The Non-existence of Perfect Cuboid

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Abstract

A perfect cuboid, popularly known as a perfect Euler brick/a perfect box, is a cuboid having integer side lengths, integer face diagonals and an integer space diagonal. Euler provided an example where only the body diagonal became deficient for an integer value but it is known as an Euler brick. Nobody has discovered any perfect cuboid, however many of us have tried it. The results of this research paper prove that there exists no perfect cuboid.

Keywords: Perfect Cuboid; Perfect Box; Perfect Euler Brick; Diophantine equation.

1 Introduction

A cuboid, an Euler brick, is a rectangular parallelepiped with integer side dimensions together with the face diagonals also as integers. The earliest time of the problem of finding the rational cuboids can go back to unknown time, however its existence can be found even before Euler's work. The definition of an Euler brick in geometric terms can be formulated mathematically which equivalent to a solution to the following system of Diophantine equations:

$$a^2 + b^2 = d^2,\ a, b, d \in \mathbb{N}; \tag{1}$$

$$b^2 + c^2 = e^2,\ b, c, e \in \mathbb{N}; \tag{2}$$

$$a^2 + c^2 = f^2,\ a, c, f \in \mathbb{N}; \tag{3}$$

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where a, b, c are the edges and d, e, f are the face diagonals. In 1719, Paul Halke [1] found the first known and smallest Euler Brick with side lengths $\{a, b, c\} = (44, 117, 240)$ and face diagonals $\{e, f, g\} = (125, 244, 267)$. Nicholas Saunderson [2], who was blind from the age of one and the fourth Lucasian Professor at Cambridge, obtained a parametric solution to the Euler Brick. Already, families of Euler bricks was announced in 1740 and Euler himself constructed more families of Euler bricks.

Perfect Cuboid is defined as a cuboid where the space diagonal also has integer length. In other words, the following Diophantine equation

$$a^2 + b^2 + c^2 = g^2;\ a, b, c, g \in \mathbb{N} \tag{4}$$

is included to the system of Diophantine equations (2)-(3) defining an Euler brick.

There is a question in everyone's mind: "does a perfect cuboid exist"? Nobody has discovered any perfect cuboid nor has it been established that one does not exist, however many of us have tried it. This problem was in great attention during the 18th century and Saunderson [2] reported a parametric solution

$$(a, b, c) = (x(4y^2 - z^2), y(x^2 - z^2), 4xyz) \text{ (if (x,y,z) be a Pythagorean tripple)}, \tag{5}$$

known as Euler cuboids which always provide Euler bricks although it does not deliver all possible Euler bricks. We could hope that some of these Euler cuboids are perfect, but Spohn [3] demonstrated that no Euler cuboid can give a perfect cuboid. Althogh, Spohn [4] was not completely proved that a derived cuboid of an Euler cuboid failed to be perfect either, but Chein [5] and Lagrange [6] both established that this could indeed never happen. Leech [7] provided a one page proof that no Euler cuboid nor its derived cuboid can be perfect. In 1770 and 1772, Euler introduced at least two parametric solutions. Euler produced an example where only the body diagonal falled short of an integer value (Euler brick). Colman [8] shows infinitely many two-parameter parametrizations of rational cuboids (whose all of the seven lengths are integers except possibly for one edge (called an edge cuboid) or face diagonal (called a face cuboid)) with rapidly increasing degree.

With the help of elementary analysis of the equations for a rational cuboid modulo for some small primes, Kraitchik [10] reported that at least one of the sides of a rational cuboid has a divisor 4 and another one is divided by 16. Moreover, the sides has divisors as different powers of 3 and at least one of the sides is divisible by both the primes 5 and 11. The equally elementary extension of this result was carried out by Horst Bergmann, whereas Leech showed that the product of all the sides and diagonals (edge and face) of a perfect cuboid is divisible by $2^8 \times 3^4 \times 5^3 \times 7 \times 11 \times 13 \times 17 \times 19 \times 29 \times 37$ (cf. [9], Problem D18). Kraitchik [10] also rediscovered the Euler cuboids of (5) and provided a list of 50 rational cuboids which are not Euler cuboids by using some ad hoc methods. He extended his classical list to 241 cuboids having the odd sides less than 10^6 in [11] and found 18 more in [12] out of these 16 were new.

