On the Complex Zeros of the Riemann Zeta Function

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Abstract: A mathematical proof is only true if the proof can be reproducible, and perhaps by alternative means than that employed in the first proof. A proof of the Riemann Hypothesis should be generalizable because there exists zeta functions such as the Dedekind zeta function, Dirichlet series, generalized zeta functions, and L-Functions. Although we do not consider here the generalized zeta functions, it is my goal to show the reader that this proof of the Riemann Hypothesis is generalizable in that the method can be extended to analyze general function expansions, and that a complete understanding of the RZF's connection to the Fourier Transform could one day yield a better understanding of generalized signal distributions.

I define the RZF here:

$$\zeta(s) \equiv \left(\prod_{n=1}^{\infty} \frac{1}{\left(1 - \frac{1}{{p_n}^s}\right)} = \sum_{n=1}^{\infty} n^{-s} \right)$$

Where $p_n$is the nth prime number. The proof of the Hypothesis is located at: § 4.1

Introduction:

Like most proofs of the Riemann Hypothesis which have existed until today's date which are available on the internet, the aim of the current is to develop a function which has the same zeros of the RZF so that we can compare the two functions in such a manner that we may deduce that the critical value x = 1/2 is the coordinate from which the only line in a (x,iy) coordinate plane which extends towards positive and negative infinity may be the locations by which the Real and Imaginary components of ζ, U(x,y) and V(x,y) respectively, simultaneously intersect zero on a 3-D graph. Very obviously ζ's zeros are a function of four coordinates (U,iV,x,y), and so this is graphically truly a 5-D analysis. In § 1, there will be a research section concerning the current attempts at proofs, in addition to general research about the RZF. In § 2, a discussion of the Fourier and Laplace transformation for linear differential equations and the relationship to the RZF. The RZF is an entire, meromorphic function and therefore is an analytic function for all values with s≠ 1. Some of the infinitude of these relationships are located in § 3 which has some values, numerical relationships, and analytic methods are tabulated within. In § 4**,** we assume the knowledge of the previous four sections to be true and develop an argument of symmetry for the proof in § 4.1, that the Critical line is the only location that the complex zeros of the RZF could lie.

We seek to produce an algorithm which, given the numerical operations that define a "truth," which makes the respective "distance" between the definitions equal to zero. The concept of "Analytic continuation" is a mathematical definition which is a numerical transformation from one definition to another. By defining all variables of the RZF with respect to one another, an image can be drawn, and in the case of imagining the complex $\zeta(x + iy)$, a truly 5-D dimensional plot is necessary to understand the function. The way I imagined a 5-D plot is discussed here, as nothing more than the simultaneous comparison of other 3-D graphs. The fact that ζ has these properties implies that the properties formed within the derivation of the function compose those properties, for example, analyticity, implies that the set of functions which is defined to form the proof that are those definitions must also be constituent components of the definition of the RZF. By extending this gate of logic and defining well as many of the properties of the RZF, we ask ourselves what the definition of analyticity means, supposing that the RZF is defined as the convergence of two functions, namely, the analytically continued (AC[]) Euler Product (EP), the function which is defined as the set of operations that is $\zeta(x + iy)$, and the original function before numerical transformation, the EP, $\zeta(x)$. There is a notational confliction here, and it is because the EP is defined over ℝ and the RZF is defined over ℂ. This shows us that simply speaking, analytic continuation is the transformation of a function which is defined with only ℝ numbers to a function defined over a ℂ domain. Thus, there are one more coordinates from which the RZF is defined than with respect to the EP(x). There is in a sense, a loss of information, it would seem, as one transforms from the EP to the RZF through the operation that is analytic continuation. However, this is not true because the RZF possesses specific symmetric properties due to the path taken in order to define the RZF. The RZF is known to obey the functional equation, as it is defined though the functional equation, $\xi(s) = \frac{1}{2}\pi^{-\frac{x+iy}{2}}\Gamma\left(\frac{x+iy}{2}\right)\zeta(x+iy) = \frac{1}{2}\pi^{-\frac{(1-x+iy)}{2}}\Gamma\left(\frac{1-x+iy}{2}\right)\zeta(1-x+iy) = \xi((s') = 1-s))$.

This definition holds that s $= x + iy$ and for that reason, I define the primed values, s' = = x' + iy' $= 1 - x - iy$. The primed coordinate is not a definition, as the functional equation was derived, that is, it is manufactured through manipulation of the integrand which defines the RZF. This inherently requires that there exists a symmetric set of primed functions, functions which are symmetric with respect to the unprimed functions.

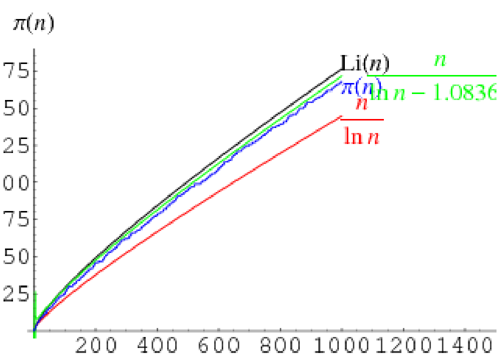
§ 1

RZF Research Section

Using the numerical properties of n^{-s}, Riemann in his final manuscript writes how to Analytically continue the EP $(\sum_{n=1}^{\infty} n^{-x})$; a function with x ϵ ℝ such that $-\infty \leq x \leq \infty$, which is divergent ∀ x < 0, convergent for 0 < x < 1, divergent at the singularity at x = 1, and convergent ∀ x < 1; to a function which is absolutely convergent for all Complex values s(x', y') = x(x', y') + iy(x', y'), other than the singularity at s = 1, where the line s = 1 + iy does not make the RZF zero. With the functional equation, it can be shown that the "Trivial Zeros" are located at s_k = (-2,-4,-6,-8…) + i0. The values of the RZF for x > 1 will not be zero for all y because the analytically continuous RZF has been transformed to by the EP continuously. The RZF [1] for s = 2 +i0 is $\frac{\pi^2}{6}$ [2], for s = 3 +i0 ≈ 1.202 and decreases in value with increasingly higher x values. As x ⟶ ∞, ζ ⟶ 1, $\zeta(x_k + iy_k) \neq 0 \;\; \forall \;\; x_k > 1, \; y_k > 0$. Also, because $\zeta(x) = \infty$ ∀ x < 0, all of the Complex zeros of the RZF, $x_k + iy_k$, must lie within the Critical Strip.

Riemann constructs the functional equation in his 1859 manuscript through the integral definition of n^{-s}, and the use of the geometric series. He also discusses what it means to define the integral in the "positive sense" versus the "negative sense," where he is seemingly implying with these terms, the direction of integration over the contour which has an infinitely large imaginary value. One of the main goals it seems of Riemann's final paper was to investigate the distribution of the Prime Numbers and prove Gauss' hypothesis, namely that the Prime Number Counting function, $\pi(\sigma) \approx \frac{\sigma}{\log(\sigma)} as\ \sigma \longrightarrow \infty$, which he obtained from the tabulation of the frequencies of prime numbers within intervals of a thousand by hand. From Riemann's work, the Prime Number Theorem [3], see figure one for a visual. $\lim_{x \to \infty} \frac{\pi(x)}{li(x)} = 1$, has apparently been proven by Hadamard and de la Vallée Poussin, due to his exploration of the Logarithmic Integral, Li(x^s)= $\lim_{\alpha \to 0}\left(\int_0^{1-\alpha} \frac{\sigma^{s-1} d\sigma}{\ln(\sigma)} + \int_{1+\alpha}^{x} \frac{\sigma^{s-1} d\sigma}{\ln(\sigma)}\right)$, (This definition can be found on Page two of "Spectral Analysis and the Riemann Hypothesis," written by Gilles Lachaud, and another Math world search [4,5]) by proving that $\zeta(1 + y)$ is not equal to zero for any y.

