AN INEQUALITY APPROACH OF THE COLLATZ CONJECTURE

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Abstract:

We define the Collatz Function $F_{col}: \mathbb{N} \to \mathbb{N}$ as follows-

$$F_{col}(N) \coloneqq \frac{N}{2}, if \;\; N \; is \; even$$

$$F_{col}(N) \coloneqq 3N + 1, if \; N \; is \; odd$$

We define the two branches of the above function $f: \mathbb{N} \to \mathbb{N}$ and $g: \mathbb{N} \to \mathbb{N}$ as follows-

$$f: \mathbb{N} \to \mathbb{N}(N) \coloneqq \frac{N}{2}, if \;\; N \; is \; even$$

$$g: \mathbb{N} \to \mathbb{N}(N) \coloneqq 3N + 1, if \; N \; is \; odd$$

We also define the functional sequence of a number as the set of functions applied consecutively on a certain natural number until the number 1 is obtained, and then prove that any two consecutive $g's$ are separated by at least one f . We then define $\Psi_k(n)$ as the value obtained by the execution of the functional sequence:

$$S(k) \equiv \{gf^{\alpha_1} ... gf^{\alpha_k}\}$$

on n.

We then show that if $\Psi_x(n) > n$ for all natural values of x, then there exists a function $f_k(x, n)$ such that

$$\Psi_x(n) > f_k(x, n) \; for \; all \; x, k \in \mathbb{N}, and \lim_{k \to \infty} f_k(x, n) \to \infty$$

This implies that $\Psi_1(n) \to \infty$, which is possible if and only if $n \to \infty$, providing a contradiction. Hence, $\Psi_x(n) < n$ for some $x \in \mathbb{N}$, implying that the Collatz Conjecture is true

INTRODUCTION

The Collatz Conjecture, also referred to as the Ulam Conjecture, the Kakutani Problem, the Thwaites Conjecture, Hasse's Algorithm or the Syracuse Problem, was proposed in 1937 by German Mathematician Lothar Collatz.

The conjecture states that if we define the function

$F_{col}: \mathbb{N} \rightarrow \mathbb{N}$ such that-

$$F_{col}(N) := \frac{N}{2}, if \;\; N \; is \; even$$

$$F_{col}(N) := 3N + 1, if \; N \; is \; odd$$

Then $\forall \; N \in \mathbb{N}, \exists \; k \in \; \mathbb{N}$ such that ${F_{col}}^k(N) = 1$, where

${F_{col}}^k(N) = F_{col}(F_{col}(F_{col}(...(N))...)$, where F_{col} is repeated k times.

This conjecture has been verified to be true for all natural numbers till an approximate value of 2^{68}

Since the posing of the problem, there have been many partial results on it, the most recent of which is the partial result established by mathematician Terence Tao, stating that 'almost' all numbers, under repetitive execution of the function $F_{col}: \mathbb{N} \rightarrow \mathbb{N}$, attain almost bounded values. Before that, partial results were established by mathematician Riho Terras in 1976 that 'almost' all numbers x yield an $\Omega < x$, under repetitive execution of the function $F_{col}: \mathbb{N} \rightarrow \mathbb{N}$. This upper bound was later improved to $x^{0.869}$ in 1979, and then it was further improved to $x^{0.7925}$ in 1994.

1. SOME NOTATIONS

Let us define $f: \mathbb{N} \to \mathbb{N}(N) := \frac{N}{2}, if \;\; N \; is \; even$

$$g: \mathbb{N} \to \mathbb{N}(N) := 3N + 1, if \; N \; is \; odd$$

Also, let us define the 'functional sequence' for any $N \in \mathbb{N}$ as the set of functions applied consecutively on a certain natural number until the number 1 is obtained, and let S_N denote the functional sequence of N.

For instance, if we have the natural number 5, we have the following continuous mapping obtained by repetitive execution of the functions $f: \mathbb{N} \to \mathbb{N}$ and $g: \mathbb{N} \to \mathbb{N}$, in obedience with the obtained parities:

$$5 \to 16 \to 8 \to 4 \to 2 \to 1$$

Note that the functions applied, in consecutive order, are $g, f, f, f \; and \; f$. Hence, the functional sequence of 5 is:

$$S_5 \equiv \{gffff\}$$

We can also shorten the sequence to:

$$S_5 \equiv \{gf^4\}$$

Since there are 4 consecutive repetitions of $f: \mathbb{N} \to \mathbb{N}$.

2. SOME IMPORTANT RESULTS

Theorem 2.1

$For \; any \; n \in \mathbb{N}, S_n \; does \; not \; contain \; two \; consecutive \; g's.$

Proof

Let us assume contrarily that $\exists \, n \in \mathbb{N}$ such that S_n contains two consecutive $g's$. Hence, a certain number of executions of $F_{col}: \mathbb{N} \to \mathbb{N}$ yields a certain $\Omega \in \mathbb{N}$ such that the function $g: \mathbb{N} \to \mathbb{N}$ is applicable twice. This is possible if and only if both Ω and $g(\Omega) = 3\Omega + 1$ are odd. We know that the subtraction of two odd numbers is always even. Hence, $(3\Omega + 1) - \Omega = 2\Omega + 1$ is even, $\Omega \in \mathbb{N}$. Evidently, this is a contradiction. Thus, the assumption must be incorrect and hence,

$$For\ any\ n \in \mathbb{N}, S_n\ does\ not\ contain\ two\ consecutive\ g's.$$

This conclude the proof □

The above theorem implies that any two consecutive $g's$ are separated by at least one f. Thus, for any odd n, S_n is of the form:

$$S_n \equiv \{gf^{\alpha_1}gf^{\alpha_2}gf^{\alpha_3}gf^{\alpha_4}...\}, where\ \alpha_i \in \mathbb{N}, \forall i \in \mathbb{N}$$

<u>Theorem 2.2</u>

$$\forall\ N \in \mathbb{N}_{\geq \mathbf{2}}, \exists\ k \in \mathbb{N}\ \underline{such\ that}\ {F_{col}}^k(N) < N$$

<u>Proof</u>

The result is obvious for even values of N. This is because if N is even, then $f(N) = \frac{N}{2} < N$. Hence, we can concern ourselves with the odd values of N only.