Lal and Blundon [13] pointed out that for integers $m,\ n,\ p$ and q; the cuboid having sides $x = |2mnpq|,\ y = |mn(p^2 - q^2)|$ and $z = |pq(m^2 - n^2)|$ has at least two face diagonals as integers and is rational cuboid if and only if $y^2 + z^2 = \square$. Using the symmetry, they aimed for a computer search through all the quadruples $(m,\ n,\ p,\ q)$ satisfying $1 \leq m, n, p, q \leq 70$ to check if $y^2 + z^2 = \square$ and reported 130 rational cuboids, out of which none are perfect, however Shanks [14] pointed out some corrigenda about their paper.

Korec [15] found no perfect cuboids having the least side smaller than 10000 by the consideration as follows: let $x,\ y,\ z$ are the sides of a perfect cuboid, then we can find natural numbers $a,\ b,\ c$ all dividing x and $t = \sqrt{y^2 + z^2}$ such that $y = \frac{1}{2}\left(\frac{x^2}{a} - a\right),\ z = \frac{1}{2}\left(\frac{x^2}{b} - b\right),\ t = \frac{1}{2}\left(\frac{x^2}{c} - c\right)$. Korec [16] extended his result and found no perfect cuboid if the least edge smaller than 10^6. In another research paper, Korec [17] not found any perfect cuboid if the full diagonal of a perfect cuboid is $< 8 \times 10^9$. Moreover, if x and z are the maximal edge and the full diagonal respectively of a perfect cuboid, then $z \leq x\sqrt{3}$.

Rathbun [18] found 6800 body, 6749 face, 6380 edge and no perfect cuboids by a computer search if a side $x \leq 333750000$, while in his another research paper [19], he reported that 4839 of the 6800 body or rational cuboids contain an odd side less than 333750000 which is a extention and correction of Kraitchiks classical table [10, 11, 12]. If all "odd sides" $\leq 10^{10}$, then Butler [20] found no perfect cuboids despite an thorough computer search.

This article is dedicated for the answer of the question "are there perfect cuboids?". It has been discovered that there is no perfect cuboid.

2 Results and Discussion

2.1 Pythagorean quadruple

A set of four natural numbers $(a,\ b,\ c,\ d)$ is well known as a Pythagorean quadruple if the equation $a^2 + b^2 + c^2 = d^2$ satisfies. The simplest example of a quadruple is $(1,\ 2,\ 2,\ 3)$ as $1^2 + 2^2 + 2^2 = 3^2$ and $(2,\ 3,\ 6,\ 7)$ is the next simplest (primitive) example as $2^2 + 3^2 + 6^2 = 7^2$.

All the primitive quadruples [21] can be generated by the equation

$$(m^2+n^2+p^2+q^2)^2 = (m^2+n^2-p^2-q^2)^2 + 4(mp+nq)^2 + 4(mq-np)^2;\ m,n,p,q \in \mathbb{N}. \quad (6)$$

We can find a primitive perfect cuboids iff the following equations are also true.

$$4(mp+nq)^2 + 4(mq-np)^2 = A^2,\ A \in \mathbb{N}; \quad (7)$$

$$(m^2+n^2-p^2-q^2)^2 + 4(mp+nq)^2 = B^2,\ B \in \mathbb{N}; \quad (8)$$

$$(m^2+n^2-p^2-q^2)^2 + 4(mq-np)^2 = C^2,\ C \in \mathbb{N}. \quad (9)$$

2.2 Theorem

For the natural numbers $(a,\ b,\ c,\ d,\ e)$; let $(a,\ b,\ d)$, $(a,\ c,\ e)$ be the Pythagorean tripples (three natural numbers)

such that $a^2+b^2=d^2$ and $a^2+c^2=e^2$, then $a^2 = \frac{(m_2^2-m_1^2)(n_1^2-n_2^2)}{4}$, $b = \frac{m_1n_1+m_2n_2}{2}$, $c = \frac{m_2n_2-m_1n_1}{2}$, $d = \frac{m_2n_1+m_1n_2}{2}$, $e = \frac{m_1n_2-m_2n_1}{2}$; where $m_1, m_2, n_1, n_2 \in \mathbb{N}$. (10)

Proof: From the equations (10), we can obtain $b^2-c^2=d^2-e^2$.

