac{\pi}{2} \tag{A2.3.5}$$

The corresponding second degree split polynoms are:

$$Q_{p\_p\_e}(\zeta, C_2) = \prod_{j=1}^{\infty} \left[1 - \frac{\zeta}{c_{2_{(j)}}}\right] \qquad R_{p\_p\_e}(\zeta, D_2) = \sqrt{\zeta} \cdot \prod_{j=1}^{\infty} \left[1 - \frac{\zeta}{d_{2_{(j)}}}\right]$$

$$Q_{p\_n\_e}(\zeta, C_2) = \prod_{j=1}^{\infty} \left[1 - \frac{\zeta}{-c_{2_{(j)}}}\right] \qquad R_{p\_n\_e}(\zeta, D_2) = \sqrt{\zeta} \cdot \prod_{j=1}^{\infty} \left[1 - \frac{\zeta}{-d_{2_{(j)}}}\right] \qquad \text{(A2.3.6)}$$

$$Q_{n\_p\_e}(\zeta, C_2) = \prod_{j=1}^{\infty} \left[1 - \frac{\zeta}{i \cdot c_{2_{(j)}}}\right] \qquad R_{n\_p\_e}(\zeta, D_2) = \sqrt{\zeta} \cdot \prod_{j=1}^{\infty} \left[1 - \frac{\zeta}{i \cdot d_{2_{(j)}}}\right]$$

$$Q_{n\_n\_e}(\zeta, C_2) = \prod_{j=1}^{\infty} \left[1 - \frac{\zeta}{-i \cdot c_{2_{(j)}}}\right] \qquad R_{n\_n\_e}(\zeta, D_2) = \sqrt{\zeta} \cdot \prod_{j=1}^{\infty} \left[1 - \frac{\zeta}{-i \cdot d_{2_{(j)}}}\right]$$

The corresponding first degree split functions gives with (A1.5.2) and (A1.5.3) the cosine resp. the sine functions:

$$Q_{p\_e}(\sigma, C_1) = Q_{p\_p\_e}(\sigma, C_2) \cdot Q_{p\_n\_e}(\sigma, C_2) =$$

$$= \prod_{j=1}^{\infty} \left[1 - \frac{\sigma}{c_{2_{(j)}}}\right] \cdot \prod_{j=1}^{\infty} \left[1 + \frac{\sigma}{c_{2_{(j)}}}\right] = \prod_{j=1}^{\infty} \left[1 - \left(\frac{\sigma}{2 \cdot j - 1} \cdot \frac{2}{\pi}\right)^2\right] = \cos(\sigma)$$

$$Q_{n\_e}(\tau, C_1) = Q_{n\_p\_e}(\tau, C_2) \cdot Q_{n\_n\_e}(\tau, C_2) = \prod_{j=1}^{\infty} \left[1 + \left(\frac{\tau}{2 \cdot j - 1} \cdot \frac{2}{\pi}\right)^2\right] = \cos(i \cdot \tau)$$

$$\text{(A2.3.7)}$$

$$R_{p\_e}(\sigma, D_1) = R_{p\_p\_e}(\sigma, D_2) \cdot R_{p\_n\_e}(\sigma, D_2) = \sigma \cdot \prod_{j=1}^{\infty} \left[1 - \left(\frac{\sigma}{2 \cdot j} \cdot \frac{2}{\pi}\right)^2\right] = \sin(\sigma)$$

$$R_{n\_e}(\tau, D_1) = R_{n\_p\_e}(\tau, D_2) \cdot R_{n\_n\_e}(\tau, D_2) = \tau \cdot \left[\prod_{j=1}^{\infty} \left[1 + \left(\frac{\tau}{2 \cdot j} \cdot \frac{2}{\pi}\right)^2\right]\right] = \sin(i \cdot \tau)$$