FIGURE ONE



(FIGURE ONE) This image is from Wolfram Math world [3])

Evidently, Riemann defined the RZF with $\Pi(s-1)$ in his 1859 manuscript with: $\Pi(s-1)\zeta(s) = \int_0^{\infty} \sigma^{s-1} e^{-s} d\sigma$ and shows that because $\int_{\infty}^{\infty} \frac{(-\sigma)^{s-1}}{e^{\sigma}-1} d\sigma = (e^{-\pi s i} - e^{\pi s i})\Gamma(s)$, the Trivial Zeros must be located at the negative even integers because then, $2\sin(\pi s)\,\Pi(s-1)\zeta(s) = i\int_{\infty}^{\infty} \frac{(-\sigma)^{s-1}}{e^{\sigma}-1} d\sigma$. Due to the symmetry embedded in

Classical and Quantum Systems; like Spring Forces (-kx) of R. Hookes Spring Experiments which have squared functions of velocity for the Lagrangian that describes the sinusoidally varying solutions to the resulting second order differential equation from the application of the Euler-Lagrange minimization techniques, and the Harmonic Oscillator which requires discrete increments of energy ($\hbar\omega$) to go from one state of the wave function to the next combined with the fact that there are analytic and algebraic methods to arrive to the same solution (e.g, as shown by David J Griffiths in "Introduction to Quantum Mechanics," Chapter 2.3) [6]); it seems that manufacturing the RZF in such a manner that it follows the mathematical rules within an arbitrarily defined Pseudo-Physics analysis would be easier to exploit the properties of symmetry of the RZF. This is true, it seems, that the symmetry of the RZF becomes more apparent, but also more apparent is the fact that the ideas become much more difficult to comprehend at times because the notations themselves are defined to be contrary to the known working laws of mathematics, but are in some way related. The notation in the fields, when comparing the methods in any two fields effectively amplifies the difficulty of analysis, as inherent in expanding on previously nonexistent thoughts, so the route, may be good if one wants to see beauty, but if one wishes to show colleagues, it is best, it seems, to keep separate the fields and talk about one field at once. It is quickly clear the relationship of the symmetries of QM to the RZF, so it is rather possible that a quantum physicist will find some technique to locate the zeros of the RZF or any other complex function, but my focus here is not specifically the precise location of the zeros, but rather the location of the real part of the zeros with respect to the imaginary part of the zeros. Viewing this hypothetical system in the language of Physics is known to be done with Spectral Analysis, and similarly, we will work to develop an intuitive image of the RZF henceforth.

Upon analyzing the functional equation of $\zeta(x+iy)$ with respect to $\zeta\big((x'+iy') = (1-x-iy)\big)$ which defines ξ for derived in B. Riemann's 1859 manuscript: $\xi(y) = \frac{1}{2}\pi^{-\frac{x_k+iy}{2}}\Gamma\left(\frac{x_k+iy}{2}\right)\zeta(x_k+iy) = \frac{1}{2}\pi^{-\frac{(1-x_k+iy)}{2}}\Gamma\left(\frac{1-x_k+iy}{2}\right)\zeta(1-x_k+iy) = \xi(1-y)$, two functions manufactured through the definitions of the RZF, the Critical Line seemed to suggest a certain symmetry about it, and I previously knew from some route I created that the zeros are symmetric with respect to the Critical Line, and that the Riemann Hypothesis necessitates that $\zeta(x_k+iy_k)$ has all $\mathbb{R}eal$ y_k, or that $y_k = y'_k + i(x'_k = 0)$. Before finding this fact to be true, I was considering functional equations for Xi such as $\xi(s(x,y)) = \xi(s'(x',y'))$, for s' which is some coordinate transformation of s because the primed coordinates x' and y' are with respect to x and y. From this conception of coordinate transformation, the RZF's functional equation can be written as $\zeta(s(x,y)) = F(x,y,x',y')\zeta(s'(x',y'))$. If we view the complex numbers as two-vectors, we can see that $\zeta(s(x,y))$ is a ∞ x ∞ Matrix of numbers (see fact #8) which transforms the vector <x, iy> to a new two vector, and that if we consider coordinate transformations of the unprimed coordinate system to the primed coordinate system the functional equation provides a $F(x,y,x',y')$ such that $\zeta(s'(x',y')$ can be related to the infinite matrix of numbers representative of the unprimed coordinates' frame. This is exceedingly obvious because like the Taylor expansion, there exist infinitely many values of s to write the power series, with respect to.

FIGURE TWO

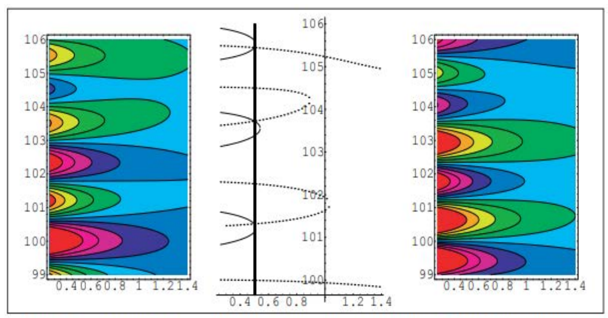


Figure 2. Contour plot of $\Re\zeta(s)$, the curves $\Re\zeta(s) = 0$ (solid) and $\Im\zeta(s) = 0$ (dotted), contour plot of $\Im\zeta(s)$.

The above and below image of the Real and Imaginary components of the RZF contour and 3-d plots are beautifully depicted from Brian Conrey's work [7].

FIGURE THREE

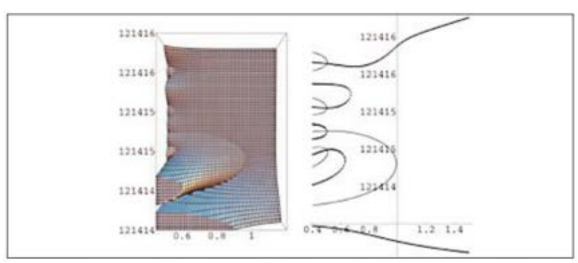


Figure 3. 3-D plot of $|\Re\zeta(s)|$, and the curves $\Re\zeta(s) = 0$ (solid) and $\Im\zeta(s) = 0$ (dotted). This may be the first place in the critical strip where the curves $\Re\zeta(s) = 0$ loop around each other.

Firstly, this is a short paper which attempts to describe a difficult and lengthy topic which has still yet much room to expand. This presentation could not possibly contain all of the possible relationships to the RZF because of the mathematical concepts necessary to build the function, but I will present a new outlook on the theory of Numbers from a perspective of Physics in hope that other researchers will find this method advantageous or supplementary. From such knowledge, relationships to physics concepts, engineering concepts, and concepts in virtually every mathematical field can be seen with this single function, where it is not explicitly from the connection to the distribution of Prime numbers, but because of the advanced mathematics, and years of mathematical thought necessary to even consider thinking about the function. With this being said, it has been more challenging to discipline myself not think about other various mathematical relationships to the analysis of the entire class of functions known as "zeta functions," than it has been to consider how to approach a proof that the zeros of the RZF all share the mutual Real value , $x_k = \frac{1}{2}$. I have searched for and reviewed many failed proofs, attempted proofs, and alternate ways of viewing the RZF, like that of the Todd Function introduced from the very knowledgeable Sir Michael Atiyah, a fields medalist who has contributed to Number Theory and Topology [8], individuals -whose lifelong pursuit of mathematical learning whom I have contacted via email- who have worked many years on an approach to the RH with the residue theorem (which seemed like an interesting approach, however, which is outside of my knowledge), and the re-description of the RH with Jensen Polynomials by Michael Griffin, Ken Ono, Larry Rolen, and Don Zagier [9]. Atiyah looked to build separate a function to compare the RZF to, and similarly I do that here. Griffin et. al. have shown that the RH can be defined in a way which could also exploit the symmetry properties of the RZF. According to their work, "Expanding on notes of Jensen, P´olya proved that the Riemann Hypothesis (RH) is equivalent to the hyperbolicity of the Jensen polynomials for the Riemann zeta function ζ(s) at its point of symmetry." This approach, more formally looks to defined a new function which can be compared to the RZF, where the property which is to be investigated is hyperbolicity because the hyperbolicity evidently tells one about the zeros of the RZF. Other generalized RZF analyses such as author Brandon Fodden has done in his research article on the Diophantine Equations [10].

The RH is best stated by Enrico Bombieri [11], a Fields Medalist who has work in many fields of mathematics, and member of Clay Mathematics Institute and who is extraordinarily knowledgeable about the RH, discusses B. Riemann's 1859 manuscript available on Clay Mathematics' Website [12]; this single webpage has Riemann's original manuscript [13], an English and German Translation by David Wilkins [14,14.1] and an investigation of the fair copy of Riemann's 1859 manuscript [15] found in "Riemann's Nachlass [16]," by Wolfgang Gabcke. Bombieri discusses the history of the RZF, its applications, its generalizations (L-Functions), and many of its relations throughout mathematics, and evidence of the RH. After discussing the first series of proofs with my friend, and first Calculus professor, Dr. Zachlin, he pointed out a vital flaw in my logic. The first proofs I had constructed using Physics were not general enough, and although they applied in specific instances, could not apply in all instances, and could not satisfy the RH due to the specific assumptions I made (analytically continuous Complex functions can be written as a product of their zeros, so infinitely many complex functions that are transformations of the RZF might have its zeros). I have trialed a wide selection of coordinate bases, and how the RZF operates to transform from one Complex coordinate definition to another in addition to performing various different polynomial and function expansions of the terms of the RZF such as the Taylor Theorem. I have sampled various different families of functions similar to the RZF, some which are symmetric and antisymmetric, and some which are inverse functions, in addition to applying many different aspects of the Physics, Mathematics, and Engineering toolkit from my undergraduate courses at Lakeland Community College, Cleveland State University, and multiple different Calculus textbooks and additional textbooks within S.T.E.M, which I read only for my joy of mathematics thus far, but I can safely say that none of the advanced and undergraduate material I have seen could be rigidly applied to formulate a proof of the RH. It took for me to forward my understanding of space-time to see further the greater implications of the RH. With this being said, I believe I have sourced *why* the RZF has zeros at the Critical Line ([17] illustration and information by Eric Weinstein) only, and my claim is that within the Critical Strip, it is the only x coordinate which allots for the symmetry condition that the zeros obey.