Or, $\frac{b-c}{d-e} = \frac{d+e}{b+c} = \frac{m_1}{m_2}$ where $m_1,\ m_2\ (\in \mathbb{N})$ are relatively prime numbers. (11)

Then $b-c = \frac{m_1}{m_2}(d-e),\ b+c = \frac{m_2}{m_1}(d+e)$ (12)

Since $b-c$ and $b+c$ are integers and $m_1,\ m_2\ (\in \mathbb{N})$ are relatively prime numbers,

$d-e=m_2n_1$ and $d+e=m_1n_2$. Thus $d = \frac{m_2n_1+m_1n_2}{2}$, $e = \frac{m_1n_2-m_2n_1}{2}$. (13)

From equations (12) and (13), we get

$$b-c=m_1n_1,\ b+c=m_2n_2 \text{ i.e. } b = \frac{m_1n_1+m_2n_2}{2},\ c = \frac{m_2n_2-m_1n_1}{2}. \quad (14)$$

Then from equation (10), (13) and (14) we get

$$a^2 = d^2-b^2 = \frac{(m_2^2-m_1^2)(n_1^2-n_2^2)}{4}. \quad (15)$$

2.3 Lemma

For the natural numbers $m, n, p, q \in \mathbb{N}$ if $m^2 + n^2 = k_1^2 r,\ p^2 + q^2 = k_2^2 r$ then there exists no perfect cuboid.

Proof: From the equation (7), if

$$4(mp+nq)^2 + 4(mq-np)^2 = 4(m^2+n^2)(p^2+q^2) = A^2,\ A \in \mathbb{N} \tag{16}$$

$$\text{with } m^2+n^2 = k_1^2 r,\ p^2+q^2 = k_2^2 r \text{ i.e. } m^2p^2 + m^2q^2 + n^2p^2 + n^2q^2 = k_1^2k_2^2r^2, \tag{17}$$

$$\text{then } 4(mp+nq)^2 + 4(mq-np)^2 = 4k_1^2k_2^2r^2 = A^2,\ A \in \mathbb{N}. \tag{18}$$

Case (i):

$$\text{If } mp+nq = rk_1k_2 \text{ i.e. } m^2p^2 + n^2q^2 + 2mnpq = r^2k_1^2k_2^2, \text{ then from equations} \tag{19}$$

$$\text{(8) and (17), we get } r^2(k_1^2-k_2^2)^2 + 4r^2k_1^2k_2^2 = B^2 = r^2(k_1^2+k_2^2)^2,\ B \in \mathbb{N}. \tag{20}$$

$$\text{Also from equation (17) and (19), } mq-np = 0. \tag{21}$$

Thus the equation (9) has no Pythagorean tripple and hence, there exists no perfect cuboid for this case.

Case (ii):

$$\text{If } mq-np = rk_1k_2 \text{ i.e. } m^2q^2 + n^2p^2 - 2mnpq = r^2k_1^2k_2^2 \text{ then from equations} \tag{22}$$

$$\text{(17) and (9), we get } r^2(k_1^2-k_2^2)^2 + 4r^2k_1^2k_2^2 = C^2 = r^2(k_1^2+k_2^2)^2,\ C \in \mathbb{N}. \tag{23}$$

$$\text{Also from equation (17) and (22), } mp+nq = 0. \tag{24}$$

Hence the equation (8) has no Pythagorean tripple and hence, there exists no perfect cuboid for this case.

Case (iii):

Let us consider $m_1 = k_1$, $m_2 = k_2$, $n_1 = 2rk_2$ and $n_2 = 2rk_1$ in the equations (13), (14) and (15); then $a^2 = r^2(k_2^2 - k_1^2)^2$, $b = 2rk_1k_2$, $c = 0$. Then the equation (9) has no Pythagorean tripple and hence, there exists no perfect cuboid for this case.

2.4 Lemma

For the natural numbers $m, n, p, q \in \mathbb{N}$ if $m^2+n^2 = k_1^2,\ p^2+q^2 = k_2^2$; then there exists no perfect cuboid.

