Proof that no odd perfect number exists

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Abstract: Using properties of the sum of divisors function, which are derived from its definition, it is proved that every perfect number is even. The general form of perfect numbers is then inferred, and expressed in terms of the sum of divisors function.

Useful formula for the sum of divisors

The divisors of a prime power p^x are the successive powers from 0 to x, so its sum is:

$$\sigma(p^x) = 1 + p + \ldots + p^x \tag{1}$$

It should be easy to see that the one for the next power can be expressed as a function of it in two equivalent recursive ways:

$$\sigma(p^{x+1}) = 1 + p \cdot \sigma(p^x) = \sigma(p^x) + p^{x+1} \tag{2}$$

Grouping terms, the following parametric expression is obtained, which is a usual formula for calculation:

$$\sigma(p^x) = \frac{p^{x+1} - 1}{p - 1} \tag{3}$$

Definition of perfect number

A perfect number N is defined as a natural number whose sum of divisors, $\sigma(N)$, is equal to the double of itself. This can be expressed by the next recursive equation:

$$N = \sigma(N)/2 \tag{4}$$

Extraction of one prime power

Every natural number can be decomposed into the product of powers of prime numbers. A perfect number can not be the power of just one prime, because the only solution then would be $N=2^{\infty}$, as can be shown from Eq.(3). So we can choose to decompose it into the power of one prime, p^x, and its cofactor, a, which may contain one or more prime powers:

$$N = p^x \cdot a \tag{5}$$

As the sum of divisors function is multiplicative for coprime factors, $\sigma(N)$ can also be decomposed into the corresponding two sums of divisors. Further, at least one of them has to be even, because of the 2 in the definition of perfect number. We choose the coprime factor p^x such that $\sigma(p^x)$ is even. As a consequence, $p>2$. Then the original relation, Eq. (4), becomes:

(6)

$$p^x \cdot a = \sigma(a) \cdot \frac{\sigma(p^x)}{2}$$

Separation in two equations

Let's now fully factor *a* into prime powers. If *σ(a)* and *σ(p^x)/2* are not coprime they will have one or more common prime factors. If q is a common prime factor, it will appear with an exponent *s* in σ(p^x)/2, and with a different exponent *r* in σ(a). If *q* is not a common factor, either *s* or *r* is 0. So, in the more general way, the previous equation can be factored into:

$$p^x \cdot \prod q^{s+r} = \sigma\left(\prod q^{s+r}\right) \cdot \prod q^s \tag{7}$$

Note: a simplified notation is being used on this text, in which the product symbol goes over all the different prime factors *q* and their corresponding exponents *r* and *s*, which are different for each *q*.

After rearranging components, and using the multiplicative property of the sum of divisors, it becomes:

$$p^x = \prod \frac{\sigma(q^{s+r})}{q^r} = \frac{\prod(1+q+\ldots+q^{s+r})}{\prod q^r} \tag{8}$$

It should be obvious in the later form that the product in the numerator can not be divided by the product in the denominator unless the denominator is 1. So it is inferred that *r=0* for each *r*.

As a consequence *σ(a)* has to be coprime to *a*. Then the Eq.(6) can be decomposed into the following two coupled relations:

$$\begin{cases} \prod q^s = \dfrac{\sigma(p^x)}{2} \\ p^x = \prod \sigma(q^s) \end{cases} \tag{9}$$

Two coprime factors

The first relation contains the sum of divisors of p^x, which is even. The number of divisors of p^x is *(x+1)*, and it has to be also even because every divisor is odd. Then another variable can be used to simplify expressions, *2·y=x+1*, so that the right part of the first equation can be decomposed from the calculation formula (3) into the product of two factors:

$$\frac{\sigma(p^x)}{2} = \left(\frac{1+p^y}{2}\right) \cdot \left(\frac{p^y-1}{p-1}\right) \tag{10}$$

It can be seen that the difference between numerators is 2, so the greatest common divisor of numerators is 2, but both of them are divided by an even quantity, therefore the two factors are coprime.

Only one remains

Let's call them *b* and *c* respectively. Now, for some *z*, and noting that the second factor is σ(p^{y-1}), the equations (9) can be further separated into four equations:

$$\begin{cases} b = \dfrac{1+p^y}{2}; \qquad c = \sigma(p^{y-1}) \\ p^{2\cdot y-z} = \sigma(b); \qquad p^{z-1} = \sigma(c) \end{cases} \tag{11}$$

The sum of divisors of any number is always greater than the number itself, unless they both are 1. Therefore, the following two inequalities can be established from the previous equations:

$$\begin{cases} 1 < \dfrac{\sigma(b)}{b} = \dfrac{2 \cdot p^{2 \cdot y - z}}{1 + p^y} \\ 1 \leq \dfrac{\sigma(c)}{c} = \dfrac{p^{z-1}}{\sigma(p^{y-1})} \end{cases} \tag{12}$$

The first one leads to $z \leq y$. But this result would be impossible in the second one if $y>1$, so the only possibility is that $z=y=1$, which implies that the second factor in Eq.(10) disappears, $c=1$.

Bound condition

Since $p^x=p$, no more distinct prime factors q are possible, as Eq.(9) then would imply the contradiction that a prime is a product. That equation is then reduced to:

$$\begin{cases} q^s = \dfrac{1 + p}{2} \\ p = \sigma(q^s) \end{cases} \tag{13}$$

From the calculation formula Eq.(3) taken in the limit $s \to \infty$, the following upper bound condition can be established:

$$\frac{q}{q-1} > \frac{\sigma(q^s)}{q^s} = \frac{2 \cdot p}{p+1} \tag{14}$$

Conclusion

The minimum possible value of the term on the right is 3/2, because $p > 2$. Then, as the inequality is not inclusive, it is needed that $q < 3$, so the only possibility is

$$q = 2 \tag{15}$$

Thus 2^s is the only possible factor of $(p + 1)/2$, which is a factor of a, which is a factor of N, so it is concluded that **all perfect numbers are even**.

General form of a perfect number

To complete the case, it can be observed that the two relations of Eq.(13) are really the same one when $q=2$, because the sum of divisors of a power of 2 is:

$$\sigma(2^s) = 2^{s+1} - 1 \tag{16}$$

Then we arrive to the general form of every perfect number N:

$$N = 2^s \cdot \sigma(2^s) \mid \sigma(2^s)\, prime \tag{17}$$

which means that **any perfect number must be a power of 2 multiplied by its sum of divisors, which must be a prime number**.