Taking the roots ($c_{2_{(j)}}$) resp. ($d_{2_{(j)}}$) composing the vectors (C_2) and (D_2) as adjoint vectors of the vector (A_2), results with the equations of lemma 2.1 the relations of Euler specially for the exponential function:

$$2 \cdot Q_{n\_e}(\tau, C_1) = 2 \cdot \prod_{j=1}^{\infty} \left[1 + \left[\frac{\tau}{c_{2_{(j)}}}\right]^2\right] = 2 \cdot \cos(i \cdot \tau) = 2 \cdot \cosh(\sigma) = e^{\sigma} + e^{-\sigma} =$$

$$= \lim_{n \to \infty} \left[\prod_{k=0}^{n} \left(1 + \frac{\sigma}{a_2(k,n)}\right) + \prod_{k=0}^{n} \left(1 - \frac{\sigma}{a_2(k,n)}\right)\right] = P_{p\_n\_e}(\sigma, A_2) + P_{p\_p\_e}(\sigma, A_2)$$

$$2 \cdot Q_{p\_e}(\sigma, C_1) = 2 \cdot \prod_{j=1}^{\infty} \left[1 - \left[\frac{\sigma}{c_{2_{(j)}}}\right]^2\right] = 2 \cdot \cos(\sigma) = 2 \cdot \cosh(i \cdot \tau) = e^{i \cdot \tau} + e^{-i \cdot \tau} = \qquad \text{(A2.3.8)}$$

$$= \lim_{n \to \infty} \left[\prod_{k=0}^{n} \left(1 + i \cdot \frac{\tau}{a_2(k,n)}\right) + \prod_{k=0}^{n} \left(1 - i \cdot \frac{\tau}{a_2(k,n)}\right)\right] = P_{n\_n\_e}(\tau, A_2) + P_{n\_p\_e}(\tau, A_2)$$

$$2\cdot i\cdot R_{n\_e}(\tau, D_1) = 2\cdot i\cdot\tau\cdot\prod_{j=1}^{\infty}\left[1+\left[\frac{\tau}{d_{2_{(j)}}}\right]^2\right] = 2\cdot i\cdot\sin(i\cdot\tau) = 2\cdot\sinh(\sigma) = e^{\sigma} - e^{-\sigma} =$$

$$= \lim_{n\to\infty}\left[\prod_{k=0}^{n}\left(1+\frac{\sigma}{a_2(k,n)}\right) - \prod_{k=0}^{n}\left(1-\frac{\sigma}{a_2(k,n)}\right)\right] = P_{p\_n\_e}(\sigma, A_2) - P_{p\_p\_e}(\sigma, A_2)$$

(A2.3.8)

$$2\cdot i\cdot R_{p\_e}(\sigma, D_1) = 2\cdot i\cdot\sigma\cdot\prod_{j=1}^{\infty}\left[1-\left[\frac{\sigma}{d_{2_{(j)}}}\right]^2\right] = 2\cdot i\cdot\sin(\sigma) = 2\cdot\sinh(i\cdot\tau) = e^{i\cdot\tau} - e^{-i\cdot\tau} =$$

$$= \lim_{n\to\infty}\left[\prod_{k=0}^{n}\left(1+i\cdot\frac{\tau}{a_2(k,n)}\right) - \prod_{k=0}^{n}\left(1-i\cdot\frac{\tau}{a_2(k,n)}\right)\right] = P_{n\_n\_e}(\tau, A_2) - P_{n\_p\_e}(\tau, A_2)$$

The trigonometric functions are therefore the adjoint functions for the exponential function. The other relations of Euler resulting from equations (A2.2.10) result the exponential function as the sum of the adjoint split polynoms:

$$Q_{n\_e}(\tau, C_1) - i\cdot R_{n\_e}(\tau, D_1) = \prod_{j=1}^{\infty}\left[1+\left[\frac{\tau}{c_{2_{(j)}}}\right]^2\right] - i\cdot\tau\cdot\prod_{j=1}^{\infty}\left[1+\left[\frac{\tau}{d_{2_{(j)}}}\right]^2\right] =$$

$$= \cos(i\cdot\tau) - i\cdot\sin(i\cdot\tau) = \cosh(\sigma) + \sinh(\sigma) = e^{\sigma} = \lim_{n\to\infty}\prod_{k=0}^{n}\left(1+\frac{\sigma}{a_2(k,n)}\right) = P_{p\_n\_e}(\sigma, A_2)$$

$$Q_{n\_e}(\tau, C_1) + i\cdot R_{n\_e}(\tau, D_1) = \prod_{j=1}^{\infty}\left[1+\left(\frac{\tau}{c_{2_j}}\right)^2\right] + i\cdot\tau\cdot\prod_{j=1}^{\infty}\left[1+\left[\frac{\tau}{d_{2_{(j)}}}\right]^2\right] =$$