Proof: From the equation (7), if

$$4(mp+nq)^2+4(mq-np)^2 = 4(m^2+n^2)(p^2+q^2) = A^2,\ A\in\mathbb{N} \tag{25}$$

$$\text{with } m^2+n^2 = k_1^2,\ p^2+q^2 = k_2^2 \text{ i.e. } m^2p^2+m^2q^2+n^2p^2+n^2q^2 = k_1^2k_2^2, \tag{26}$$

$$\text{then } 4(mp+nq)^2+4(mq-np)^2 = 4k_1^2k_2^2 = A^2,\ A\in\mathbb{N} \tag{27}$$

Case (i):

$$\text{If } mp+nq = k_1k_2 \text{ i.e. } m^2p^2+n^2q^2+2mnpq = k_1^2k_2^2, \text{ then from equations} \tag{28}$$

$$\text{(8) and (26), we get } (k_1^2-k_2^2)^2+4k_1^2k_2^2 = B^2 = (k_1^2+k_2^2)^2,\ B\in\mathbb{N}. \tag{29}$$

$$\text{Also from equation (26) and (28), } mq-np = 0. \tag{30}$$

Thus the equation (9) has no Pythagorean tripple and hence, there exists no perfect cuboid for this case.

Case (ii):

$$\text{If } mq-np = k_1k_2 \text{ i.e. } m^2q^2+n^2p^2-2mnpq = k_1^2k_2^2 \text{ then from equations} \tag{31}$$

$$\text{(26) and (9), we get } (k_1^2-k_2^2)^2+4k_1^2k_2^2 = C^2 = (k_1^2+k_2^2)^2,\ C\in\mathbb{N}. \tag{32}$$

$$\text{Also from equation (26) and (31), } mp+nq = 0. \tag{33}$$

Hence the equation (8) has no Pythagorean tripple and hence, there exists no perfect cuboid for this case.

Case (iii):

Let us consider $m_1 = k_1$, $m_2 = k_2$, $n_1 = 2k_2$ and $n_2 = 2k_1$ in the equations (13), (14) and (15); then $a^2 = (k_2^2-k_1^2)^2$, $b = 2k_1k_2$, $c = 0$. Then the equation (9) has no Pythagorean tripple and hence, there exists no perfect cuboid for this case.

2.5 Lemma

For the natural numbers $m, n, p, q \in \mathbb{N}$ if $m^2+n^2 = r,\ p^2+q^2 = k_2^2r$ then there exists no perfect cuboid.

Proof: From the equation (7), if

$$4(mp+nq)^2+4(mq-np)^2 = 4(m^2+n^2)(p^2+q^2) = A^2\ (A\in\mathbb{N}), \tag{34}$$

$$\text{with } m^2+n^2 = r,\ p^2+q^2 = k_2^2r \text{ i.e. } m^2p^2+m^2q^2+n^2p^2+n^2q^2 = r^2k_2^2, \tag{35}$$

$$\text{then } 4(mp+nq)^2+4(mq-np)^2 = 4k_2^2r^2 = A^2,\ A\in\mathbb{N} \tag{36}$$

Case (i):

If $mp + nq = rk_2$ i.e. $m^2p^2 + n^2q^2 + 2mnpq = r^2k_2^2$, then from equations (37)

(8) and (35), we get $r^2(1 - k_2^2)^2 + 4k_1^2k_2^2 = B^2 = r^2(1 + k_2^2)^2,\ B \in \mathbb{N}$. (38)

Also from equation (35) and (37), $mq - np = 0$. (39)

Thus the equation (9) has no Pythagorean tripple and hence, there exists no perfect cuboid for this case.

Case (ii):

If $mq - np = rk_2$ i.e. $m^2q^2 + n^2p^2 - 2mnpq = r^2k_2^2$ then from equations (40)

(35) and (9), we get $r^2(1 - k_2^2)^2 + 4r^2k_2^2 = C^2 = r^2(1 + k_2^2)^2,\ C \in \mathbb{N}$. (41)

Also from equation (35) and (40), $mp + nq = 0$. (42)

Hence the equation (8) has no Pythagorean tripple and hence, there exists no perfect cuboid for this case.