Authors such as Tatenda Kubalalika [18] have devised approaches similar in nature to that which I had originally developed in 2018, and which are similar to what sir Michael Atiyah tried to do with the development of a weakly analytic function he called the "Todd Function." Many of the attempts I have seen otherwise seem to try to manufacture a function, as Tatenda did with the "Prime zeta Function," however, as Dr. Zachlin easily pointed out to me with my logical fallacy with such a construction, there exist infinitely many functions which would satisfy the general criteria, and so the function's vital properties are assumptions in the line of logic thereafter, and therefore, is not proving anything. It became immediately obvious to me then that the only way that a symmetric function which could be analyzed in this respect would be one which is arrived to after a derivation of the application of n mathematical theorems or operations. With this route, one does not assume a function meets the properties, but rather one builds a function with logic by changing the respective meanings in the describing differential equations and polynomial expressions of the functions. In a sense, along the lines of deriving the RZF from the relations which underlie the EP, it is like there is a sort of algebraic normalization which defines the RZF.

It is this experience which led me to try to simply physically double the number of equations, perform a Taylor Series Expansion on both the primed equations and the unprimed equations, and what you realize quickly is that something does not seem right, and that would be correct. With such a derivation, you find that you are comparing the EP(x) to the RZF(x,y), where clearly you lack the full amount of variables to transport the surface plot of RZF(x,y) "backwards" to the EP(x), all you can do is traverse along the real line with y' =0, x' =x, however, what happens when y' $\neq$0? Very obviously the analysis becomes much more complicated because EP(x) does not lie within ℂ, while RZF(x,y) ∈ ℂ for infinitely many values. Therefore, in some respect, the primed variables are defined as ℂ variable with respect to the unprimed variables. It becomes clear with this simple thought experiment, the comparison of the RZF being a transformation of the EP after operations have been performed, that one needs to extend the parameterized case, by adding a balancing function. Thus, by comparing the definitions of the primed and unprimed coordinates which we can define from Xi(s) = Xi(1-s), the functional equation. Immediately obvious is that because of our notational symmetry (primed equations and unprimed equations), a symmetric equation (describing a function) such as the functional equation which has local minimal values at s = 0, and 1, we should find that due to the fact that the points of symmetry must lie at extremal values, so if the function we define is notationally symmetric with respect to the RZF, the points of symmetry will "break" our analysis, and force us to change definitions because those points, are points where the originally ℝ and 𝕀 components become 𝕀 and ℝ respectively.

Very luckily, when I first began studying the RZF, and analytic continuation in general, which took me quite a while to grasp –and there are connections as an undergraduate, I known I miss still-, I had developed an equation for x^x, which relates the function $(-x)^x$ and the complex exponential $e^{i\pi x}$, which where x^x has discontinuities when x < 0 are some of the same places that $\mathbb{R}(x^x)$ =- $x^x\sin(\pi x)$, as opposed to the original definition from Eq 1.2) in which $\mathbb{R}(x^x)$ = $(-1)^x x^x\cos(\pi x)$. So, the mechanics which describe the sign swapping require respective definitions, assuming the derivation of Eq 1.2, where ln(-x) ∈ ℂ, as opposed to the derivation of the functional equation by which Riemann explicitly writes that ln(-x) ∈ ℝ. We know therefore when we insert Eq 1.2 into the definitions along the way of the RZF's derivation, that then the definitions are only respectively defined.

By comparing the definition of RZF to the primed RZF, RZF', and inserting that one of the defining statements of the RZF, are intentionally different, we will obtain, with respect to the EP(x) mapping, an entirely different graph, but which due to the fact that this insertion, this new defining rule changes the definition of the RZF, to RZF', the desirable property, remains; namely, that the points of symmetry will be defined with respect to the coordinates, which in the case of the RZF are defined with respect to the XI function because it is defined as the functional equation which is in effect the result of the analytic continuation of the EP.

Further, by directly comparing the Polynomial expression of the RZF, and asserting that y = x, one obtains a very clear numerical image of a parameterization in which RZF(x,y)⟶RZF(x) and ϵ ℂ. Combined with its symmetric equation gives us a way to obtain EP(x) ϵ ℂ, which otherwise is not possible with multiple variables due to the lack of equality of information (if EP were defined as a Real output). Importantly, sign swapping, in defining RZF', not only is defined respectively to the coordinates, but too are defined with respect to the intentional incorrect definition of n^{-s} with $(-n)^{-s}e^{-i\pi s}$ which defines to be true, ln(-x) ∈ ℂ, as opposed to ln(x) ∈ ℝ, which is defined along the derivation of RZF.

The RZF is analytic, so it can be expressed as a complex polynomial as through Laruent's Theorem. This is also the case for RZF', so the application of Laurent's Theorem does not interfere with the desired properties. There is a simple pole of the RZF at s =1, and so there is a simple pole at RZF' = RZF'(1-s), which because if y = 0, the EP(x) = ∞ at x =0 and x =1. The symmetry in a sense is bounded by the infinities of the EP, and the zeros of Xi(s), from which, due to the functional equation, RZF's zeros obey the symmetric property that $\zeta\left(\frac{1}{2}+iy\right)=\zeta(\frac{1}{2}-iy)$. It is satisfactory, therefore, to find that the residue of the RZF is 1.

There have been other attempts of a proof of the RH by authors using undergraduate techniques such as the mean value theorem of integrals, and by looking at the polynomial expressions of the RZF, but thus far such approaches have been unable to attack the main problem: a deeper understanding of the properties of analytically continued functions. Using the logic presented above, and working through many research articles, I believe I was able to manufacture a way to build a notationally symmetric, RZF' which is related to the coordinates of the RZF through a linear transformation, namely due to the fact that the functional equation is an equation which is invariant when s = 1-s. In § 4, I use the Laurent expansion, and Eq 1.2 to show why the symmetry property is implicative of the RZF's zeros only being at the half line.

There have been authors who have devised spectral interpretations of the RZF's zeros such as those by Dr. Ralph Meyer and Andre Voros [19,20]. Meyer's work was inspired by A. Connes [21] who has developed much work on the understanding of generalized theorems such as has been developed by A. Weil, and has shown relationships to the atomic absorption spectrum. Additionally, researchers such as Michel Planat, Patrick Sole, and Omar Sami have shown connections to the mathematical framework of Quantum Mechanics [22]. It is often the case that the gamma function occurs in physics analysis, for example, in the mathematical description of scattering such as that in the work of my Optics professor from Cleveland State University, Dr. Streletsky and CSU graduate, George D. J. Phillies "Dynamics of Semirigid Rod Polymers From Experimental Studies" [23].

Additionally, A. M. Odlyzyko and A. Schonhage have investigated algorithms for calculating values of the RZF [24]. Mathematicians such as William D. Banks, Ahmet M. Güloglu, C. Wesley Nevans have analyzed primitive Dirichlet characters and their relation to the RZF [25].

The relation of selberg classes to the RZF and alternate descriptions of the RH wih Li's criterion has been researched by Lejla Smajlovic´ [26]. There are also research articles such as by Shaoji Feng, and Xiaosheng Wu, who analyze the gaps between the RZF's zeros [27]. Mathematicians such as Yunus Karabulut and Cem Yalçın Yıldırım, have investigated the zeros of the derivatives of the RZF [28]. Author Rober A. Van Gorder, has investigated as to whether the RZF obeys a differential equation, to which he finds this true [29]. Additionally, scientists such as Antanas Laurincikas have investigated the universality of the RZF [30].

Expanding the work of Conrey and Bomberi, mathematicians Xia-Qi Ding, Shao-Ji Feng have worked out a variation of the mean value theorem of the Riemann Zeta Function [31]. Additionally, J. B. Conrey, A. Ghosh and S. M. Gonek have worked out that almost all of the zeros of the RZF possess the property known as simplicity [32].

Authors such as K. Chandrasekharan, have written books which derive the basics of the RZF and mathematicians such as G.H Hardy and J. E. Littlewood who have multiple volumes on the subject of the RZF [33, 34].