$$= \cos(i\cdot\tau) + i\cdot\sin(i\cdot\tau) = \cosh(\sigma) - \sinh(\sigma) = e^{-\sigma} = \lim_{n\to\infty}\prod_{k=0}^{n}\left(1-\frac{\sigma}{a_2(k,n)}\right) = P_{p\_p\_e}(\sigma, A_2)$$

$$Q_{p\_e}(\sigma, C_1) + i\cdot R_{p\_e}(\sigma, D_1) = \prod_{j=1}^{\infty}\left[1-\left[\frac{\sigma}{c_{2_{(j)}}}\right]^2\right] + i\cdot\sigma\cdot\prod_{j=1}^{\infty}\left[1-\left[\frac{\sigma}{d_{2_{(j)}}}\right]^2\right] =$$ (A2.3.9)

$$= \cos(\sigma) + i\cdot\sin(\sigma) = \cosh(i\cdot\tau) + \sinh(i\cdot\tau) = e^{i\cdot\tau} = \lim_{n\to\infty}\prod_{k=0}^{n}\left(1+i\cdot\frac{\tau}{a_2(k,n)}\right) = P_{n\_n\_e}(\tau, A_2)$$

$$Q_{p\_e}(\sigma, C_1) - i\cdot R_{p\_e}(\sigma, D_1) = \prod_{j=1}^{\infty}\left[1-\left[\frac{\sigma}{c_{2_{(j)}}}\right]^2\right] - i\cdot\sigma\cdot\prod_{j=1}^{\infty}\left[1-\left[\frac{\sigma}{d_{2_{(j)}}}\right]^2\right] =$$

$$= \cos(\sigma) - i\cdot\sin(\sigma) = \cosh(i\cdot\tau) - \sinh(i\cdot\tau) = e^{-i\cdot\tau} = \lim_{n\to\infty}\prod_{k=0}^{n}\left(1-i\cdot\frac{\tau}{a_2(k,n)}\right) = P_{n\_p\_e}(\tau, A_2)$$

The definitions of the trigonometric functions and equation (A2.3.3) may be generalized for functions of the adjoint variables too. Taking as imaginary roots $(i\cdot f(\tau)/c_{(j)})$, resp. $(i\cdot f(\tau)/d_{(j)})$ they correspond with (A2.3.7) to the following relations of the trigonometric resp. of the exponential function:

$$\cosh(f(\tau)) = \cos(i\cdot f(\tau)) = \prod_{j=0}^{\infty}\left[1+\left(\frac{f(\tau)}{2\cdot j-1}\cdot\frac{2}{\pi}\right)^2\right] = 0 \qquad \frac{e^{f(\zeta)}}{e^{-f(\zeta)}} = -1$$

(A2.3.10)

$$\sinh(f(\tau)) = -i\cdot\sin(i\cdot f(\tau)) = -i\cdot f(\tau)\cdot\prod_{j=1}^{\infty}\left[1+\left(\frac{f(\tau)}{2\cdot j}\cdot\frac{2}{\pi}\right)^2\right] = 0 \qquad \frac{e^{f(\zeta)}}{e^{-f(\zeta)}} = 1$$

With (A2.2.12) the following quotient equations of the first degree split polynoms ($Q_{n\_e}(\sigma, C_1)$) and ($R_{n\_e}(\sigma, D_1)$) with imaginary roots define real roots for the original exponential function ($P_{p\_n\_e}(\sigma, A_2)$) and ($P_{p\_p\_e}(\sigma, A_2)$):

$$\frac{Q_{n\_e}(s)}{R_{n\_e}(s)} = \frac{1}{s}\cdot\prod_{j=1}^{\infty}\frac{1+\left[\frac{s}{c_{2(j)}}\right]^2}{1+\left[\frac{s}{d_{2(j)}}\right]^2} = -i \qquad \frac{Q_{n\_e}(s)}{R_{n\_e}(s)} = \frac{1}{s}\cdot\prod_{j=1}^{\infty}\frac{1+\left[\frac{s}{c_{2(j)}}\right]^2}{1+\left[\frac{s}{d_{2(j)}}\right]^2} = i \qquad \text{(A2.3.11)}$$