Let us now consider a certain odd natural number $n \in \mathbb{N}$. Hence, the functional sequence of n must be of the form:

$$S_n \equiv \{gf^{\alpha_1}gf^{\alpha_2}gf^{\alpha_3}gf^{\alpha_4}...\}, where\ \alpha_i \in \mathbb{N}, \forall\ i \in \mathbb{N}$$

Let us define $\Psi_k(n)$ as the value obtained by the execution of the functional sequence:

$$S(k) \equiv \{gf^{\alpha_1}...gf^{\alpha_k}\}$$

on n.

Now, let us assume contrarily that

$$\forall\, k \in \mathbb{N}, {F_{col}}^{k}(n) \geq n$$

Hence, it is evident that

$$\Psi_k(n) \geq n, \forall\, k \in \mathbb{N} \qquad\qquad (*\,)$$

Claim 2.2.1

$$\Psi_x(n)\ is\ recursive\ and\ follows\ the\ identity:$$

$$\Psi_{k+1}(n) = \frac{3\Psi_k(n)+1}{2^{\alpha_{k+1}}}, \forall\, k \in \mathbb{N}$$

Proof

Note that, by definition, we can write

$$\Psi_{k+1}(n) = f^{\alpha_{k+1}}(g(f^{\alpha_k}(g(\ldots(g\big(f^{\alpha_1}(n)\big)\ldots)$$

and, $$\Psi_k(n) = (f^{\alpha_k}(g(\ldots(g\big(f^{\alpha_1}(n)\big)\ldots)$$

The substitution of the second equation into the first gives-

$$\Psi_{k+1}(n) = f^{\alpha_{k+1}}\left(g\big(\Psi_k(n)\big)\right) = f^{\alpha_{k+1}}(3\Psi_k(n) + 1)$$

Hence,

$$\Psi_{k+1}(n) = \frac{3\Psi_k(n)+1}{2^{\alpha_{k+1}}}$$

This concludes the proof □

Claim 2.2.2

$$\Psi_k(n)\ is\ given\ by -$$

$$\Psi_k(n) = \frac{3^k n + 3^{k-1} + \sum_{i=2}^{k} 3^{k-i} \cdot 2^{\alpha_1 + \ldots + \alpha_{i-1}}}{2^{\alpha_1 + \ldots + \alpha_k}}$$

Proof

Notice that for $k = 1$, $\Psi_1(n)$ is the value obtained by the execution of the functional sequence

$$S \equiv \{g f^{\alpha_1}\}$$

on n.

Hence,

$$\Psi_1(n) = f^{\alpha_1}\big(g(n)\big) = f^{\alpha_1}(3n+1) = \frac{3n+1}{2^{\alpha_1}}$$

which is in accordance with our claim.

This serves as the base for an inductive process.

Let us now assume that

$$\Psi_x(n) = \frac{3^x n + 3^{x-1} + \sum_{i=2}^{x} 3^{x-i} \cdot 2^{\alpha_1 + \ldots + \alpha_{i-1}}}{2^{\alpha_1 + \ldots + \alpha_x}}$$

for a certain $x \in \mathbb{N}$.

Hence, using Claim 2.2.1, we can write $\Psi_{x+1}(n)$ as-

$$\Psi_{x+1}(n) = \frac{3\Psi_k(n)+1}{2^{\alpha_{k+1}}} = \frac{3\left(\frac{3^x n + 3^{x-1} + \sum_{i=2}^{x-1} 3^{x-i} . 2^{\alpha_1 + \ldots + \alpha_{i-1}}}{2^{\alpha_1 + \ldots + \alpha_x}}\right)+1}{2^{\alpha_{x+1}}}$$

Hence, $\Psi_{x+1}(n) = \dfrac{\left(\dfrac{3^{x+1}n + 3^{x} + \sum_{i=2}^{x} 3^{(x+1)-i} . 2^{\alpha_1+\ldots+\alpha_{i-1}} + 2^{\alpha_1+\ldots+\alpha_x}}{2^{\alpha_1+\ldots+\alpha_x}}\right)}{2^{\alpha_{x+1}}}$

Note that,

$$\sum_{i=2}^{x} 3^{(x+1)-i} \cdot 2^{\alpha_1+\ldots+\alpha_{i-1}} + 2^{\alpha_1+\ldots+\alpha_x} = \sum_{i=2}^{x+1} 3^{(x+1)-i} \cdot 2^{\alpha_1+\ldots+\alpha_{i-1}}$$

Hence,

$$\Psi_{x+1}(n) = \frac{\left(\dfrac{3^{x+1}n + 3^{x} + \sum_{i=2}^{x+1} 3^{(x+1)-i} \cdot 2^{\alpha_1+\ldots+\alpha_{i-1}}}{2^{\alpha_1+\ldots+\alpha_x}}\right)}{2^{\alpha_{x+1}}}$$

Which implies,

$$\Psi_{x+1}(n) = \frac{3^{x+1}n + 3^{x} + \sum_{i=2}^{x+1} 3^{(x+1)-i} \cdot 2^{\alpha_1+\ldots+\alpha_{i-1}}}{2^{\alpha_1+\ldots+\alpha_{x+1}}}$$

Thus, our claim holds true for $(x+1)$.

Hence, by the principle of mathematical induction,

$$\forall\, k \in \mathbb{N},\ \ \Psi_k(n) = \frac{3^{k}n + 3^{k-1} + \sum_{i=2}^{k} 3^{k-i} \cdot 2^{\alpha_1+\ldots+\alpha_{i-1}}}{2^{\alpha_1+\ldots+\alpha_k}}$$

This concludes the proof □

Claim 2.2.3

Let $\Phi: \mathbb{N} \to \mathbb{N}$ *be any arbitrary function such that*

$$\Psi_x(n) \geq \Phi(n), \forall\, x \in \mathbb{N}$$

Then,

$$\Psi_x(n) \geq \frac{2^{\alpha_{x+1}} \cdot \Phi(n) - 1}{3}$$

Proof

We have,

$$\Psi_x(n) \geq \Phi(n), \forall\, x \in \mathbb{N}$$

Hence,

$$\Psi_x(n) \geq \Phi(n) \text{ and } \Psi_{x+1}(n) \geq \Phi(n), \forall\, x \in \mathbb{N}$$