Relationships to other generalized RZF's such as the Dedekind RZF and Dirichlet L-Functions have been devised by mathematicians such as Frimpong A. Baidoo [35]. The zeros of the RZF have further been investigated by Littlewood independently such as in his paper, "On the zeros of the Riemann Zeta Function," and G. Hardy proved using contour integration that there exists infinitely many zeros on the critical line in his paper, "The zeros of Riemann's Zeta function on the critical line." With another connection to physics, authors Germ´an Sierra1 and Paul K. Townsend show how Landau Levels are related to the RZF's zeros [36]. The symmetry of the zeros have been investigated by Nicholas M. Katz and Peter Sarnak [37]. Black hole mechanics and the RZF are discussed by S. Hawking [38].

§ 2

On the Fourier Series Relationship to the Riemann Zeta Function

The RZF can be written generally as:

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-x} Cos(yln(n)) + i \sum_{n=1}^{\infty} n^{-x} Sin(yln(n)).$$

With the complex variable being equal to s = x+iy. The Fourier Series can be written as:

$$f(y) = a_0 + \sum_{n=1}^{\infty} A_n \cos(ky) + \sum_{n=1}^{\infty} B_n \sin(ky)$$

Where f is a periodic function of y. An F(y) can be formed such that F obeys Poisson's equation, in such a way that F is representative of the electric potential of a system [39]. When the Charge density, which is a function of four variables is zero, F obeys Laplace's equation assuming separation of variables, and the completeness of the constituent functions has a general solutions of the form:

$$V(x,y) = (Ae^{x} + Be^{-x})\big(Csin(ky) + Dcos(ky)\big)$$

a more general form would be an infinite sum of the linear combinations:

$$V(x,y) = \sum_{n=1}^{\infty} (a_n e^{x} + b_n e^{-x}) D_n \sin(ky) + \sum_{n=1}^{\infty} (a_n e^{x} + b_n e^{-x}) C_n \cos(ky).$$

Laplace's 2-D partial differential equation is similar in form to the Wave Equation [40] which provides general solutions of the form:

$$\psi(x,t) = C_1 f(ky - \omega t) + C_2 f(ky + \omega t)$$

The Underlying governing differential equation, or rule which allows for the construction of the function which satisfies this rule, is fundamentally the same differential equation studied with different parameters, aside from complex constants. Remarkably similar even the solution family appears, but they have radically different numerical values and form. With, however, numerical transformations of the variables, the original definitions could be transformed to mean something new, respectively. Allow, for example, the x of the RZF to be zero, and we have something very similar to the function written with a Fourier series, other than the primary difference that the B_n are not complex. This single difference in equation form, causes for an entirely different form of solution because the roots of the analytic function must satisfy the roots of two functions, namely,

$$\sum_{n=1}^{\infty} Sin\big(yln(n)\big) \; and \; \sum_{n=1}^{\infty} Cos(yln(n))$$

(the real and imaginary parts of the RZF)

Whereas the function, $f(y)$ is clearly a single variable function which has only one output. A generalization, however, of this series such as:

$$f(x,y) = a_0(x,y) + \sum_{n=1}^{\infty} A_n(\mathrm{x,y}) \cos(k_n(x,y)y) + \sum_{n=1}^{\infty} B_n(\mathrm{x,y}) \sin(k_n(x,y)y)$$

leads to amplitudes of the sine and cosine waves which are functions of the coordinates, and also infinitely many families of infinitely many frequencies that the summed waveforms propagate with. This function, which is a generalization of the function which satisfies the Fourier transformation methodology for differential equation solutions, because it is a function of multiple variables as opposed to one and has roots described by two values and not one (a real value and an imaginary value), although with respect to the y variable, is only one function a function of one variable. Thus, f(x,y) can satisfy a new differential equation which depends on x and y, and still yet, provided a further simplification of the frequencies be a solution to the original differential equation.

Laplace's equation which acts on the function V(x,y) is thus a coordinate transformation of the wave equation which acts on the function $\psi(x,t)$, with the coordinate transformation y = it, and a parameterization of the wave velocity. So, certain properties of the functions which obey the Laplace equation also change when we change the form of the differential equation because we have redefined the variables, and in some way related to the solution of the Wave Equation and the functions which are solutions to them. These functions are called Harmonic functions. Very similarly, the Riemann-Cauchy Equations which are effectively the differential equations by which analytic functions such as the RZF must obey, can be compared under a similar coordinate transformation and respective redefinition of the variables, as just discussed with the V(x,y) and $\psi(x,t)$. When compared to the Wave Equation, I find that the Riemann-Cauchy conditions are different from the wave equation depending on whether the wave velocity is Imaginary or Real. The hyperbolic equations which the Lorentz equations of SR can be arrived to, produce a singularity when the velocity of the problem is equal to c, and effectively we obtain an infinite value for the coordinates, and hence energies, which is the mathematical reason that the speed of light is a universal speed limit. Letting the velocity of the observer be imaginary in this instance avoids this catastrophe is the observer traveled at light speed and instead provides a complex solution, however, very clearly, such an insertion changes the definitions of the original variables. Much should be expected when someone redefines the variables in a relative way which is contrary to experience. This is because the hypotenuse of the measured triangle which obeys $a^2 + b^2 = c^2$ when either leg is an imaginary distance, changes meaning. Consider b = ik, then we have a new triangle equation with hypotenuse a.

Following this line of logic, it is quite clear that the RZF is in some way a particular generalization of the Fourier series which is related to the distribution of prime numbers. Particularly due to the exponential nature of the amplitude of the generalized Fourier series is n^-x, that at x = 1/2, the complex zeros of the RZF must lie, because the family of amplitudes which the RZF is defined with force a sign change of the Real and Imaginary components of the function, for when one considers the symmetric family of equations which must exist because of the symmetry properties of the describing differential equations that must be obeyed, we find duplicate equations for which a linear coordinate transformation must obey, and this is particularly exhibited by the existence of the symmetry of the functional equation. The particular x value which makes the form of the functional equation satisfy the sign swapping which is a condition for zeros to occur is the value of x which describes the longitudinal coordinate of the line which extends towards infinity on the latitude.

Due to the fact that poisons equation describes electromagnetic phenomena, it appears that a better understanding of the frequency analysis could possibly produce methodology to describe solutions of other similar differential equations.

§ 3

Facts, Definitions, and Relationships to the RZF

- It is known that the nontrivial zeros of the RZF lie within the Critical Strip, 0 < x < 1.

1. G.H Hardy proved using contour integration and the properties of the Gamma function, $\Gamma(s) = \int_0^\infty \sigma^{s-1} e^{-s} d\sigma$, to prove the infinitude of zeros at the critical strip, the line $s_j = (x_j = \frac{1}{2}) + iy_j$. Littlewood and Hardy proved that $\zeta\left(\frac{1}{2} + iy\right) = O(y^\epsilon)\ as\ y \longrightarrow \infty$.

2. The Functional Equation implies that the zeros are symmetric with respect to the Critical Line..

3. Computational Projects by van de Lune [41] has shown that the first 10 billion zeros are on the Critical Line, and Andrew Odlyzko [42] (on his website) has shown that the first 100 billion zeros are on the critical line.

4. In Riemann's 1859 manuscript because of the fact that the functional equation is an equation which is invariant with the replacement of all s by 1-s that the remainder of Riemann's manuscript which is designed to find true Gauss' conjecture, hinges on the truth of the Riemann Hypothesis: "This property of the function induced me to introduce, in place of

$\Pi(s-1), the\ integral\ \Pi(\frac{s}{2}-1)$ into the general term of the series $\sum_{n=1}^\infty n^{-s}$, whereby one obtains a very convenient expression for $\zeta(s)$. In fact:

$$\frac{1}{n^s}\left(\Pi\left(\frac{s}{2}-1\right)\right)\pi^{-\frac{s}{2}} = \int_0^\infty e^{-n^2\sigma}\sigma^{\frac{s}{2}-1}d\sigma\text{,"}$$

Where after some steps, he evaluates this to (in the notation used from this current paper):

$$\zeta(s)\left(\Pi\left(\frac{s}{2}-1\right)\right)\pi^{-\frac{s}{2}} = \frac{1}{s(s-1)} + \int_1^\infty \sum_{n=1}^\infty e^{-n^2\sigma}\left(\sigma^{\frac{s}{2}-1} + \sigma^{-\left(\frac{(1+s)}{2}\right)}\right).$$

From this point, he introduces the new variable $t \in \mathbb{C}$ such that t = $\frac{i}{2} - is$. And because for every variable introduced, there is another to keep track of due to the functional equation, which is the variable to consider for the symmetric case, t' = $\frac{i}{2} - i(1-s)$. After this, Riemann then defines the functional equation:

$$\xi(t) = \zeta(s)\Pi\left(\frac{S}{2}\right)(s-1)\pi^{-\frac{s}{2}}$$

Which can be written in integral form as:

$$\xi(t) = \frac{1}{2} - \left(t^2+\frac{1}{4}\right)\int_1^\infty \sum_{n=1}^\infty e^{-n^2\sigma}\left(\sigma^{-\frac{3}{4}}\cos\left(\frac{t}{2}ln(\sigma)\right)\right).$$

Riemann then shows how if $\xi(t_k) = 0$, that $-\frac{i}{2} < t_k < \frac{i}{2}$, from which he finds that the number of roots up to (k = T) of $\xi(t)$ is approximately equal to: $\frac{T}{2\pi}\ln\left(\frac{T}{2\pi}\right) - \frac{T}{2\pi}$.