and similarly the first degree split polynoms ($Q_{p\_e}(\sigma, C_1)$) and ($R_{p\_e}(\sigma, D_1)$) with real roots define imaginary roots for the original exponential function ($P_{n\_n\_e}(\sigma, A_2)$) and ($P_{n\_p\_e}(\sigma, A_2)$):

$$\frac{Q_{p\_e}(s)}{R_{p\_e}(s)} = \frac{1}{s}\cdot\prod_{j=1}^{\infty}\frac{1-\left[\frac{s}{c_{2(j)}}\right]^2}{1-\left[\frac{s}{d_{2(j)}}\right]^2} = -i \qquad \frac{Q_{p\_e}(s)}{R_{p\_e}(s)} = \frac{1}{s}\cdot\prod_{j=1}^{\infty}\frac{1-\left[\frac{s}{c_{2(j)}}\right]^2}{1-\left[\frac{s}{d_{2(j)}}\right]^2} = i \qquad \text{(A2.3.12)}$$

These equations (A2.3.11) define real roots in the infinite positive and negative, resp. equations (A2.3.12) define imaginary roots for the second degree split functions (A2.3.2), for the exponential functions.

With the relation (A1.5.3) the sines function may be expressed as shifted cosines functions:

$$\cos[i\cdot(\tau-1)] = \prod_{j=1}^{\infty}\left[1+\left(\frac{\tau-1}{2\cdot j-1}\cdot\frac{2}{\pi}\right)^2\right] = i\cdot t\cdot\prod_{j=1}^{\infty}\left[1+\left(\frac{\tau}{2\cdot j}\cdot\frac{2}{\pi}\right)^2\right] = \sin(i\cdot\tau)$$

$$\cos(\sigma-1) = \prod_{j=1}^{\infty}\left[1-\left(\frac{\sigma-1}{2\cdot j-1}\cdot\frac{2}{\pi}\right)^2\right] = \sigma\cdot\prod_{j=1}^{\infty}\left[1-\left(\frac{\sigma}{2\cdot j}\cdot\frac{2}{\pi}\right)^2\right] = \sin(\sigma) \qquad \text{(A2.3.13)}$$

Therefore the sines function may be expressed by the shifted cosines function for real arguments:

$$e^{\sigma} = \cos(i\cdot\sigma) - i\cdot\cos[i\cdot(\sigma-1)] = \prod_{j=1}^{\infty}\left[1+\left[\frac{\sigma}{c_{2(j)}}\right]^2\right] - i\cdot\prod_{j=1}^{\infty}\left[1+\left[\frac{\sigma-1}{c_{2(j)}}\right]^2\right]$$

$$e^{-\sigma} = \cos(i\cdot\sigma) + i\cdot\cos[i\cdot(\sigma-1)] = \prod_{j=1}^{\infty}\left[1+\left[\frac{\sigma}{c_{2(j)}}\right]^2\right] + i\cdot\prod_{j=1}^{\infty}\left[1+\left[\frac{\sigma-1}{c_{2(j)}}\right]^2\right] \qquad \text{(A2.3.14)}$$

and for imaginary arguments similarly:

$$e^{i\cdot\tau} = \cos(\tau) + i\cdot\cos(\tau-1) = \prod_{j=1}^{\infty}\left[1-\left[\frac{\tau}{c_{2(j)}}\right]^2\right] + i\cdot\prod_{j=1}^{\infty}\left[1-\left[\frac{\tau-1}{c_{2(j)}}\right]^2\right]$$

$$e^{-i\cdot\tau} = \cos(\tau) - i\cdot\cos(\tau-1) = \prod_{j=1}^{\infty}\left[1-\left[\frac{\tau}{c_{2(j)}}\right]^2\right] - i\cdot\prod_{j=1}^{\infty}\left[1-\left[\frac{\tau-1}{c_{2(j)}}\right]^2\right] \qquad \text{(A2.3.15)}$$