This, along with Claim 2.2.2 gives,

$$\Psi_x(n) = \frac{3^x n + 3^{x-1} + \sum_{i=2}^{x} 3^{x-i} . 2^{\alpha_1 + \ldots + \alpha_{i-1}}}{2^{\alpha_1 + \ldots + \alpha_x}} \geq \Phi(n) \qquad (**)$$

$$\Psi_{x+1}(n) = \frac{3^{x+1} n + 3^{x} + \sum_{i=2}^{x+1} 3^{(x+1)-i} . 2^{\alpha_1 + \ldots + \alpha_{i-1}}}{2^{\alpha_1 + \ldots + \alpha_{x+1}}} \geq \Phi(n) \qquad (***)$$

Multiplying both sides of the inequality established in $(***)$ by $\frac{2^{\alpha_{x+1}}}{3}$;

$$\frac{3^x n + 3^{x-1} + \sum_{i=2}^{x+1} 3^{x-i} \cdot 2^{\alpha_1 + \ldots + \alpha_{i-1}}}{2^{\alpha_1 + \ldots + \alpha_x}} \geq \frac{2^{\alpha_{x+1}} \cdot \Phi(n)}{3}$$

Which implies,

$$\frac{3^x n + 3^{x-1} + \sum_{i=2}^{x} 3^{x-i} \cdot 2^{\alpha_1 + \ldots + \alpha_{i-1}} + 3^{-1} 2^{\alpha_1 + \ldots + \alpha_x}}{2^{\alpha_1 + \ldots + \alpha_x}} \geq \frac{2^{\alpha_{x+1}} \cdot \Phi(n)}{3}$$

Thus,

$$\frac{3^x n + 3^{x-1} + \sum_{i=2}^{x} 3^{x-i} \cdot 2^{\alpha_1 + \ldots + \alpha_{i-1}}}{2^{\alpha_1 + \ldots + \alpha_x}} + 3^{-1} \geq \frac{2^{\alpha_{x+1}} \cdot \Phi(n)}{3}$$

Hence,

$$\Psi_x(n) + \frac{1}{3} \geq \frac{2^{\alpha_{x+1}} \cdot \Phi(n)}{3}$$

And thus,

$$\Psi_x(n) \geq \frac{2^{\alpha_{x+1}} \cdot \Phi(n) - 1}{3}$$

Hence,

If $\Phi: \mathbb{N} \to \mathbb{N}$ *is an arbitrary function such that*

$$\Psi_x(n) \geq \Phi(n), \forall\, x \in \mathbb{N}$$

Then,

$$\Psi_x(n) \geq \frac{2^{\alpha_{x+1}} \cdot \Phi(n) - 1}{3}$$

This concludes the proof □

Let us choose the function $\Phi(n) = n$. We can make this selection because-

$$\Psi_k(n) \geq n, \forall\, k \in \mathbb{N}$$

Also, let us define the function

$$\varphi(x,\eta) = \frac{2^{\alpha_{x+1}} \cdot \eta - 1}{3}$$

Hence, <u>Claim 2.2.3</u> implies-

$$\Psi_x(n) \geq \varphi(x,n)$$

<u>Claim 2.2.4</u>

$$\Psi_x(n) \geq \varphi\left(x, \varphi(x+1, \varphi\big(x+2, \ldots, \varphi(x+k-1, n)..)\big)\right),$$

$\forall\, k, x \in \mathbb{N}$

<u>*Proof*</u>

From <u>Claim 2.2.3</u>, we obtain-

$$\Psi_x(n) \geq \varphi(x,n), \forall\, x \in \mathbb{N}$$

Implying that the claimed result holds true for $k = 1$. This serves as the base for an inductive process.

Let us now assume that the claimed result is true for $k = \mu, \mu \in \mathbb{N}$

Hence,

$$\Psi_x(n) \geq \varphi\left(x, \varphi(x+1, \varphi\big(x+2, \ldots, \varphi(x+\mu-1, n)...)\big)\right), \forall\, x \in \mathbb{N}$$

Thus,

$$\Psi_{x+1}(n) \geq \varphi\left(x+1, \varphi(x+2, \varphi\big(x+3, \ldots, \varphi(x+\mu, n)..)\big)\right), \forall\, x \in \mathbb{N}$$

Which implies,

$$\frac{3\Psi_x(n)+1}{2^{\alpha_{x+1}}} \geq \varphi\left(x+1, \varphi(x+2, \varphi\big(x+3, .., \varphi(x+\mu, n)...)\big)\right),$$

$\forall\, x \in \mathbb{N}$

Thus,

$$\Psi_x(n) \geq \frac{2^{\alpha_{x+1}}\varphi\left(x+1, \varphi(x+2, \varphi\big(x+3, \dots, \varphi(x+\mu, n)\dots)\big)\right) - 1}{3}$$

But,

$$\frac{2^{\alpha_{x+1}}\varphi\left(x+1, \varphi(x+2, \varphi\big(x+3, \dots, \varphi(x+\mu, n)\dots)\big)\right) - 1}{3}$$
$$= \varphi(x, \varphi\big(x+1, \varphi(x+2, \varphi(x+3, \dots, \varphi(x+\mu, n)\dots))\big)$$

Implying that the claimed result holds true for $k = \mu + 1, \mu \in \mathbb{N}$.

Hence, by the principle of mathematical induction,

$$\Psi_x(n) \geq \varphi\left(x, \varphi(x+1, \varphi\big(x+2, \dots, \varphi(x+k-1, n)\dots)\big)\right)$$

$\forall\, k, x \in \mathbb{N}$

This concludes the proof □

Let us now define the function $f: \mathbb{N}^3 \to \mathbb{N}$ as follows-

$$f_k(x, n) \coloneqq \varphi\left(x, \varphi(x+1, \varphi\big(x+2, \dots, \varphi(x+k-1, n)\dots)\big)\right)$$

Claim 2.2.5

$f_k(x, n)$ is recursive and obeys the equality-

$$f_{k+1}(x, n) = \frac{2^{\alpha_{x+1}} f_k(x+1, n) - 1}{3}, \forall\, k, x \in \mathbb{N}$$

Proof

We have,

$$f_k(x, n) = \varphi\left(x, \varphi(x+1, \varphi\big(x+2, \dots, \varphi(x+k, n)\dots)\big)\right) \forall\, k, x \in \mathbb{N}$$

But,

$$f_k(x+1,n) = \varphi\left(x+1, \varphi(x+2, \varphi\left(x+3, \ldots, \varphi(x+k,n)\ldots\right)\right)\right)$$