5. "The functional equation shows that the complex zeros are symmetric with respect to the half-line," B. Conrey [7], can proven by direct substitution of the definition of t for s: $\xi(t) = \xi(1-t)$, or $\xi\left(\frac{\frac{1}{2}-s}{i}\right) = \xi\left(\frac{\frac{1}{2}+s}{i}\right)$.

6. The RZF, $\zeta(s)$, is a Complex function which is analytically continuous. When $\mathbb{I}m(s) = 0$, the RZF is the Real defined Euler Product, however, for every other value $\mathbb{I}m(s) \neq 0$, we have a complex function described by $\sum_{n=1}^\infty n^{-s}$, which converges for ($\mathbb{R}(s)$ = x) > 1 if $\mathbb{R}(s) = 0$.

7. The analytically continued RZF is finite and one valued for all s except s = 1, where there exists a pole. Riemann proved this in his 1859 manuscript during the derivation of the functional equation with contour integration.

8. The power series of $\zeta(s)$ is defined with Taylor's theorem:

$$\zeta(s) = \sum_{k=0}^\infty \frac{\frac{d^k\zeta(s_0)}{ds^k}(s-s_0)^k}{k!}.$$

With

$$\frac{d^k\zeta(s)}{ds^k} = \sum_{n=1}^\infty (-1)^n(\ln(n))^k n^{-s}, \qquad Eq\ (1)$$

Eq (1) becomes:

$$\zeta(s) = \sum_{k=0}^\infty \frac{\sum_{n=1}^\infty (-1)^n(\ln(n))^k n^{-s_0}(s-s_0)^k}{k!},$$

And letting s_0 = 0,

$$\zeta(s) = \left(\sum_{n=1}^\infty (-1)^n n^{-s_0}\left(\sum_{k=0}^\infty \frac{\ln^k(n)(s)^k}{k!}\right)\right) =$$

$$\left(\sum_{n=1}^\infty (-1)^n n^{-0}\left(\frac{\ln(n)(s)}{1!} + \frac{\ln^2(n)(s)^2}{2!} + \frac{\ln^3(n)(s)^3}{3!} + \frac{\ln^4(n)(s)^4}{4!} + \frac{\ln^5(n)(s)^5}{5!}\ldots\right)\right)$$

Which if we expand down the page,

$$\zeta(s) =$$

$$-\left(\frac{\ln(1)(s)}{1!} + \frac{\ln^2(1)(s)^2}{2!} + \frac{\ln^3(1)(s)^3}{3!} + \frac{\ln^4(1)(s)^4}{4!} + \frac{\ln^5(1)(s)^5}{5!}\ldots\right)$$

$$+\left(\frac{\ln(2)(s)}{1!} + \frac{\ln^2(2)(s)^2}{2!} + \frac{\ln^3(2)(s)^3}{3!} + \frac{\ln^4(2)(s)^4}{4!} + \frac{\ln^5(2)(s)^5}{5!}\ldots\right)$$

$$-\left(\frac{\ln(3)(s)}{1!}+\frac{\ln^2(3)(s)^2}{2!}+\frac{\ln^3(3)(s)^3}{3!}+\frac{\ln^4(3)(s)^4}{4!}+\frac{\ln^5(3)(s)^5}{5!}\dots\right)$$

$$+\left(\frac{\ln(4)(s)}{1!}+\frac{\ln^2(4)(s)^2}{2!}+\frac{\ln^3(4)(s)^3}{3!}+\frac{\ln^4(4)(s)^4}{4!}+\frac{\ln^5(4)(s)^5}{5!}\dots\right)$$

$$-\left(\frac{\ln(5)(s)}{1!}+\frac{\ln^2(5)(s)^2}{2!}+\frac{\ln^3(5)(s)^3}{3!}+\frac{\ln^4(5)(s)^4}{4!}+\frac{\ln^5(5)(s)^5}{5!}\dots\right)$$

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Which one could write in matrix notation by taking the product of a $\gamma\ x\ \gamma$ matrix and a γ vector with $\gamma = \infty$, which would look like:

$$\zeta(s) = \begin{pmatrix} \frac{\ln(2)(s)}{1!} & \frac{\ln(3)(s)}{1!} & \frac{\ln(4)(s)}{1!} & . & . \\ \frac{\ln^2(2)(s)^2}{2!} & \frac{\ln^2(3)(s)^2}{2!} & \frac{\ln^2(4)(s)^2}{2!} & . & . \\ \frac{\ln^3(2)(s)^3}{3!} & \frac{\ln^3(3)(s)^3}{3!} & \frac{\ln^3(4)(s)^3}{3!} & . & . \\ \frac{\ln^4(2)(s)^4}{4!} & \frac{\ln^4(3)(s)^4}{4!} & \frac{\ln^4(4)(s)^4}{4!} & . & . \\ . & . & . & . & . \\ . & . & . & . & . \end{pmatrix} \cdot \begin{pmatrix} - \\ + \\ - \\ + \\ . \\ : \end{pmatrix}$$

9. The Gamma Function is defined as:

$$\Gamma(s) = \int_{\sigma=0}^{\sigma=\infty} e^{-\sigma}\sigma^{s-1}d\sigma = n^s\int_{\sigma=0}^{\sigma=\infty} e^{-n\sigma}\sigma^{s-1}d\sigma.$$

The Gamma Function follows the identity:

$$\Gamma(s+1) = s\Gamma(s)$$

Which can be proven with integration by parts.

In Riemann's 1859 manuscript, Riemann uses

$\Pi\left(\frac{s}{2}-1\right)$, however, this is the Gamma function. In modern language, Riemann uses the definition of $\Gamma(s)$ to show that:

$$\Gamma(s)\zeta(s) = \sum_{n=1}^{\infty}\left(\int_0^\infty e^{-n\sigma}\sigma^{s-1}d\sigma\right) = \int_0^\infty \frac{\sigma^{s-1}d\sigma}{e^{\sigma-1}} \qquad Eq\ (8)$$

- Additionally, Zhen-Hang Yang and Jing-Feng Tian have formulated approximate formulas for the Gamma Function [43].

- Also Necdet Batir has archived some properties of the Gamma Function [44]

10. A Functional Equation serves the purpose of relating the outputs of a function to different inputs of that function. In his 1859 manuscript, Riemann derives the functional equation through the definition of the RZF in to construct the functional equation by considering a similar integral and its relationship to the integral which defined the RZF:

$$\int_\infty^\infty \frac{(-\sigma)^{s-1}d\sigma}{e^\sigma - 1} = (e^{-i\pi s} - e^{i\pi s})\left(\int_0^\infty \frac{\sigma^{s-1}d\sigma}{e^{\sigma-1}}\right), \qquad Eq\ (6)$$

Where through this fact, the RZF, as shown by Riemann, can be extended to all values of s, with the equation:

$$2\sin(\pi s)\,\Gamma(s)\zeta(s) = i\left((e^{-i\pi s} - e^{i\pi s})\left(\int_0^\infty \frac{\sigma^{s-1}d\sigma}{e^{\sigma-1}}\right)\right) = i\int_0^\infty \frac{(-\sigma)^{s-1}d\sigma}{e^\sigma - 1}. \qquad Eq\ (7)$$

The RZF's functional equation is defined as:

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-\frac{(1-s)}{2}}\Gamma\left(\frac{(1-s)}{2}\right)\zeta(1-s).$$

The RZF's functional equation can be written as:

$$\zeta(s) = 2^s\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\zeta(1-s)$$

and

$$\zeta(1-s) = 2^{1-s}\pi^{-s}\cos\left(\frac{\pi s}{2}\right)\Gamma(s)\zeta(s).$$

This equation shows why the Trivial Zeros lie at the negative even integers of x. This is because

$$\sin\left(\frac{\pi s}{2}\right) =$$

$$\cosh\left(\frac{\pi ct}{2}\right)\sin\left(\frac{\pi x}{2}\right) + isinh\left(\frac{\pi ct}{2}\right)\cos\left(\frac{\pi x}{2}\right),$$

And similarly for $\cos\left(\frac{\pi s}{2}\right)$,

$$\cos\left(\frac{\pi s}{2}\right) =$$

$$\cosh\left(\frac{\pi ct}{2}\right)\cos\left(\frac{\pi x}{2}\right) + isinh\left(\frac{\pi ct}{2}\right)\sin\left(\frac{\pi x}{2}\right).$$

11. Additionally, mathematicians arvin Knopp and Sinai Robins have collected easy proofs of the functional equation [45].