With (A2.3.13) and (A2.3.14) the quotient equations (A2.3.11), (A2.3.12) defining the real roots in the infinite and the imaginary roots in the infinite for the exponential function are:

$$\prod_{j=1}^{\infty}\frac{1+\left[\frac{\sigma}{c_{2(j)}}\right]^2}{1+\left[\frac{\sigma-1}{c_{2(j)}}\right]^2} = i \quad \text{resp.} \quad = -i \quad \text{and} \quad \prod_{j=1}^{\infty}\frac{1-\left[\frac{\tau}{c_{2(j)}}\right]^2}{1-\left[\frac{\tau-1}{c_{2(j)}}\right]^2} = -i \quad \text{resp.} \quad = i \qquad \text{(A2.3.16)}$$

According to (A2.2.11) the relations of Pythagoras are with (A2.3.9):

$$P_{n\_e}(\sigma, A_1) = P_{n\_p\_e}(\sigma, A_2) \cdot P_{n\_n\_e}(\sigma, A_2) = Q_{p\_e}(\sigma, C_1)^2 + R_{p\_e}(\sigma, D_1)^2$$

$$1 = e^{i \cdot \sigma} \cdot e^{-i \cdot \sigma} = \cos(\sigma)^2 + \sin(\sigma)^2 \qquad \text{(A2.3.17)}$$

$$P_{p\_e}(\tau, A_1) = P_{p\_p\_e}(\tau, A_2) \cdot P_{p\_n\_e}(\tau, A_2) = Q_{n\_e}(\tau, C_1)^2 + R_{n\_e}(\tau, D_1)^2$$

$$1 = e^{\tau} \cdot e^{-\tau} = \cos(i \cdot \tau)^2 + \sin(i \cdot \tau)^2$$

Shifting the roots of the exponential function ($i \cdot a_2(k, n)$) and ($-i \cdot a_2(k, n)$) from the imaginary axes to the critical line ($r_{tr_{(k)}} = \frac{1}{2} + i \cdot a_2(k, n)$), ($r_{tr_{(k)}} = \frac{1}{2} - i \cdot a_2(k, n)$), than the polynom quotient equation (A2.3.4) of the exponential function, - which is defining the roots of the adjoint functions on the real axes - will be with (A2.2.19):

$$\lim_{n \to \infty} \prod_{k=0}^{n} \left[\frac{1 - \frac{s}{r_{tr_{(k)}}}}{1 - \frac{1 - s}{r_{tr_{(k)}}}} \right] = \lim_{n \to \infty} \prod_{k=0}^{n} \left(\frac{1 - \frac{s}{\frac{1}{2} + i \cdot a_2(k, n)}}{1 - \frac{1 - s}{\frac{1}{2} - i \cdot a_2(k, n)}} \right) = -1 \quad \text{or} \quad = 1 \qquad \text{(A2.3.18)}$$

The constants being with (A2.3.1):

$$a_2(k, n) = \frac{\left(2 \cdot k + \sqrt{n}\right)^2}{2 \cdot \sqrt{n}}$$

Similarly the polynom quotient equation (A2.3.12), - which is defining the roots of the exponential function on the critical line - will be:

$$\frac{1}{1 - s} \cdot \prod_{j=1}^{\infty} \frac{1 - \left[\frac{s}{\frac{1}{2} + c_{2_{(j)}}} \right]^2}{1 - \left[\frac{1 - s}{\frac{1}{2} + d_{2_{(j)}}} \right]^2} = \frac{1}{1 - s} \cdot \prod_{j=1}^{\infty} \frac{1 - \left[\frac{s}{\frac{1}{2} + c_{2_{(j)}}} \right]^2}{1 - \left[\frac{1 - s}{\frac{1}{2} + d_{2_{(j)}}} \right]^2} = -i \quad \text{or} = 1 \qquad \text{(A2.3.19)}$$

The constants being with (A2.3.5):

$$c_{2_{(j)}} = (2 \cdot j - 1) \cdot \frac{\pi}{2} \qquad d_{2_{(j)}} = (2 \cdot j) \cdot \frac{\pi}{2} = c_{2_{(j)}} + \frac{\pi}{2}$$