Thus,

$$f_k(x,n) = \varphi\left(x, f_k(x+1,n)\right) = \frac{2^{\alpha_{x+1}} f_k(x+1,n) - 1}{3}, \forall\, k, x \in \mathbb{N}$$

This concludes the proof □

<u>Claim 2.2.6</u>

$$f_k(x,n) = \frac{\left(2^{\sum_{m=1}^{k}(\alpha_{x+m})}\right)n - \left(\sum_{m=0}^{k-1} 3^m 2^{\alpha_{x+1}+\alpha_{x+2}+\ldots+\alpha_{x+k-m-1}}\right)}{3^k},$$

$\forall\, k, x \in \mathbb{N}$

<u>*Proof*</u>

Note that

$$f_1(x,n) = \frac{2^{\alpha_{x+1}} n - 1}{3}$$

Implying that the claimed result holds true for $k = 1$

Let us now assume that the claimed result is true for $k = \rho, \rho \in \mathbb{N}$.

Thus,

$$f_\rho(x,n) = \frac{\left(2^{\sum_{m=1}^{\rho}(\alpha_{x+m})}\right)n - \left(\sum_{m=0}^{\rho-1} 3^m 2^{\alpha_{x+1}+\alpha_{x+2}+\ldots+\alpha_{x+\rho-m-1}}\right)}{3^\rho}$$

Therefore, from <u>Claim 2.2.5</u>, we obtain-

$$f_{\rho+1}(x,n) = \varphi\left(x, f_\rho(x+1,n)\right)$$

$$= \frac{2^{\alpha_{x+1}}\left(\frac{\left(2^{\sum_{m=1}^{\rho}(\alpha_{x+1+m})}\right)n - (\sum_{m=0}^{\rho-1} 3^m 2^{\alpha_{x+2}+\alpha_{x+3}+\ldots+\alpha_{x+\rho-m}})}{3^\rho}\right) - 1}{3}$$

$$= \frac{2^{\alpha_{x+1}}\left(\left(2^{\sum_{m=1}^{\rho}(\alpha_{x+1+m})}\right)n - (\sum_{m=0}^{\rho-1} 3^m 2^{\alpha_{x+2}+\alpha_{x+3}+\ldots+\alpha_{x+\rho-m}})\right) - 3^\rho}{3^{\rho+1}}$$

$$= \frac{2^{\alpha_{x+1}}\left(\left(2^{\sum_{m=2}^{\rho+1}(\alpha_{x+m})}\right)n - (\sum_{m=0}^{\rho-1} 3^m 2^{\alpha_{x+2}+\alpha_{x+3}+\ldots+\alpha_{x+\rho-m}})\right) - 3^\rho}{3^{\rho+1}}$$

$$= \frac{\left(\left(2^{\sum_{m=1}^{\rho+1}(\alpha_{x+m})}\right)n - (\sum_{m=0}^{\rho-1} 3^m 2^{\alpha_{x+1}+\alpha_{x+2}+\alpha_{x+3}+\ldots+\alpha_{x+\rho-m}})\right) - 3^\rho}{3^{\rho+1}}$$

$$= \frac{\left(\left(2^{\sum_{m=1}^{\rho+1}(\alpha_{x+m})}\right)n - (\sum_{m=0}^{\rho} 3^m 2^{\alpha_{x+1}+\alpha_{x+2}+\alpha_{x+3}+\ldots+\alpha_{x+\rho-m}})\right)}{3^{\rho+1}}$$

Implying that the claimed result holds true for $k = \rho + 1, \rho \in \mathbb{N}$.

Hence, by the principle of mathematical induction,

$$f_k(x,n) = \frac{\left(2^{\sum_{m=1}^{k}(\alpha_{x+m})}\right)n - (\sum_{m=0}^{k-1} 3^m 2^{\alpha_{x+1}+\alpha_{x+2}+\ldots+\alpha_{x+k-m-1}})}{3^k},$$

$\forall\, k, x \in \mathbb{N}$

This concludes the proof □

Claim 2.2.7

$$f_{k+\varepsilon}(x,n) - f_k(x,n) \geq \left(\frac{2}{3}\right)^k \left(\sum_{m=0}^{\varepsilon-1} \left(\frac{2}{3}\right)^m \left(\left(\frac{2^{\alpha_{x+k+m+1}}}{3} - 1 \right) n - \frac{1}{3} \right) \right),$$

$\forall\ \varepsilon, k \in \mathbb{N}$

Proof

Note that,

$$f_{k+1}(x,n) - f_k(x,n)$$

$$= \frac{\left(2^{\sum_{m=1}^{k+1}(\alpha_{x+m})}\right) n - \left(\sum_{m=0}^{k} 3^m 2^{\alpha_{x+1}+\alpha_{x+2}+\ldots+\alpha_{x+k-m}}\right)}{3^{k+1}} - \frac{\left(2^{\sum_{m=1}^{k}(\alpha_{x+m})}\right) n - \left(\sum_{m=0}^{k-1} 3^m 2^{\alpha_{x+1}+\alpha_{x+2}+\ldots+\alpha_{x+k-m-1}}\right)}{3^k}$$

$$= \left(\frac{\left(2^{\sum_{m=1}^{k}(\alpha_{x+m})}\right) 2^{\alpha_{x+k+1}} n}{3^{k+1}} - \frac{\left(\sum_{m=0}^{k} 3^m 2^{\alpha_{x+1}+\ldots+\alpha_{x+k-m}}\right)}{3^{k+1}} \right) - \left(\frac{\left(2^{\sum_{m=1}^{k}(\alpha_{x+m})}\right) n}{3^k} - \frac{\sum_{m=0}^{k-1} 3^m 2^{\alpha_{x+1}+\ldots+\alpha_{x+k-m-1}}}{3^k} \right)$$

$$= \frac{\left(2^{\sum_{m=1}^{k}(\alpha_{x+m})}\right)\left((2^{\alpha_{x+k+1}} - 3)n\right)}{3^{k+1}} - \frac{\left(\sum_{m=0}^{k} 3^m 2^{\alpha_{x+1}+\alpha_{x+2}+\ldots+\alpha_{x+k-m}}\right) - 3\left(\sum_{m=0}^{k-1} 3^m 2^{\alpha_{x+1}+\alpha_{x+2}+\ldots+\alpha_{x+k-m-1}}\right)}{3^{k+1}}$$

$$= \frac{\left(2^{\sum_{m=1}^{k}(\alpha_{x+m})}\right)((2^{\alpha_{x+k+1}} - 3)n)}{3^{k+1}}$$
$$- \frac{(\sum_{m=0}^{k} 3^m 2^{\alpha_{x+1}+\alpha_{x+2}+\ldots+\alpha_{x+k-m}}) - (\sum_{m=0}^{k-1} 3^{m+1} 2^{\alpha_{x+1}+\alpha_{x+2}+\ldots+\alpha_{x+k-m-1}})}{3^{k+1}}$$