12. The definition of $\xi(t)$ is the equation which satisfies the condition that one may substitute 1-s for s in the RZF's functional equation:

$$\xi(t) = \pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)\,,$$

Which due to the fact that ξ is defined with respect to the functional equation, ξ's functional equation is:

$$\xi(t) = \pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)$$
$$= \pi^{-\frac{(1-s)}{2}}\Gamma\left(\frac{(1-s)}{2}\right)\zeta(1-s) = \xi(t').$$

There are graphical images of ξ provided by Eric Weisstein [46], again on Wolfram Math.

13. An interesting Numerical relationship:

k	Number of Prime Numbers up to t_k	Imaginary part of RZF zeros, t_k
1	6	14.134725
2	8	21.022040
3	9	25.010858
4	10	30.424876
5	11	32.935062
6	12	37.586178

By the time the sixth zero has been inputted into the zeta function, twelve prime numbers have been surpassed.

14. An equivalent statement the RH, discussed by Bombieri [11] is that for every $\epsilon > 0$, there exists positive constant $C(\epsilon)$ such that

$$\left|\left(li(x) = \lim_{\alpha\to 0}\left(\int_0^{1-\alpha}\frac{dz}{\ln(z)} + \int_{1+\alpha}^{\sigma}\frac{dz}{\ln(z)}\right)\right) - \pi(\sigma)\right| \le C(\epsilon)\sigma^{\frac{1}{2}+\epsilon}$$

Note that li(x) is defined with the Cauchy Principal Value integral due to the fact that log(1) = 0.

$li(x) - \pi(x)$ has infinitely many fluctuations in sign and is known to change sign at around 10^{371}.

15. The prime number counting function has been discussed by many authors, such as Tadej Kotnik [47].

16. Miscellaneously, the functional equation can be proven by alternate routes, for example, see Harvard Mathematics, "Proof of Functional Equation by Contour Integral and Residues" [48].

17. Jeff Valer discusses the history of the functions and some of the relations to the theory of Number "The Riemann Hypothesis, Millennium Prize Problem," in a Lecture Video on the topic [49],.

18. Peter Sarnak discusses the more general class of functions known as L-Functions, and the more general cases of the RH in his paper, "Problems of the Millennium: The Riemann Hypothesis (2004)," which [50].

§ 4: Justification for § 4.1

According to Laurent's Theorem, analytic functions may be expanded by a power series with

$$\zeta(s) = \sum_{n=-\infty}^{\infty} A_n(s-s_0)^n$$

With coefficients calculated by

$$A_n = \frac{1}{2\pi i}\oint \frac{\zeta(k)}{(s' - s_0)^{n+1}} dk$$

Where through the Cauchy-Goursat theorem [51], we guarantee that Taylor's Theorem is applicable to ζ,

$$\zeta(s) = \sum_{n=0}^{\infty} \frac{\frac{d^n\zeta(s_0)}{ds^n}(s - s_0)^n}{n!},$$

And so if we wish to introduce functions of s' that are analytic, we must define the equations which do so with respect to that variable, and parameterizing the definition of s' to make it with respect to s, and to arrive to the power series of ζ', we think that if ζ is analytic with respect to s, the equations for ζ' will similarly follow with respect to s':

$$\zeta'(s') = \sum_{n=0}^{\infty} \frac{\frac{d^n\zeta(s_0')}{ds^{*n}}(s' - s_0')^n}{n!}.$$

Thus far, due to the evidence which points towards the necessity of a symmetry condition, the RH becomes proving that the only places which exhibit the required symmetry for the x values is at the Critical Line, because symmetry Condition Met at |x| $= \frac{1}{2}$ Implies $x_k = \frac{1}{2}$. With a phasor interpretation, a similar argument is to justify that when comparing the RZF and RZF underneath parameterization, the points of symmetry are the places where the primed and unprimed coordinates are π radians out of phase. As opposed to this approach, I have been lucky enough to develop an equation (Eq 1.2), with the insertion of a change of definition of the natural logarithm's domain, which seems to be able to provide satisfactory evidence of the symmetry we seek to prove.

The multivalued nature of the logarithm (when selecting the log of a negative number to be Real or Complex) causes for additional imaginary terms to be able to be considered in problems, with the modification of the rule that the Log can be Complex as opposed to real. Consider:

$$f^k = (-f)^k e^{-i\pi k} \qquad Eq\ (1.2)$$

Proof of Eq 1.2):

$x \in Re; such\ that\ f, g, k \longrightarrow f(x), g(x), k(x)\ |$ $f, g, k \in Re$, and

$$ln(fg) = ln(f) + ln(g);$$

with $f = f^k, g = (-f)^k$,

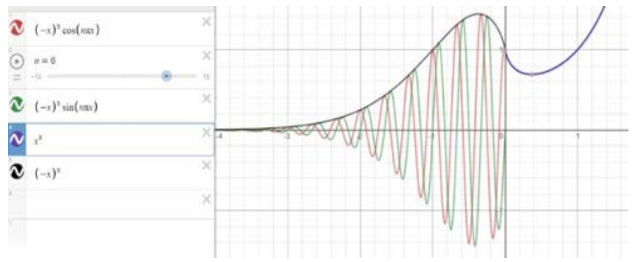
$$\Longrightarrow ln(f^k(-f)^k) = kln(f) + k(ln(f) + ln(-1));$$

but, $e^{i\pi} = -1$, so

$$\Longrightarrow f^k = (-f)^k e^{-i\pi k} \quad \blacksquare$$

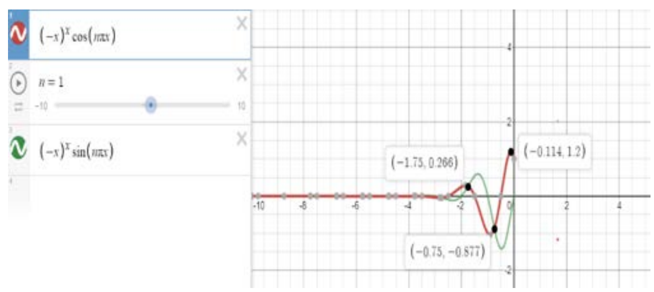
I have created the following plots using Desmos.com, a free online graphing utility, to provide graphical plots of some instances of Equation 1.2: Plotting the real and imaginary parts of x^x as an example, n = 6 (note that n is the n for $e^{i\pi n} = -1$, which for this case breaks the equality)

FIGURE FOUR



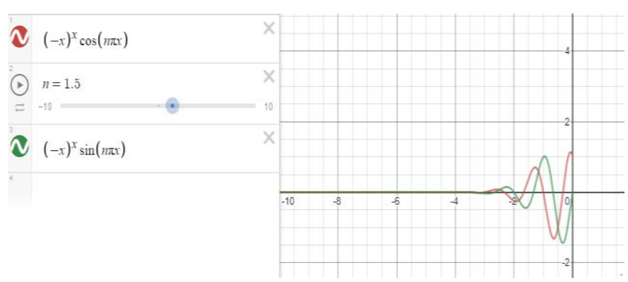
The following image is the same plot for n = 1:

FIGURE FIVE



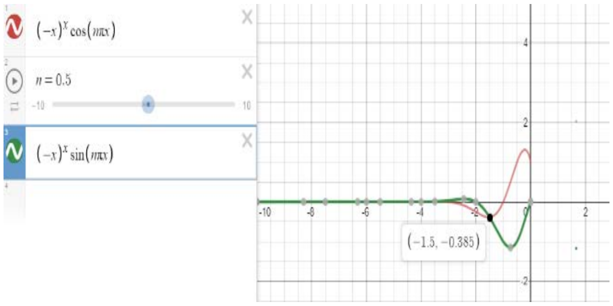
The plot is done again for n = 1.5:

FIGURE SIX



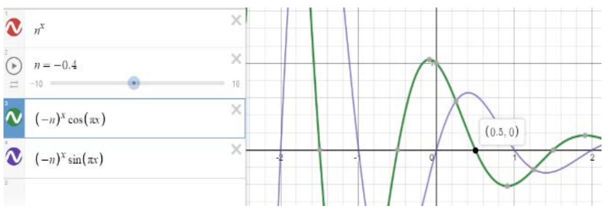
The plot is done again for n = .5:

FIGURE SEVEN



The next figure now shows a different, but similar function, n^x, and its real and imaginary components (again using Eq 1.2) where n is selected as -.4 to give a visualization that for all negative n of n^x, the real part intersects zero at x = 1/2:

FIGURE EIGHT



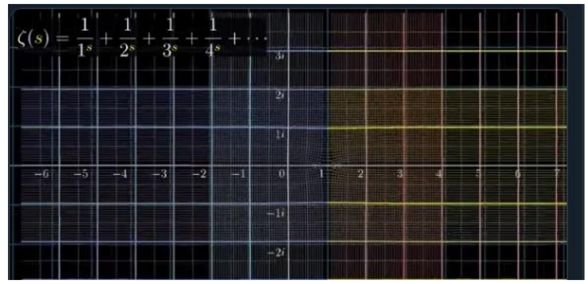
The points symmetry we are looking for are the numbers where the Real parts and Imaginary parts swap location, that is, whether the imaginary numbers are on the right hand side of the page or left side once we performed the operations. These points are the points where the real components and imaginary components of complex two vectors swap places. The RZF is a linear sum of infinitely many functions of this nature, so to examine further the symmetry properties of a function compared with respect to n^{-x} is to examine the symmetry properties of the RZF because of the respective linearity of the terms. By using Eq 1.2, we have inherently compared two such functions.