A2.4 The gamma function

The definition of the gamma function as infinite product (A1.5.7) with roots at odd integers is:

$$P_{p\_p\_\Gamma}(\zeta) = \lim_{n\to\infty}\left[\prod_{k=0}^{n}\left(1+\frac{\zeta\cdot\ln(\sqrt{n})}{2\cdot a(k,n)}\right)\cdot\prod_{j=0}^{\sqrt{n}}\frac{2\cdot j+1-\zeta}{2\cdot j+1}\right] =$$

$$= P_{p\_n\_e}\left(\frac{\zeta\cdot\ln(\sqrt{n})}{2},A\right)\cdot Q_{p\_p\_e}\left(\zeta\cdot\frac{\pi}{2},C\right) = \lim_{n\to\infty}\left(e^{\frac{\zeta\cdot\ln(\sqrt{n})}{2}}\cdot\prod_{j=0}^{\sqrt{n}}\frac{2\cdot j+1-\zeta}{2\cdot j+1}\right) = \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1-\zeta}{2}\right)}$$

$$P_{p\_n\_\Gamma}(\zeta) = \lim_{n\to\infty}\left[\prod_{k=0}^{n}\left(1-\frac{\zeta\cdot\ln(\sqrt{n})}{2\cdot a(k,n)}\right)\cdot\prod_{j=0}^{\sqrt{n}}\frac{2\cdot j+1+\zeta}{2\cdot j+1}\right] = \qquad \text{(A2.4.1)}$$

$$= P_{p\_p\_e}\left(\frac{\zeta\cdot\ln(\sqrt{n})}{2},A\right)\cdot Q_{p\_n\_e}\left(\zeta\cdot\frac{\pi}{2},C\right) = \lim_{n\to\infty}\left(e^{-\frac{\zeta\cdot\ln(\sqrt{n})}{2}}\cdot\prod_{j=0}^{\sqrt{n}}\frac{2\cdot j+1+\zeta}{2\cdot j+1}\right) = \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1+\zeta}{2}\right)}$$

and of the delta function as required by symmetry for roots at even integers and with (A1.5.13) replacing the split sine function by the shifted split cosine function:

$$P_{p\_p\_\Delta}(\zeta) = \lim_{n\to\infty}\left[\left[\prod_{k=0}^{n}\left(1+\frac{\zeta\cdot\ln(\sqrt{n})}{2\cdot a(k,n)}\right)\right]\cdot\left(\prod_{j=0}^{\sqrt{n}}\frac{2\cdot j+1-\zeta}{2\cdot j+1}\right)\right] =$$

$$= P_{p\_n\_e}\left(\frac{\zeta\cdot\ln(\sqrt{n})}{2},A\right)\cdot R_{p\_p\_e}\left(\zeta\cdot\frac{\pi}{2},C\right) = \lim_{n\to\infty}\left(e^{\frac{\zeta\cdot\ln(\sqrt{n})}{2}}\cdot\prod_{j=0}^{\sqrt{n}}\frac{2\cdot j+1-\zeta}{2\cdot j+1}\right) = \frac{\Delta\left(\frac{1}{2}\right)}{\Delta\left(\frac{1-\zeta}{2}\right)}$$

$$P_{p\_n\_\Delta}(\zeta) = \lim_{n\to\infty}\left[\prod_{k=0}^{n}\left(1-\frac{\zeta\cdot\ln(\sqrt{n})}{2\cdot a(k,n)}\right)\cdot\left(\prod_{j=0}^{\sqrt{n}}\frac{2\cdot j+1+\zeta}{2\cdot j+1}\right)\right] = \qquad \text{(A2.4.2)}$$

$$= P_{p\_p\_e}\left(\frac{\zeta\cdot\ln(\sqrt{n})}{2},A\right)\cdot R_{p\_n\_e}\left(\zeta\cdot\frac{\pi}{2},C\right) = \lim_{n\to\infty}\left(e^{-\frac{\zeta\cdot\ln(\sqrt{n})}{2}}\cdot\prod_{j=0}^{\sqrt{n}}\frac{2\cdot j+1+\zeta}{2\cdot j+1}\right) = \frac{\Delta\left(\frac{1}{2}\right)}{\Delta\left(\frac{1+\zeta}{2}\right)}$$