Now, note that-

$$\left(\sum_{m=0}^{k-1} 3^{m+1} 2^{\alpha_{x+1}+\alpha_{x+2}+\ldots+\alpha_{x+k-m-1}}\right) = \sum_{m=1}^{k} 3^m 2^{\alpha_{x+1}+\alpha_{x+2}+\ldots+\alpha_{x+k-m}}$$

Hence,

$$f_{k+1}(x,n) - f_k(x,n)$$

$$= \frac{\left(2^{\sum_{m=1}^{k}(\alpha_{x+m})}\right)((2^{\alpha_{x+k+1}} - 3)n)}{3^{k+1}}$$
$$- \frac{(\sum_{m=0}^{k} 3^m 2^{\alpha_{x+1}+\alpha_{x+2}+\ldots+\alpha_{x+k-m}}) - \sum_{m=1}^{k} 3^m 2^{\alpha_{x+1}+\alpha_{x+2}+\ldots+\alpha_{x+k-m}}}{3^{k+1}}$$

$$= \frac{\left(2^{\sum_{m=1}^{k}(\alpha_{x+m})}\right)((2^{\alpha_{x+k+1}} - 3)n)}{3^{k+1}} - \frac{3^0 2^{\alpha_{x+1}+\alpha_{x+2}+\ldots+\alpha_{x+k}}}{3^{k+1}}$$

$$= \frac{\left(2^{\sum_{m=1}^{k}(\alpha_{x+m})}\right)((2^{\alpha_{x+k+1}} - 3)n) - 2^{\alpha_{x+1}+\alpha_{x+2}+\ldots+\alpha_{x+k}}}{3^{k+1}}$$

$$= \frac{\left(2^{\sum_{m=1}^{k}(\alpha_{x+m})}\right)((2^{\alpha_{x+k+1}} - 3)n - 1)}{3^{k+1}}$$

Now, note that $\alpha_{x+m} \geq 1, \forall\, m \in \mathbb{N}_{\leq k}$. Hence,

$$\left(2^{\sum_{m=1}^{k}(\alpha_{x+m})}\right) \geq \left(2^{\sum_{m=1}^{k}1}\right) = 2^k$$

This implies,

$$f_{k+1}(x,n) - f_k(x,n) \geq \frac{2^k\left((2^{\alpha_{x+k+1}} - 3)n - 1\right)}{3^{k+1}}$$
$$= \left(\frac{2}{3}\right)^k \left(\left(\frac{2^{\alpha_{x+k+1}}}{3} - 1\right)n - \frac{1}{3}\right), \forall\, k \in \mathbb{N}$$

The above implies that the claimed result holds true for $\varepsilon = 1$. This serves as the base for an inductive process.

Let us now assume that the claimed result holds true for $\varepsilon = \psi$, $\psi \in \mathbb{N}$. Thus-

$$f_{k+\psi}(x,n) - f_k(x,n) \geq \left(\frac{2}{3}\right)^k \left(\sum_{m=0}^{\psi-1} \left(\frac{2}{3}\right)^m \left(\left(\frac{2^{\alpha_{x+k+m+1}}}{3} - 1\right)n - \frac{1}{3}\right)\right)$$

Also, note that by virtue of the arguments made initially, we can state-

$$f_{k+\psi+1}(x,n) - f_{k+\psi}(x,n) \geq \left(\frac{2}{3}\right)^{k+\psi} \left(\left(\frac{2^{\alpha_{x+k+\psi+1}}}{3} - 1\right)n - \frac{1}{3}\right)$$

$$\left(\text{As } f_{k+1}(x,n) - f_k(x,n) \geq \left(\frac{2}{3}\right)^k \left(\left(\frac{2^{\alpha_{x+k+1}}}{3} - 1\right)n - \frac{1}{3}\right), \forall\, k \in \mathbb{N}\right)$$

The addition of the above two inequalities yields-

$$f_{k+\psi+1}(x,n) - f_k(x,n) \geq \left(\frac{2}{3}\right)^{k+\psi}\left(\left(\frac{2^{\alpha_{x+k+\psi+1}}}{3} - 1\right)n - \frac{1}{3}\right) + \left(\frac{2}{3}\right)^{k}\left(\sum_{m=0}^{\psi-1}\left(\frac{2}{3}\right)^{m}\left(\left(\frac{2^{\alpha_{x+k+m+1}}}{3} - 1\right)n - \frac{1}{3}\right)\right)$$

Which gives-

$$f_{k+\psi+1}(x,n) - f_k(x,n) \geq \left(\frac{2}{3}\right)^{k}\left(\sum_{m=0}^{\psi-1}\left(\frac{2}{3}\right)^{m}\left(\left(\frac{2^{\alpha_{x+k+m+1}}}{3} - 1\right)n - \frac{1}{3}\right) + \left(\frac{2}{3}\right)^{\psi}\left(\frac{2^{\alpha_{x+k+\psi+1}}}{3} - 1\right)n - \frac{1}{3}\right)$$
$$= \left(\frac{2}{3}\right)^{k}\left(\sum_{m=0}^{\psi}\left(\frac{2}{3}\right)^{m}\left(\left(\frac{2^{\alpha_{x+k+m+1}}}{3} - 1\right)n - \frac{1}{3}\right)\right), \forall\, k \in \mathbb{N}$$

Implying that the claimed result holds true for $\varepsilon = \psi + 1, \psi \in \mathbb{N}$.