Consider further a simulation of analytic continuation of the RZF by Grant Sanderson, a youtuber who has created a simulation to depict analytic continuation of the RZF [52]. I have taken screenshots of the simulation because it depicts the symmetry properties just discussed.

Here, I screenshotted images of the simulation. The left side is the continuation of the RZF with respect to the half line, and the right side is the original set of values the RZF is defined for:

The lines begin flat,

FIGURE NINE



And then start to curve, and have maximal values at the critical line.

FIGURE TEN

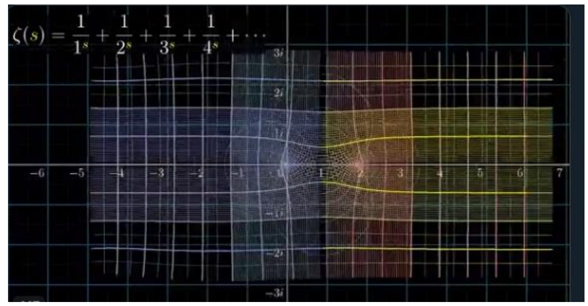


FIGURE ELEVEN

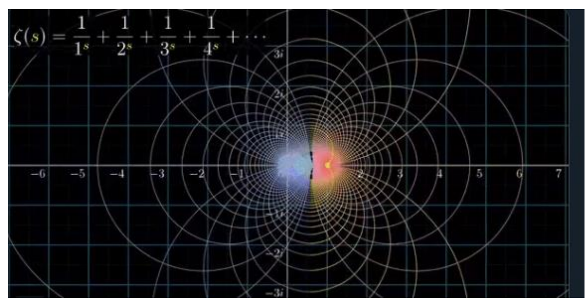
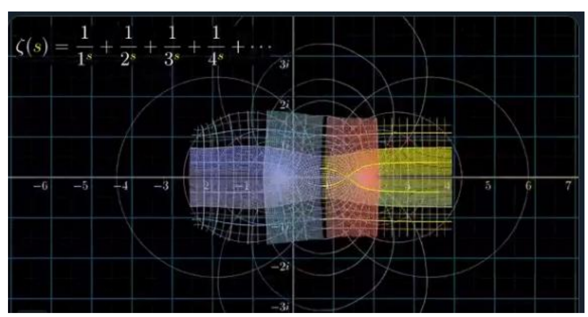


FIGURE TWELVE



Additional simulations have been performed by bloggers such as "Christian S. Perone", who plotted the real and imaginary components of the RZF in time, where clearly, as Re(s)= .5, the Real and Imaginary graphs simultaneously intersect zero. The following images show this, where with $\mathbb{R}$ (s) <.5, $\mathbb{R}(\zeta)$> 0, intersects the graph at ℝ (s), and ℝ(ζ)<0 for values ℝ(s)>.5 [53].

I have screen captured this simulation also because it shows visually with the parameterization just discussed that the Real and Imaginary parts of the RZF intersect zero simultaneously only at Re(s)= .5:

FIGURE THIRTEEN

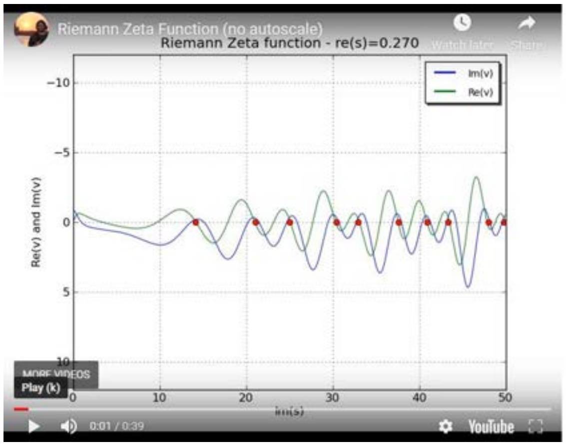


FIGURE FOURTEEN

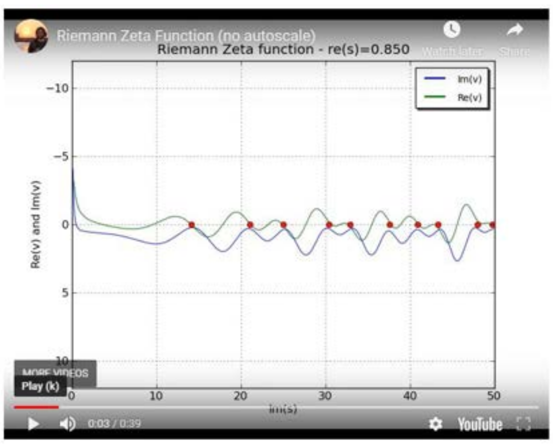
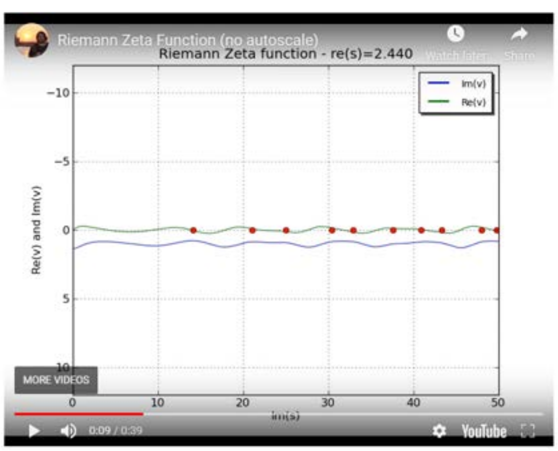


FIGURE FIFTEEN



Finally, with now, the understanding of the parameterization and supplementary visualization, knowledge of the functions which are related to the RZF, different expressions of the RZF, we can devise symmetric RZF which stems from the functional equation, and have through looking at the composing functions of the sum (n^-x), we are able to justify as to why the complex zeros lie along the critical line.

We will build a symmetric RZF and compare it to the RZF by assigning a primed RZF, a function of primed variables which are in some way related to the unprimed variables. We will consider linear numerical transformations, because the definition of the functional equation has s = 1-s, which is a linear transformation. The symmetric function must therefore have a polynomial expression, so the "trick" I have developed to manufacture the polynomial equation which is notationally symmetric to the RZF is nothing more than the application of Eq 1.2) to the RZF.

The RZF is defined for:

19. $\zeta(s) < \pm\infty \;\forall\; s \neq 1$ (Convergence)
20. $\zeta(s) = \infty \; at \; s = 1$ (Divergence)
21. $\zeta'(s') < \pm\infty \;\forall\; s' \neq 1$ (Convergence)
22. $\zeta'(s') = \infty \; at \; s' = 1$ (Divergence)
23. $\zeta(x) < \infty \; for \; x > 1$ (Convergence)
24. $\zeta(x) = \infty \; at \; x = 1$ (Divergence)
25. $\zeta(x) < \infty \; for \; 0 < x < 1$ (Convergence)
26. $\zeta(x) = \infty \;\forall\; x < 0$ (Divergence)

$$\xi(s) = F(s)\zeta(s) = F(s')\zeta(s') = \xi(s')$$

Which occurs for s' = 1 = s, a linear numerical transformation. x ϵ ℝ $\leftrightarrow \zeta(x)$ ϵ ℝ; so, how could the real values of this function possibly be related to the complex values after the operation known as "analytic continuation has been performed?" The numerical transformation in a sense would look like $AC[\zeta(x)] = \zeta(x' + iy')$. Under the parameterization such that x' is linear in x, it becomes obvious that there are locations of symmetry embedded in the transformation due to the fact that the constituting functions of the RZF, namely, U(x',y') and V(x',y') are harmonic, and follow the Cauchy-Riemann differential equations, which in their second order are effectively the Laplace transformation differential equation. With nothing more than the insertion of the unit velocity to be an imaginary unit in the wave equation, one can obtain the Laplace equation, which would therefore give solutions of the same form as that of the Laplace equation, but under the relative definitions imposed. The functional equation is derived by Riemann supposing that ln(-x) $\epsilon\ \mathbb{R}$, so my idea is to extend the analysis by asserting that ln(-x') $\epsilon\ \mathbb{C}$. This gives us double the variables to –with respect to the change of definition- compare $\zeta'(s')$ and $\zeta(s)$.