Herewith the gamma and the delta functions are combined functions: they have real roots at odd, resp. even, positive resp. negative integers like the trigonometric functions, and further real roots in the infinite like the exponential function. The two split components in (A2.4.1) resp. (A2.4.2) multiplied give the known relation for the cosines function, resp. a corresponding relation for the sines function:

$$\cos\left(\zeta\cdot\frac{\pi}{2}\right) = P_{p\_p\_\Gamma}(\zeta)\cdot P_{p\_n\_\Gamma}(\zeta) = \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1+\zeta}{2}\right)}\cdot\frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1-\zeta}{2}\right)} \qquad \text{(A2.4.3)}$$

$$\sin\left(\zeta\cdot\frac{\pi}{2}\right) = P_{p\_p\_\Delta}(\zeta)\cdot P_{p\_n\_\Delta}(\zeta) = \frac{\Delta\left(\frac{1}{2}\right)}{\Delta\left(\frac{1+\zeta}{2}\right)}\cdot\frac{\Delta\left(\frac{1}{2}\right)}{\Delta\left(\frac{1-\zeta}{2}\right)}$$

Taking the real roots of the gamma functions, with (A2.2.8) their quotient equations define roots on the adjoint, on the imaginary axes for the adjoint functions ($R_{n\_\Gamma}(\zeta)$) and ($Q_{n\_\Gamma}(\zeta)$):

$$\frac{P_{p\_p\_\Gamma}(\zeta)}{P_{p\_n\_\Gamma}(\zeta)} = \frac{\Gamma\left(\frac{1+\zeta}{2}\right)}{\Gamma\left(\frac{1-\zeta}{2}\right)} = 1 \qquad \frac{P_{p\_p\_\Gamma}(\zeta)}{P_{p\_n\_\Gamma}(\zeta)} = \frac{\Gamma\left(\frac{1+\zeta}{2}\right)}{\Gamma\left(\frac{1-\zeta}{2}\right)} = -1 \qquad \text{(A2.4.4)}$$

and similarly of the quotient equations of the delta functions define roots on the adjoint, on the imaginary axes for the adjoint functions ($Q_{n\_\Delta}(\zeta)$) and ($R_{n\_\Delta}(\zeta)$):

$$\frac{P_{p\_p\_\Delta}(\zeta)}{P_{p\_n\_\Delta}(\zeta)} = \frac{\Delta\left(\frac{1+\zeta}{2}\right)}{\Delta\left(\frac{1-\zeta}{2}\right)} = -1 \qquad \frac{P_{p\_p\_\Delta}(\zeta)}{P_{p\_n\_\Delta}(\zeta)} = \frac{\Delta\left(\frac{1+\zeta}{2}\right)}{\Delta\left(\frac{1-\zeta}{2}\right)} = 1 \qquad \text{(A2.4.5)}$$

With lemma A2.2 the polynom quotient equations of these adjoint functions define the roots of the original functions.

With (A2.4.1) it may be written for the polynom quotient equation:

$$\frac{\Gamma\left(\frac{1+\zeta}{2}\right)}{\Gamma\left(\frac{1-\zeta}{2}\right)} = \lim_{n\to\infty}\left[\prod_{k=0}^{n}\left(\frac{\frac{2\cdot a(k,n)}{\ln(\sqrt{n})}+\zeta}{\frac{2\cdot a(k,n)}{\ln(\sqrt{n})}-\zeta}\right)\cdot\prod_{j=0}^{\sqrt{n}}\frac{2\cdot j+1-\zeta}{2\cdot j+1+\zeta}\right] \qquad \text{(A2.4.6)}$$

In case the roots on the imaginary axes are all shifted to the critical line, with (A2.2.19) and with lemma A2.4. the polynom quotient equation takes the form:

$$\frac{\Gamma\left[\frac{1-(1-s)}{2}\right]}{\Gamma\left(\frac{1-s}{2}\right)} = \lim_{n\to\infty}\left[\prod_{k=0}^{n}\left(\frac{1-\frac{s}{\frac{1}{2}+i\cdot\frac{2\cdot a(k,n)}{\ln(\sqrt{n})}}}{1-\frac{1-s}{\frac{1}{2}-i\cdot\frac{2\cdot a(k,n)}{\ln(\sqrt{n})}}}\right)\cdot\prod_{j=0}^{\sqrt{n}}\frac{1-\frac{1-s}{\frac{1}{2}-i\cdot(2\cdot j+1)}}{1-\frac{s}{\frac{1}{2}+i\cdot(2\cdot j+1)}}\right] \qquad \text{(A2.4.7)}$$