Hence, by the principle of mathematical induction,

$$f_{k+\varepsilon}(x,n) - f_k(x,n) \geq \left(\frac{2}{3}\right)^{k}\left(\sum_{m=0}^{\varepsilon-1}\left(\frac{2}{3}\right)^{m}\left(\left(\frac{2^{\alpha_{x+k+m+1}}}{3} - 1\right)n - \frac{1}{3}\right)\right),$$

$\forall\, \varepsilon, k \in \mathbb{N}$

This concludes the proof □

Let us now consider the two sets-

$$\boldsymbol{S_1} \equiv \{m \mid 0 \leq m \leq \varepsilon - 1, \alpha_{x+m+k+1} = 1\}, and$$

$$\boldsymbol{S_2} \equiv \{m \mid 0 \leq m \leq \varepsilon - 1, \alpha_{x+m+k+1} \geq 2\}$$

Note that $\boldsymbol{S_1}$ and $\boldsymbol{S_2}$ are disjoint sets ($\boldsymbol{S_1} \cap \boldsymbol{S_2} \equiv \phi$). Thus, every $m \in \mathbb{N}_0$, $0 \leq m \leq \varepsilon - 1$ can be casted into either of the two sets $\boldsymbol{S_1}$ or $\boldsymbol{S_2}$. Hence,

$$\left(\frac{2}{3}\right)^k \left(\sum_{m=0}^{\varepsilon-1} \left(\frac{2}{3}\right)^m \left(\left(\frac{2^{\alpha_{x+k+m+1}}}{3} - 1\right) n - \frac{1}{3}\right)\right)$$

$$= \left(\frac{2}{3}\right)^k \left(\sum_{m=0}^{\varepsilon-1} \left(\frac{2}{3}\right)^m \left(\left(\frac{2^{\alpha_{x+k+m+1}}}{3}\right) n - n - \frac{1}{3}\right)\right)$$

$$= \left(\frac{2}{3}\right)^k \left(\sum_{m=0}^{\varepsilon-1} \left(\frac{2}{3}\right)^m \left(\left(\frac{2^{\alpha_{x+k+m+1}}}{3}\right) n - \frac{3n+1}{3}\right)\right)$$

$$= \frac{2^k}{3^{k+1}} \left(\sum_{m=0}^{\varepsilon-1} \left(\frac{2}{3}\right)^m (2^{\alpha_{x+k+m+1}} n - (3n+1))\right)$$

$$= \frac{2^k}{3^{k+1}} \left(\sum_{m=0}^{\varepsilon-1} \left(\frac{2}{3}\right)^m 2^{\alpha_{x+k+m+1}} n - (3n+1) \sum_{m=0}^{\varepsilon-1} \left(\frac{2}{3}\right)^m\right)$$

Note that the sum-

$$S = \sum_{m=0}^{\varepsilon-1} \left(\frac{2}{3}\right)^m$$

is the sum of the first ε terms of a Geometric Progression, starting with 1 and having a common ratio $\left(\frac{2}{3}\right)$. Hence,

$$S = \left(\frac{1-\left(\frac{2}{3}\right)^{\varepsilon}}{1-\frac{2}{3}}\right) = \left(\frac{1-\left(\frac{2}{3}\right)^{\varepsilon}}{\frac{1}{3}}\right) = 3\left(1-\left(\frac{2}{3}\right)^{\varepsilon}\right)$$

$$\left(\frac{2}{3}\right)^{k}\left(\sum_{m=0}^{\varepsilon-1}\left(\frac{2}{3}\right)^{m}\left(\left(\frac{2^{\alpha_{x+k+m+1}}}{3}-1\right)n-\frac{1}{3}\right)\right)$$

$$= \frac{2^{k}}{3^{k+1}}\left(\sum_{m=0}^{\varepsilon-1}\left(\frac{2}{3}\right)^{m}2^{\alpha_{x+k+m+1}}n-(3n+1)\left(3\left(1-\left(\frac{2}{3}\right)^{\varepsilon}\right)\right)\right)$$

$$= \frac{2^{k}}{3^{k+1}}\left(\sum_{m=0}^{\varepsilon-1}\left(\frac{2}{3}\right)^{m}2^{\alpha_{x+k+m+1}}n-(9n+3)\left(1-\left(\frac{2}{3}\right)^{\varepsilon}\right)\right)$$

Let us now consider the sum-

$$\varSigma = \sum_{m=0}^{\varepsilon-1}\left(\frac{2}{3}\right)^{m}2^{\alpha_{x+k+m+1}}n$$

Note that the above sum can be rewritten as follows-

$$\varSigma = \left(\sum_{m\in\boldsymbol{S_1}}\left(\left(\frac{2}{3}\right)^{m}2^{\alpha_{x+k+m+1}}n\right)\right)+\left(\sum_{m\in\boldsymbol{S_2}}\left(\left(\frac{2}{3}\right)^{m}2^{\alpha_{x+k+m+1}}n\right)\right)$$

Now, note that $\forall\, m \in \boldsymbol{S_1}$, $2^{\alpha_{x+k+m+1}} = 2^1 = 2$, and $\forall\, m \in \boldsymbol{S_2}$, $2^{\alpha_{x+k+m+1}} \geq 2^2 = 4$.

Thus,

$$\Sigma = \left(\sum_{m \in \boldsymbol{S_1}} \left(\left(\frac{2}{3}\right)^m 2^{\alpha_{x+k+m+1}} n \right) \right) + \left(\sum_{m \in \boldsymbol{S_2}} \left(\left(\frac{2}{3}\right)^m 2^{\alpha_{x+k+m+1}} n \right) \right)$$

$$\geq \left(\sum_{m \in \boldsymbol{S_1}} \left(\left(\frac{2}{3}\right)^m 2n \right) \right) + \left(\sum_{m \in \boldsymbol{S_2}} \left(\left(\frac{2}{3}\right)^m 4n \right) \right)$$

$$= \left(\sum_{m \in \boldsymbol{S_1}} \left(\frac{2^{m+1}}{3^m} n \right) \right) + 2 \left(\sum_{m \in \boldsymbol{S_2}} \frac{2^{m+1}}{3^m} (n) \right)$$

$$= \left\{ \left(\sum_{m \in \boldsymbol{S_1}} \left(\frac{2^{m+1}}{3^m} n \right) \right) + \left(\sum_{m \in \boldsymbol{S_2}} \frac{2^{m+1}}{3^m} (n) \right) \right\} + \left(\sum_{m \in \boldsymbol{S_2}} \frac{2^{m+1}}{3^m} (n) \right)$$

$$= \left\{ \sum_{m \in \boldsymbol{S_1} \cup \boldsymbol{S_2}} \left(\frac{2^{m+1}}{3^m} n \right) \right\} + \left(\sum_{m \in \boldsymbol{S_2}} \frac{2^{m+1}}{3^m} (n) \right)$$

$$= \left\{ \sum_{m=0}^{\varepsilon-1} \left(\frac{2^{m+1}}{3^m} n \right) \right\} + \left(\sum_{m \in \boldsymbol{S_2}} \frac{2^{m+1}}{3^m} (n) \right)$$