We know that forming a parameterization should best be done in the case that x sweeps from negative to positive infinity, we will see the regions of continuity all the way up to where the definitions of the RZF change, due to convergence and divergence at the values 0 and 1. This is better exemplified by the fact that Xi(s) which is defined as an analytic function multiplied into $\zeta(s)$, another analytically continuous function for all values other than one, is equal to 1 at both Xi(0) and Xi(1). This implies that there is a maximal value somewhere between these values due to the Extreme Value Theorem, and if the function is symmetric should have extrema half way between the identical endpoints. This being the case, we have inherently found that the zeros are on the critical line only, because of the fact that zeros lie between extrema, and if the extremal are at the points of symmetry, and the RZF is symmetric, then underneath the parameterization that AC[RZF(x)] = RZF(x + iy), then we can compare the new RZF(x' +iy') which has been defined in a notationally symmetric manner, which I will show with the usage of EQ 1.2 and the Laurent expansion. Again, this equation is intentionally defined by the redefinition of the logarithm's domain to instead be complex. This simple change of definition, changes respectively the definition of the RZF.

The visual experiment performed by Christian Perone shows that the RZF underneath parameterization actually intersects at a simulation at x =1/2, to meet the criterion of zeros, so the definitions of the complex valued RZF's must be relative to this criterion of intersection at zero. The construction of a new function, RZF' leads one to compare this notationally symmetric function, which where points of symmetry are, should be immediately shown by the fact that the half integers of x are very obviously in the selection of the logarithm to be complex for (n^-x), which if done correctly seems to be justifiably true with Eq 1.2, an equation explicitly designed to show this symmetry property. Further, it is clear that the points which have a change in definition should be the points at which the functions become discontinuous

The Laurent expansion gives one:

$$Eq\ 1)$$

$$\zeta'(s') = \left((\cosh(y')\cos(x') - isinh(y')\sin(x')) + \frac{i\left((\cosh(y')\sin(x') + isinh(y')\cos(x'))\right)}{2\pi i}\right)\left(\oint_C \frac{\left(\sum_{n=1}(-n)^{-s'}\right)}{(s'-s'_0)} + \oint_C \frac{\left(\sum_{n=1}(-n)^{-s'}\right)}{(s'-s'_0)^2} + \oint_C \frac{\left(\sum_{n=1}(-n)^{-s'}\right)}{(s'-s'_0)^3} + \cdots\right)$$

And the same function which has log(-x) defined as a real number:

$$\zeta(s) = \left(\oint_C \frac{\left(\sum_{n=1}(n)^{-s}\right)}{(s-s_0)} + \oint_C \frac{\left(\sum_{n=1}(n)^{-s}\right)}{(s-s_0)^2} + \oint_C \frac{\left(\sum_{n=1}(n)^{-s}\right)}{(s-s_0)^3} + \cdots\right)$$

Now, to separate these terms and to assign the Real and Imaginary parts of the respective functions, we will see very obviously that due to the fact that if we select the Real and Imaginary components now, there are infinitely many values for which the sign swapping occurs in the primed functions due to the fact that there is a multiple of $\sqrt{-1}$ now out front, which swaps the definitions in the required symmetric manner. This occurs at x = 1/2 and can be seen with the insertion of x= 1/2 within Eq 1.2[RZF(x+iy)] = RZF'(x'+iy').

Once U(x,y) and V(x,y) have been explicitly defined, we see that certain values of x, particularly values of mirror symmetry, that the definition of U(x,y), and V(x,y) change. This argument affectively boils down to showing then that the RZF readily verifies these symmetry properties, and very happily, through the definition of the coordinate transformation, s = 1-s, $\zeta\left(\frac{1}{2}+iy\right) = \zeta(\frac{1}{2}-iy)$.

Because the properties of these type of exponential functions are well known. We know that the equation we are using are symmetric, the places where the definitions of the numbers change with respect to one another.

§ 4.1 Proof of the RH:

Lemma 1): *The roots of ζ are symmetric with respect to the iy axis.*

Proof of Lemma 1) requires the RH. The Functional Equation asserts that $\xi(s)$ possesses the same value with the insertion of $\xi(s' = (1-s))$. Thus, if all of the complex roots of ζ in fact lie on the critical line, the roots must be symmetric with respect to the iy axis. Therefore, to prove Lemma 1), we have proven the RH because Lemma 1) implies that the RH is true.

Lemma 2): *$\exists$ a ζ' such that ζ' is notationally symmetric with respect to ζ.*

Proof of Lemma 2): Employing Eq 1.2 on ζ produces a function which possesses the exact values of ζ, but which has definitions that change at certain values of x. See Eq 1) for proof of analyticity. Eq 1.2 applies to the individual terms of ζ, of which obviously superimpose to produce ζ, therefore the desired properties are unaffected by the application of Eq 1.2.

Lemma 3)*: The roots of ζ must lie at points of definition swap of ζ and ζ' if Lemma 1) is true. In other words, values of definition swap imply values of symmetry.*

Proof of Lemma 3): For a single variable function, due to the Extreme Value Theorem, extrema must lie between roots. Therefore, due to continuity, the roots, if they all lie along a single line must be between extremal values. Taking x to be real, the extremal values of ζ are therefore bounded by $\xi(s = 0)\ and\ \xi(s = 1)$ because at these values (0 and 1), ξ takes the same value. Obviously $\zeta(s = 0\ or\ 1)$ is ∞, which are locations where the definition of ζ is ambiguous. Because ζ' was manufactured with respect to ζ, the functions are notationally symmetric. If one assigns coordinates (U,iV) and (U',iV') to the real and imaginary values of ζ and ζ' there are obviously values for which $\mathbb{R}(\zeta) \neq \mathbb{R}(\zeta')$ and $\mathbb{I}(\zeta) \neq \mathbb{I}(\zeta')$, even though we defined ζ' such that $\mathbb{R}(\zeta) = \mathbb{R}(\zeta')$ and $\mathbb{I}(\zeta) = \mathbb{I}(\zeta')$. This breaks the symmetry that we invoked and so therefore, these values must be values of symmetry. Due to the fact that ξ and ξ' is defined with respect to ζ and ζ', the locations of symmetry breaking (which imply symmetry) are with respect to the locations of ambiguity.

Lemma 4)*: x = $\frac{1}{2}$ is the point of symmetry with respect to the two definitions ζ and ζ'.*

Proof of Lemma 4):

The minima of ξ at x = $\frac{1}{2}$ from Lemma 3) obviously corresponds to the points of definition swap. Parameterizing ζ' such that x' = 1 – x, and y' = -y,

Note that the complex zeros of a function must occur when the real and imaginary components of that function intersect at zero (zero is with respect to choice of coordinate). The places where zeros occur thus cause the real valued operations Re($\zeta(|x|\ )$) and Im($\zeta(|x|\ )$) to themselves become imaginary in the primed function. These points are the points where thus, the arbitrary definitions of real and imaginary components swap. It is due to the fact that independent as to whether the values swap, that the function will still output the expected answer. This means that the locations of symmetry are the places where the Im() terms and the Re() terms become imaginary, are the locations which cause for the numerical operations to change definition with respect to the definitions originally assigned. The zeros of the real terms are given by $\cos(\pi x) = \cos(\pi(1-x)$ (due to Eq 1.2), which require $x = \frac{n}{2}$. Note that this location corresponds to the maxima of $\sin(\pi x)$.

So, the symmetrically defined equation also swaps the sign at x = n/2. Because the Euler Product diverges for $|x|$ < 0, we are forced to select $x = \frac{1}{2}$ as the point of symmetry.

Thus, if the facts mentioned are true, and the definitions possess that which is described, Lemma 4) implies Lemma 3) is true. The truth of Lemma 1) hinges upon that of Lemma 3) and therefore because the truth of the RH hinges on the truth of Lemma 1), the RH would become Riemann's Conjecture.

Conclusion

This article develops a symmetry breaking technique for sourcing locations of symmetry and utilizes the basic facts of the RZF to design a proof of the RH which is intended to be generalizable

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