Every perfect number has only one odd prime factor

Enrique Santos, July 2020

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Abstract: Using properties of the sum of divisors function, the general form of a perfect number is inferred from its definition, and expressed in terms of the sum of divisors function, together with the condition for a natural number of that form to be a perfect number. In short, here it is proved that every perfect number is half the product of two consecutive natural numbers, the first one being prime, and the second one being a power of 2. The reciprocal affirmation is also inferred here, although it was already proved more than a century ago, and known more than two millennia ago: whenever a power of two is the successor of a prime, the product of both is twice a perfect number.

Multiplicative property of the sum of divisors

The divisors of a natural power of a prime, p^x, are the successive powers from 0 to x, so its sum is:

$$\sigma(p^x) = 1 + p + \ldots + p^{x-1} + p^x \tag{1}$$

Multiplying two expressions like this for two different primes, p^x and q^y, the resulting set of addends would be equal to all the divisors of the product $p^x \cdot q^y$. Consequently, $\sigma(p^x) \cdot \sigma(q^y) = \sigma(p^x \cdot q^y)$.

Generalizing this result to more prime powers, it can be deduced that **the sum of divisors function is multiplicative** for co-prime factors:

$$\sigma\left(\prod_p p^{x_p}\right) = \prod_p \sigma(p^{x_p}) \tag{2}$$

It means that $\sigma(a \cdot b) = \sigma(a) \cdot \sigma(b)$ whenever a is co-prime to b. The reciprocal is also true, but it is not needed on this text anyhow.

A calculation formula for the sum of divisors

Separating in Eq.(1) just the last addend on one side, and just the first addend on the other side, two equivalent expressions of $\sigma(p^x)$ as a function of $\sigma(p^{x-1})$ can be equated:

$$\sigma(p^{x-1}) + p^x = 1 + p \cdot \sigma(p^{x-1}) \tag{3}$$

Solving for $\sigma(p^{x-1})$, the following expression is obtained, which is a usual **formula for calculation**:

$$\sigma(p^{x-1}) = \frac{p^x - 1}{p - 1} \tag{4}$$

As every number can be factored into the product of prime powers, this formula, along with the multiplicative property, allows the calculation of the sum of divisors of any natural number greater than 1, if it can be fully factored. The only exception of this formula is the obvious case, $\sigma(1) = 1$.

Definition and general expression of a perfect number

A perfect number N is defined as a natural number whose sum of divisors, $\sigma(N)$, is equal to the double of itself. This can be expressed by the following recursive equation:

$$N = \frac{\sigma(N)}{2} \tag{5}$$

Expressing a perfect number N as a product of prime powers, its sum of divisors would have the form of Eq.(2). Then, at least one of the prime powers must have an even sum of divisors, because of the 2 on the definition. Let p^x be that prime power. As a consequence, $p > 2$, because $\sigma(2^x)$ is not even, as required. Let A be the factor of N co-prime to p, so that $N = p^x \cdot A$,

$$p^x \cdot A = \frac{\sigma(p^x)}{2} \cdot \sigma(A) \tag{6}$$

Part I: Main exponent

Decomposition

The number of divisors of p^x is one more than the exponent, as can be seen from Eq.(1), and has to be even in order that its sum is even, because every divisor is odd, so it can be written as: $x + 1 = 2 \cdot y$

Using this on the previous relation, the first factor on the right can be further decomposed from the calculation formula (4):

$$\frac{\sigma(p^x)}{2} = \frac{\sigma(p^{2 \cdot y - 1})}{2} = \frac{p^{2 \cdot y} - 1}{2 \cdot (p-1)} = \left(\frac{1 + p^y}{2}\right) \cdot \left(\frac{p^y - 1}{p - 1}\right) \tag{7}$$

It can be seen that the difference between numerators on the right is 2, so its halves are consecutive numbers, and therefore, co-prime. Since both numerators are divided by an even quantity, the two factors resulting on the right are co-prime.