Note that the sum-

$$S^{'} = \sum_{m=0}^{\varepsilon-1} \left(\frac{2^{m+1}}{3^m} n \right)$$

is the sum of the first ε terms of a Geometric Progression, starting with $2n$ and having a common ratio $\left(\frac{2}{3}\right)$. Hence,

$$S^{'} = 2n\left(\frac{1-\left(\frac{2}{3}\right)^{\varepsilon}}{1-\left(\frac{2}{3}\right)}\right) = 6n\left(1-\left(\frac{2}{3}\right)^{\varepsilon}\right)$$

Therefore,

$$\Sigma \geq 6n\left(1-\left(\frac{2}{3}\right)^{\varepsilon}\right) + \left(\sum_{m\in\boldsymbol{S_2}} \frac{2^{m+1}}{3^m}(n)\right)$$

Let,

$$|\boldsymbol{S_2}| = s, and\ \boldsymbol{S_2} \equiv \{m_1, m_2, \dots, m_s\}\ (Each\ m_i\ is\ distinct)$$

Thus,

$$\left(\sum_{m\in\boldsymbol{S_2}} \frac{2^{m+1}}{3^m}(n)\right) = 2n\left(\sum_{m\in\boldsymbol{S_2}} \left(\frac{2}{3}\right)^{m}\right)$$

$$= 2n\left(\left(\frac{2}{3}\right)^{m_1} + \left(\frac{2}{3}\right)^{m_2} + \dots + \left(\frac{2}{3}\right)^{m_s}\right)$$

Note that the above sum achieves its minimum possible value when each m_i obtains its minimum value, which is when $m_1 = 0, m_2 = 1, \dots, m_s = s-1$ (Since each m_i is distinct)

Thus,

$$\left(\sum_{m\in \boldsymbol{S_2}} \frac{2^{m+1}}{3^m}(n)\right) = 2n\left(\sum_{m\in \boldsymbol{S_2}} \left(\frac{2}{3}\right)^m\right)$$

$$\geq 2n\left(\left(\frac{2}{3}\right)^0 + \left(\frac{2}{3}\right)^1 + \dots + \left(\frac{2}{3}\right)^{s-1}\right)$$

Note that the sum-

$$S'' = \left(\frac{2}{3}\right)^0 + \left(\frac{2}{3}\right)^1 + \dots + \left(\frac{2}{3}\right)^{s-1}$$

is the sum of the first s terms of a Geometric Progression, starting with 1 and having a common ratio $\left(\frac{2}{3}\right)$. Hence,

$$S' = 1\left(\frac{1-\left(\frac{2}{3}\right)^s}{1-\left(\frac{2}{3}\right)}\right) = 3\left(1-\left(\frac{2}{3}\right)^s\right)$$

Hence,

$$\Sigma \geq 6n\left(1-\left(\frac{2}{3}\right)^\varepsilon\right) + \left(\sum_{m\in \boldsymbol{S_2}} \frac{2^{m+1}}{3^m}(n)\right)$$

$$\geq 6n\left(1-\left(\frac{2}{3}\right)^\varepsilon\right) + 2n\left(3\left(1-\left(\frac{2}{3}\right)^s\right)\right)$$

$$= 6n\left(1-\left(\frac{2}{3}\right)^\varepsilon\right) + 6n\left(1-\left(\frac{2}{3}\right)^s\right)$$

Thus,

$$\Sigma = \sum_{m=0}^{\varepsilon-1}\left(\frac{2}{3}\right)^m 2^{\alpha_{x+k+m+1}}n \geq 6n\left(1-\left(\frac{2}{3}\right)^{\varepsilon}\right) + 6n\left(1-\left(\frac{2}{3}\right)^{s}\right)$$

Hence,

$$\frac{2^k}{3^{k+1}}\left(\sum_{m=0}^{\varepsilon-1}\left(\frac{2}{3}\right)^m 2^{\alpha_{x+k+m+1}}n - (9n+3)\left(1-\left(\frac{2}{3}\right)^{\varepsilon}\right)\right)$$

$$\geq \frac{2^k}{3^{k+1}}\left(6n\left(1-\left(\frac{2}{3}\right)^{\varepsilon}\right) + 6n\left(1-\left(\frac{2}{3}\right)^{s}\right) - (9n+3)\left(1-\left(\frac{2}{3}\right)^{\varepsilon}\right)\right)$$

$$= \frac{2^k}{3^k}\left(2n\left(1-\left(\frac{2}{3}\right)^{\varepsilon}\right) + 2n\left(1-\left(\frac{2}{3}\right)^{s}\right) - (3n+1)\left(1-\left(\frac{2}{3}\right)^{\varepsilon}\right)\right)$$

Now, note that $s = |\boldsymbol{S_2}| \geq 1$. Hence, $1-\left(\frac{2}{3}\right)^{s} > 1-\frac{2}{3}$

Also, note that $\varepsilon \geq 1$ and hence, $1-\left(\frac{2}{3}\right)^{\varepsilon} > 1-\frac{2}{3}$

Hence,

$$\frac{2^k}{3^{k+1}}\left(\sum_{m=0}^{\varepsilon-1}\left(\frac{2}{3}\right)^m 2^{\alpha_{x+k+m+1}}n - (9n+3)\left(1-\left(\frac{2}{3}\right)^{\varepsilon}\right)\right)$$

$$> \frac{2^k}{3^k}\left(2n\left(1-\frac{2}{3}\right) + 2n\left(1-\frac{2}{3}\right) - (3n+1)\left(1-\frac{2}{3}\right)\right)$$

$$= \frac{2^k}{3^k}\left((n-1)\left(\frac{1}{3}\right)\right) = \frac{2^k}{3^{k+1}}(n-1) > 0, \forall\, k, \varepsilon \in \mathbb{N}$$

