The critical line and the roots of the Riemann Zeta-function

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The basic idea is to express all components of the functional equation of the Riemann ζ-function as infinite products, with all components of the infinite products having roots exclusively on the critical line and on the real axes.

Starting with the functional equation of the Riemann ζ-function, it is shown, that all of its components may be written as infinite products resulting infinite polynoms. All roots of these polynoms in the central form are on the imaginary axes and on the real axes. If the roots on the imaginary axes are shifted to the critical line, than the resulting form is identical with the functional equation: Thus, the roots of all components of the components of the functional equation - written as infinite products - have roots on the critical line and/or on the real axes.

Key words: Riemann Zeta- Function

1. The functional equation and infinite product series

The critical line of the the Riemann $\zeta(\sigma)$-function is defined as follows:

$$s = \sigma + i \cdot \tau = \frac{1}{2} + i \cdot \tau \ ; \ \sigma = \frac{1}{2} \ ; \ \sigma = 1 - \sigma \tag{1.1}$$

The functional equation of the Riemann $\zeta(s)$-function /2/ results from the extension of the validity of the function by algebraic continuation. It may be written as follows:

$$\xi(s) = \Gamma\left(\frac{s}{2}\right) \cdot \pi^{-\frac{s}{2}} \cdot \zeta(s) = \Gamma\left(\frac{1-s}{2}\right) \cdot \pi^{-\frac{1-s}{2}} \cdot \zeta(1-s) = \xi(1-s)$$

$$\frac{\xi(s)}{\xi(1-s)} = \frac{\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{1-s}{2}\right)} \cdot \frac{\pi^{-\frac{s}{2}}}{\pi^{-\frac{1-s}{2}}} \cdot \frac{\zeta(s)}{\zeta(1-s)} = 1 \tag{1.2}$$

This equation states, that ($\xi(s)$) is central symmetrical over ($\sigma = 1/2$):

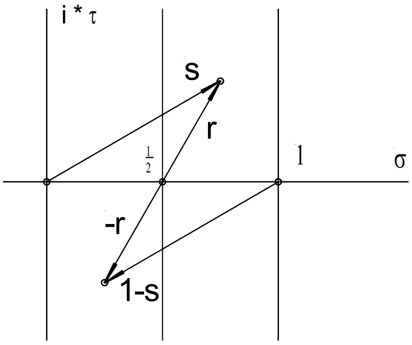

Figure 1.1: The symmetry conditions of the ξ(s)-function

It can be easily shown, that all components of the above functional equation of the Riemann Zeta function may be written as polynoms with all roots on the real and on the imaginary axes. Further, it can be shown, that any polynoms having roots on the imaginary and on the real axes have - after the coordinate transformation shifting all the roots from the imaginary axes to the critical line - the resulting form is identical with the functional equation.

In a first step it can be shown, that in case a polynom has roots on the positive imaginary axes and its conjugate complex pair has roots on the negative imaginary axes, its roots are defined by the polynom quotient equations of the form:

$$\frac{\prod_{j=1}^{\infty}\left[1-\frac{r}{i\cdot a_{(j)}}\right]}{\prod_{j=1}^{\infty}\left[1-\frac{r}{-i\cdot a_{(j)}}\right]} = -1 \text{ or } = 1 \tag{1.3}$$

Than it can be shown, that after shifting all roots of the polynoms in the polynom quotient equations to the critical line ($r_{tr_{(j)}} = \frac{1}{2} + i\cdot a_{(j)}$) respectively ($r_{tr_{(j)}} = \frac{1}{2} - i\cdot a_{(j)}$) these equations have the form:

$$\frac{\prod_{j=1}^{\infty}\left[1-\frac{s}{r_{tr_{(j)}}}\right]}{\prod_{j=1}^{\infty}\left[1+\frac{1-s}{r_{tr_{(j)}}}\right]} = \frac{\prod_{j=1}^{\infty}\left[1-\frac{s}{\frac{1}{2}+i\cdot a_{(j)}}\right]}{\prod_{j=1}^{\infty}\left[1+\frac{1-s}{\frac{1}{2}-i\cdot a_{(j)}}\right]} = 1 \text{ or } = -1 \tag{1.4}$$

Thus, if the function ($\xi(s)$) may be written as a polynom, with all roots on the imaginary and/or on the real axes, and the shift of all roots parallel to the real axes by ($\sigma = 1/2$),to the critical line, results the functional equation, than the functional equation defines all roots on the critical line and/or on the real axes, but nowhere else.