Let's call them b and c respectively to simplify expressions, noting that the second factor is $\sigma(p^{y-1})$, by Eq.(4):

$$b = \frac{1 + p^y}{2}; \qquad c = \sigma(p^{y-1}) \tag{8}$$

Now, grouping in one side the terms with p, and in the other side the terms with A, a more detailed fractional relation is obtained from Eq.(6):

$$\frac{p^{2 \cdot y - 1}}{b \cdot c} = \frac{\sigma(A)}{A} \tag{9}$$

Two bounded equations

The numerator $\sigma(A)$ can be factored into sums of divisors of the co-prime factors of A, as seen on Eq.(2). Since A is also the denominator, the whole quotient $\sigma(A)/A$ is also multiplicative, so, it can be separated according to the left denominator partition, by factoring $A = B \cdot C$ such that b is a divisor of B, c is a divisor of C, and B is co-prime to C.

Besides, the left term is an irreducible fraction, because p is co-prime to b and c, which are also co-primes. Therefore, it is possible to factor $p^{2\cdot y-1}$ into two factors in any possible way, and decompose the left term into the product of two irreducible fractions, with denominators b and c. A parameter $z < 2\cdot y$, is used to get any of the possible partitions of the exponent: $2\cdot y - 1 = (2\cdot y - z) + (z - 1)$

$$\frac{p^{2\cdot y-z}}{b}\cdot\frac{p^{z-1}}{c}=\frac{\sigma(B)}{B}\cdot\frac{\sigma(C)}{C} \tag{10}$$

In this way, there has to be some value of z for which the previous equation can be separated into two equations. After substituting b and c by its values from Eq.(8), the new system is

$$\begin{cases} \dfrac{p^{2\cdot y-z}}{(1+p^y)/2} & = \dfrac{\sigma(B)}{B} > 1 \\ \dfrac{p^{z-1}}{\sigma(p^{y-1})} & = \dfrac{\sigma(C)}{C} \geq 1 \end{cases} \tag{11}$$

The inequalities can be established because the sum of divisors of any number is greater than the number itself, or equal just in case the number is 1 (which is not possible for the upper left part).

This fact also prevents considering other possible fractions multiplying on the left sides, inverse one of the other on each equation.

A simpler relation results

The first inequality leads to $z \leq y$, in order that the numerator be greater than the denominator.

But the second inequality would require that $z > y$ when $y > 1$, for the numerator to be greater.

So the only possibility is that $z = y = 1$, which implies that p has no exponent, $p^{2\cdot y-1} = p$, and the second factor in Eq.(7) disappears, $c = 1$. Notice that this result would not change considering other possible fractions multiplying on the left sides, as previously said.

Then, only the upper equation in Eq.(11) remains, with $B = A$, which can be written as an integer relation, instead of fractional, in order to identify it with the original definition of the perfect number $N = p\cdot A$,

$$p\cdot A=\frac{1+p}{2}\cdot\sigma(A) \tag{12}$$

Part II: Solution

Equivalent system of equations, and a recombination

Because of the multiplicative property Eq.(2), there must be only one prime power divisor of A, q^r, such that $\sigma(q^r) = E\cdot p$ is a divisor of $\sigma(A)$ with factor p.

Then, taking into account that q can not be a factor of $\sigma(q^r)$, another relation can be obtained for the divisors of $(1 + p)/2$, which must contain part of q^r, let's call it q^s with $s \leq r$:

In order to have a complete system equivalent to Eq.(12) there is needed a third relation which equates the remaining part of A to the remaining part of $\sigma(A)$.

$$\begin{cases} p\cdot E=\sigma(q^r) \\ q^s\cdot F=(1+p)/2 \\ q^{r-s}\cdot G=\sigma(E\cdot F\cdot G) \end{cases} \tag{13}$$

Therefore now the perfect number N is the product of the left sides, $N = p \cdot q^r \cdot E \cdot F \cdot G$. Taking p from the second equation, and substituting it on the first one, the following expression results:

$$\sigma(q^r) = 2 \cdot E \cdot F \cdot q^s - E \tag{14}$$

It is used now, in combination with the fact that $\sigma(q^r) = \sigma(q^{r-1}) + q^r$, which is obvious from the expanded sum of powers Eq.(1), as well as $\sigma(q^r) = \sigma(q^{s-1}) + q^s \cdot \sigma(q^{r-s})$, to get the following relations:

$$\begin{aligned} \sigma(q^{r-1}) &= (\, 2 \cdot E \cdot F - q^{r-s} \qquad\qquad) \cdot q^s - E \\ \sigma(q^{s-1}) &= (\, 2 \cdot E \cdot F - \sigma(q^{r-s}) \qquad) \cdot q^s - E \end{aligned} \tag{15}$$