Hence,

$$\frac{2^k}{3^{k+1}}\left(\sum_{m=0}^{\varepsilon-1}\left(\frac{2}{3}\right)^m 2^{\alpha_{x+k+m+1}}n-(9n+3)\left(1-\left(\frac{2}{3}\right)^{\varepsilon}\right)\right)>0, \forall\, k, \varepsilon \in \mathbb{N}$$

Which implies that-

$$f_{k+\varepsilon}(x,n)-f_k(x,n)$$

$$=\left(\frac{2}{3}\right)^k\left(\sum_{m=0}^{\varepsilon-1}\left(\frac{2}{3}\right)^m\left(\left(\frac{2^{\alpha_{x+k+m+1}}}{3}-1\right)n-\frac{1}{3}\right)\right)>0, \forall\, k, \varepsilon \in \mathbb{N}$$

Hence,

$$\forall\, \varepsilon \in \mathbb{N},\ f_{k+\varepsilon}(x,n)>f_k(x,n)$$

The above implies that-

$$f_k(x,n)<f_{k+1}(x,n)<f_{k+2}(x,n)<\ldots$$

Also, if k is chosen such that-

$$k<\log_{\frac{3}{2}}\left(\frac{n-1}{3}\right)$$

Then,

$$\left(\frac{3}{2}\right)^k<\left(\frac{n-1}{3}\right)$$

Which implies,

$$3\left(\frac{3}{2}\right)^k<(n-1)$$

Thus,

$$n-1>\frac{3^{k+1}}{2^k}$$

Hence,

$$f_{k+\varepsilon}(x,n)-f_k(x,n)>\frac{2^k}{3^{k+1}}(n-1)>1, \forall\, \varepsilon \in \mathbb{N}$$

Which implies that

$$f_{k+1}(x,n)-f_k(x,n)>1$$

This ensures that $\lim_{k\to\infty} f_k(x,n)$ does not converge to a finite limit.

Thus,

$$\lim_{k\to\infty} f_k(x,n)\to\infty$$

But, Claim 2.2.4 suggests-

$$\Psi_x(n)\geq f_k(x,n), \forall\, k, x\in\mathbb{N}$$

Hence,

$$\Psi_x(n)\geq \lim_{k\to\infty} f_k(x,n), \forall\, x\in\mathbb{N}$$

Thus,

$$\Psi_x(n)\to\infty, \forall\, x\in\mathbb{N}$$

Thus,

$$\Psi_1(n)=\frac{3n+1}{2}\to\infty, which\ is\ possible\ only\ if\ n\to\infty$$

Clearly, this is a contradiction. Thus, our assumption must be incorrect, and thus,

$$\forall\, N \in \mathbb{N}_{\geq 2}, \exists\, k \in \mathbb{N}\ \underline{such\ that}\ {F_{col}}^{k}(N) < N$$

This concludes the proof □

Remark

The Collatz Conjecture follows immediately from Theorem 2.2 due to the principle of mathematical induction. Theorem 2.2 suggests that every number $n \in \mathbb{N}$ yields an $\acute{\epsilon} < n$. This argument can be used over and over again to argue that all natural numbers eventually yield 1.

3. PROOF OF THE COLLATZ CONJECTURE

Theorem 3.1

The Collatz Conjecture is true. In other words, if we define the function

F_{col}: $\mathbb{N} \rightarrow \mathbb{N}$ such that-

$$F_{col}(N) \coloneqq \frac{N}{2}, if\ \ N\ is\ even$$

$$F_{col}(N) \coloneqq 3N+1, if\ N\ is\ odd$$

Then $\forall\, N \in \mathbb{N}, \exists\, k \in\ \mathbb{N}$ such that ${F_{col}}^{k}(N) = 1$

Proof

The Collatz Conjecture has been verified to be true for all natural numbers till an approximate value of 2^{68}. This serves as an appropriate base for an inductive process.

Let us now assume that the Collatz Conjecture is true $\forall\ N \in \mathbb{N}_{\leq\beta}$, $\beta \in \mathbb{N}$.

Thus, $\forall\ m \in \mathbb{N}_{\leq\beta}, \exists\ \gamma_m \in \mathbb{N}$ such that

$$F_{col}{}^{\gamma_m}(m) = 1$$

Let us now consider the case of $N = (\beta + 1)$. Thus, Theorem 2.2 implies that

$$\exists\ \delta \in \mathbb{N}\ \underline{such\ that}\ F_{col}{}^{\delta}(\beta + 1) < \beta + 1$$

This implies that-

$$\exists\ \delta \in \mathbb{N}\ \underline{such\ that}\ F_{col}{}^{\delta}(\beta + 1) = \text{þ} \leq \beta \qquad (****)$$

Also, note that $\text{þ} \leq \beta$.Thus, $\text{þ} \in \mathbb{N}_{\leq\beta}$ and hence, $\exists \gamma_{\text{þ}} \in \mathbb{N}$ such that-

$$F_{col}{}^{\gamma_{\text{þ}}}(\text{þ}) = 1 \qquad (*****)$$

Now, from $(****)$,

$$F_{col}{}^{\delta}(\beta + 1) = \text{þ}$$

Hence,

$$F_{col}{}^{\delta+\gamma_{\text{þ}}}(\beta + 1) = F_{col}{}^{\gamma_{\text{þ}}}(\text{þ})$$

But, from $(*****)$, $F_{col}{}^{\gamma_{\text{þ}}}(\text{þ}) = 1$. Hence,

$$F_{col}{}^{\delta+\gamma_{\text{þ}}}(\beta + 1) = 1$$

Also, note that $\delta + \gamma_{\text{þ}} \in \mathbb{N}$. Thus, $\exists\ \gamma_{\beta+1} = \delta + \gamma_{\text{þ}} \in \mathbb{N}$ such that-

$$F_{col}{}^{\gamma_{\beta+1}}(\beta + 1) = 1$$

Implying that the conjecture holds true for $(\beta + 1)$.

Hence, by the principle of mathematical induction, if we define the function

F_{col}: ℕ → ℕ such that-

$$F_{col}(N) \coloneqq \frac{N}{2}, if\ \ N\ is\ even$$

$$F_{col}(N) \coloneqq 3N + 1, if\ N\ is\ odd$$

Then $\forall\ N \in \mathbb{N}, \exists\ k \in\ \mathbb{N}$ such that ${F_{col}}^{k}(N) = 1$, implying that the Collatz Conjecture is true!