It can be shown, the shifting of all roots of the gamma function to the critical line results for the first component in (1.2) as following form :

$$\frac{\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{1-s}{2}\right)} = \lim_{q\to\infty}\left[\prod_{k=0}^{n}\frac{1-\frac{s}{\frac{1}{2}+i\cdot\frac{2\cdot b(k,q)}{\ln(\sqrt{q})}}}{1-\frac{1-s}{\frac{1}{2}-i\cdot\frac{2\cdot b(k,q)}{\ln(\sqrt{q})}}}\cdot\prod_{j=0}^{\sqrt{n}}\left[\frac{1-\frac{1-s}{\frac{1}{2}-i\cdot(2\cdot j+1)}}{1-\frac{s}{\frac{1}{2}+i\cdot(2\cdot j+1)}}\right]\right] \tag{1.5}$$

with the constants ($b(k,q)$) being with (A2.3.1)) positive real numbers:

$$b(k,q) = \frac{\left(2\cdot k+\sqrt{q}\right)^2}{2\cdot\sqrt{q}} \tag{1.6}$$

It can be shown, that the roots of the exponential function - in the central form being on the imaginary axes - shifted to the critical line results for the second component in (1.2) the following form:

$$\frac{\pi^{-\frac{s}{2}}}{\pi^{-\frac{1-s}{2}}} = \frac{e^{-\frac{s\cdot\ln(\pi)}{2}}}{e^{-\frac{(1-s)\cdot\ln(\pi)}{2}}} = \lim_{q\to\infty}\prod_{k=0}^{q}\frac{1-\frac{s}{\frac{1}{2}+i\cdot\frac{2\cdot b(k,q)}{\ln(\pi)}}}{1+\frac{1-s}{\frac{1}{2}-i\cdot\frac{2\cdot b(k,q)}{\ln(\pi)}}} \tag{1.7}$$

Similarly it can be shown, that the third component in (1.2) - the Riemann Zeta function ($\zeta(s)$), in the central form having all roots on the imaginary and on the real axes - have, after the shifting of the roots from the imaginary axes to the critical line, the form of the following polynom coefficient equation:

$$\frac{\zeta(s)}{\zeta(1-s)} = \frac{\displaystyle\prod_{n=1}^{\infty} \lim_{q \to \infty} \prod_{k=0}^{q} \left[1 - \frac{s}{\frac{1}{2} + i \cdot \frac{b(k,q)}{\ln\left[P_{(n)}\right]}}\right]}{\displaystyle\prod_{n=1}^{\infty} \lim_{q \to \infty} \prod_{k=0}^{q} \left[1 - \frac{(1-s)}{\frac{1}{2} - i \cdot \frac{b(k,q)}{\ln\left[P_{(n)}\right]}}\right]} \cdot \frac{\displaystyle\prod_{n=1}^{\infty} \left[\prod_{j=1}^{\infty} \left[1 - \left(\frac{1-s}{\frac{1}{2} - i \cdot \frac{j \cdot \pi}{\ln\left(P_n\right)}}\right)^2\right]\right]}{\displaystyle\prod_{n=1}^{\infty} \left[\prod_{j=1}^{\infty} \left[1 - \left(\frac{s}{\frac{1}{2} + i \cdot \frac{j \cdot \pi}{\ln\left(P_n\right)}}\right)^2\right]\right]} = -1 \text{ or } = 1 \qquad (1.8)$$

With (1.5), (1.7) and with (1.8) the functional equation of the ζ-function defines roots exclusively on the critical line and on the real axes. Details of the transformation into the different forms are given in the annexes.

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Annexes

Annex 1: The binomial coefficients and the normal distribution

Annex 2: Splitting of polynoms

The annexes are available upon request.

References

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