Analysis of greatest common divisor

As every single quantity in Eq.(15) is greater or equal to 1, the quantity between parenthesis in any of the two equations must also be greater or equal to 1. Let's equate the upper one to $1 + k$, where $k \geq 0$ (the same can be done for the bottom one):

$$1 + k = 2 \cdot E \cdot F - q^{r-s} \tag{16}$$

Besides, since $(E \cdot F)$ is co-prime to q^{r-s}, its greatest common divisor is $1 = gcd(E \cdot F, q^{r-s})$. So there must exist two integers, u and v, such that

$$1 = u \cdot E \cdot F - v \cdot q^{r-s} \tag{17}$$

Multiplying the right side by k, and subtracting it from the right side of the previous relation, the result must be equal to 1, as the subtraction of the left sides is $(1 + k) - 1 \cdot k = 1$

$$1 = (2 - k \cdot u) \cdot E \cdot F - (1 - k \cdot v) \cdot q^{r-s} \tag{18}$$

This relation has the same form than the previous one, so that the parameters u and v can be identified

$$\begin{cases} u = 2 - k \cdot u \\ v = 1 - k \cdot v \end{cases} \tag{19}$$

which in turn implies that $k = 0$, $v = 1$ and $u = 2$.

General form of a perfect number

Then, Eq.(15) is reduced to

$$\sigma(q^{r-1}) = q^s - E \tag{20}$$

But $\sigma(q^{r-1})$ must be strictly greater than q^{r-1}, so that $q^{r-1} < q^s \leq q^r$. Therefore $s = r$, so $q^{r-s} = 1$, which implies $E \cdot F = 1$. Then, using these values on the last equation, as well as the calculation formula for the sum of divisors of a prime power, Eq.(4),

$$\frac{q^r - 1}{q - 1} = q^r - 1 \tag{21}$$

The trivial and only solution for q on this equation is $q = 2$. Then we arrive to the general form of every perfect number N:

$$N = p \cdot 2^r \tag{22}$$

which means that **any perfect number must be the product of an odd prime number and a power of 2**.

Necessary and sufficient condition to be a perfect number

The condition for a number of this form to be a perfect number is given by any of the first two equations of Eq.(13) system, which now becomes much simpler:

$$\begin{cases} p = \sigma(2^r) \\ 2^{1+r} = \sigma(p) \end{cases} \tag{23}$$

Any of them is a **necessary and sufficient condition for *$p \cdot 2^r$* to be a perfect number**. They can be read in the following way, which are two parameterization of all the perfect numbers, with parameters *r* and *p* respectively:

> Whenever the sum of divisors of a power of 2, $\sigma(2^r)$, be prime, the number $2^r \cdot \sigma(2^r)$ is a perfect number.

> Whenever the sum of divisors of a prime, $\sigma(p)$, be a power of 2, the number $p \cdot \sigma(p)$ is twice (i.e., the sum of divisors of) a perfect number.

In fact, they can be deduced from knowing that a number of the form of Eq.(22) is a perfect number, because in that case, it must fulfill the condition $\sigma(N) = 2 \cdot N$:

$$\sigma(p) \cdot \sigma(2^r) = 2 \cdot p \cdot 2^r \tag{24}$$

which gives the previous system after separating the co-prime factors, because *p* is co-prime to $\sigma(p)$, and 2 is co-prime to $\sigma(2^r)$.

Since the sum of divisors of a prime is its successor, and the sum of divisors of a power of 2 is the predecessor of its double, an alternative, may be simpler, way of expressing the form and condition of a perfect number is:

> Every perfect number is half the product of two consecutive natural numbers, the first one being prime, and the second one being a power of 2.

Conversely,

> Whenever a power of two is the successor of a prime, the product of both is twice (i.e., the sum of divisors of) a perfect number.