Case (iii):

Let us consider $m_1 = k_2$, $m_2 = 1$, $n_1 = 2r$ and $n_2 = 2rk_2$ in the equations (13), (14) and (15); then $a^2 = r^2(1 - k_2^2)^2$, $b = 2rk_2$, $c = 0$. Then the equation (9) has no Pythagorean tripple and hence, there exists no perfect cuboid for this case.

2.6 Lemma

For the natural numbers $m, n, p, q \in \mathbb{N}$ if $m^2 + n^2 = k_1^2 r,\ p^2 + q^2 = r$; then there exists no perfect cuboid.

Proof: From the equation (7), if

$$4(mp + nq)^2 + 4(mq - np)^2 = 4(m^2 + n^2)(p^2 + q^2) = A^2,\ A \in \mathbb{N} \tag{43}$$

$$\text{with } m^2 + n^2 = k_1^2 r,\ p^2 + q^2 = r \text{ i.e. } m^2p^2 + m^2q^2 + n^2p^2 + n^2q^2 = k_1^2 r^2, \tag{44}$$

$$\text{then } 4(mp + nq)^2 + 4(mq - np)^2 = 4k_1^2 r^2 = A^2,\ A \in \mathbb{N} \tag{45}$$

Case (i):

If $mp + nq = rk_1$ i.e. $m^2p^2 + n^2q^2 + 2mnpq = r^2k_1^2$, then from equations (46)

(8) and (44), we get $r^2(k_1^2 - 1)^2 + 4r^2k_1^2 = B^2 = r^2(k_1^2 + 1)^2,\ B \in \mathbb{N}$. (47)

Also from equation (44) and (46), $mq - np = 0$. (48)

Thus the equation (9) has no Pythagorean tripple and hence, there exists no perfect cuboid for this case.

Case (ii):

If $mq - np = rk_1$ i.e. $m^2q^2 + n^2p^2 - 2mnpq = r^2k_1^2$ then from equations (49)

(44) and (9), we get $r^2(k_1^2 - 1)^2 + 4r^2k_1^2 = C^2 = r^2(k_1^2 + 1)^2, \ C \in \mathbb{N}$. (50)

Also from equation (17) and (49), $mp + nq = 0$. (51)

Hence the equation (8) has no Pythagorean tripple and hence, there exists no perfect cuboid for this case.

Case (iii):

Let us consider $m_1 = 1$, $m_2 = k_1$, $n_1 = 2rk_1$ and $n_2 = 2r$ in the equations (13), (14) and (15); then $a^2 = r^2(k_1^2 - 1)^2$, $b = 2rk_1$, $c = 0$. Then the equation (9) has no Pythagorean tripple and hence, there exists no perfect cuboid for this case.

2.7 Remark

The results of other cases for the values of the m_1, m_2, n_1, n_2 regarding the Lemmas 2.3-2.6 are straight forward. For example, if you consider the values of the variables as $m_1 = rk_1$, $m_2 = rk_2$, $n_1 = 2k_2$, $n_1 = 2k_1$ in the equations (13), (14) and (15); then $a^2 = r^2(k_2^2 - k_1^2)^2$, $b = 2rk_1k_2$, $c = 0$. Then the equation (9) has no Pythagorean tripple and hence, there exists no perfect cuboid for this case.

2.8 Lemma

For the natural numbers $m, n, p, q \in \mathbb{N}$, $m^2 + n^2 = p^2 + q^2$, then there exists no perfect cuboid.

Proof: If the natural numbers $m, n, p, q \in \mathbb{N}$ such that $m^2 + n^2 = p^2 + q^2$, then from equations (6), (8) and (9) we can't get any Pythagorean quadruple and Pythagorean tripple respectively and hence, there exists no perfect cuboid for this case.

2.9 Theorem

There exists no perfect cuboid.

Proof: We can discover any perfect cuboid only when we can get any natural number solutions of equations (6)-(9). Hence by the the Lemmas 2.3-2.8, we can conclude that there exists no perfect cuboid.

Acknowledgment: *I am grateful to the University Grants Commission (UGC), New Delhi for awarding the Dr. D. S. Kothari Post Doctoral Fellowship from 9th July, 2012 to 8th July, 2015 at Indian Institute of Technology (BHU), Varanasi. The self training for this investigation was continued during the period.*

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