A2.5. Product quotient equation for the Riemann zeta function

With (A1.6.4) from annex 1 and with A2.2.7 the product polynom of the Riemann zeta function may be written with ($b(k,q) = \frac{(2\cdot k + \sqrt{q})^2}{2\cdot\sqrt{q}}$, $q = 1,2,3..\infty$, $k = 0,1,2..\infty$) as follows:

$$\zeta(s) = \prod_{n=1}^{\infty}\left[1 - \frac{1}{\left[P_{(n)}\right]^s}\right]^{-1} = \zeta_1(s)\cdot\zeta_2(s) = \frac{\prod_{n=1}^{\infty}\lim_{q\to\infty}\prod_{k=0}^{q}\left[1 + \frac{s\cdot\ln\left[P_{(n)}\right]}{b(k,q)}\right]}{\left[\prod_{n=1}^{\infty}\left[2\cdot i\cdot\sin\left[\frac{s}{2}\cdot\ln\left[P_{(n)}\right]\right]\right]\right]} \quad \text{(A2.5.1)}$$

The polynom quotient equations written for the first polynom ($\zeta_1(s)$) gives with (A2.3.4):

$$\frac{\prod_{n=1}^{\infty}\lim_{q\to\infty}\prod_{k=0}^{q}\left[1 - \frac{s\cdot\ln\left[P_{(n)}\right]}{b(k,q)}\right]}{\prod_{n=1}^{\infty}\lim_{q\to\infty}\prod_{k=0}^{q}\left[1 + \frac{s\cdot\ln\left[P_{(n)}\right]}{b(k,q)}\right]} = -1 \quad \text{or} \quad = 1 \quad \text{(A2.5.2)}$$

Similarly the polynom quotient equations written for the second polynom ($\zeta_2(s)$) gives with (A2.3.7):

$$\zeta_2(s) = \frac{1}{\left[\prod_{n=1}^{\infty}\left[2\cdot i\cdot\sin\left[\frac{s}{2}\cdot\ln\left[P_{(n)}\right]\right]\right]\right]} = \frac{1}{\left[\prod_{n=1}^{\infty}\left[2\cdot i\cdot s\cdot\ln\left(P_n\right)\cdot\left[\prod_{j=1}^{\infty}\left[1 - \left(\frac{s}{j\cdot\pi}\cdot\ln\left(P_n\right)\right)^2\right]\right]\right]\right]}$$

$$\frac{\prod_{n=1}^{\infty}\left[\prod_{j=1}^{\infty}\left[1 + \left(\frac{s}{j\cdot\pi}\cdot\ln\left(P_n\right)\right)^2\right]\right]}{\prod_{n=1}^{\infty}\left[\prod_{j=1}^{\infty}\left[1 - \left(\frac{s}{j\cdot\pi}\cdot\ln\left(P_n\right)\right)^2\right]\right]} = -1 \quad \text{or} \quad = 1 \quad \text{(A2.5.3)}$$

Both polynom quotient equations of the components of the Riemann zeta function have with lemma A2.3 roots exclusively on the adjoint axes, on the imaginary and/or on the real axes.

In case the roots on the imaginary axes are all shifted to the critical line, with (A2.2.19) and with lemma A2.4. the polynom quotient equation of the Riemann zeta function takes the form:

$$\frac{\prod_{n=1}^{\infty}\lim_{q\to\infty}\prod_{k=0}^{q}\left[1 - \frac{s}{\frac{1}{2} + i\cdot\frac{b(k,q)}{\ln\left[P_{(n)}\right]}}\right]}{\prod_{n=1}^{\infty}\lim_{q\to\infty}\prod_{k=0}^{q}\left[1 - \frac{(1-s)}{\frac{1}{2} - i\cdot\frac{b(k,q)}{\ln\left[P_{(n)}\right]}}\right]}\cdot\frac{\prod_{n=1}^{\infty}\left[\prod_{j=1}^{\infty}\left[1 - \left(\frac{1-s}{\frac{1}{2} - i\cdot\frac{j\cdot\pi}{\ln\left(P_n\right)}}\right)^2\right]\right]}{\prod_{n=1}^{\infty}\left[\prod_{j=1}^{\infty}\left[1 - \left(\frac{s}{\frac{1}{2} + i\cdot\frac{j\cdot\pi}{\ln\left(P_n\right)}}\right)^2\right]\right]} = -1 \quad \text{or} \quad = 1 \quad \text{(A2.5.4.)}$$