Affirmative resolve of the Riemann Hypothesis

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Abstract

Riemann Hypothesis has been the unsolved conjecture for 170 years. This conjecture is the last one of conjectures without proof in "Ueber die Anzahl der Primzahlen unter einer gegebenen Grosse"(B. Riemann). The statement is the real part of the non-trivial zero points of the Riemann Zeta function is 1/2. Very famous and difficult this conjecture has not been solved by many mathematicians for many years. In this paper, I try to solve the proposition about the Mobius function equivalent to the Riemann Hypothesis. First, the non-trivial formula for Mobius function is proved in theorem 1. In theorem 2 I prove the Riemann Hypothesis.

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Handles propositions equivalent to the Riemann Hypothesis. I express the Riemann Hypothesis as R.H, and the Mobius function as $\mu(n)$.

Next theorem is well-known

Theorem .

$$\sum_{n=1}^{m} \mu(n) = O(m^{\frac{1}{2}+\epsilon}) \Leftrightarrow R.H$$

I will prove Left hand formula.

The break-through for Riemann Hypothesis is, in theorem 1, the fact $\sum_{n=1}^{m} \mu(n)$ and $\sum_{n=1}^{m} \frac{m}{n}\mu(n)$ are too close. I use this fact to prove $\sum_{n=1}^{m} \frac{m}{n}\mu(n) = O(m^{\frac{1}{2}+\epsilon})$. Directly I get $\sum_{n=1}^{m} \mu(n) = O(m^{\frac{1}{2}+\epsilon})$.

Lemma 1.

$$\sum_{n=1}^{x}\mu(n) \text{ changes sign } \in [m^{(1-\epsilon')}, m](m > \exists M)$$

$$\sum_{n=1}^{x}\frac{1}{n}\mu(n) \text{ changes sign } \in [m^{(1-\epsilon')}, m](m > \exists M)$$

Proof. $\sum_{n=1}^{x}\mu(n)$ changes sign in $[m^{(1-\epsilon')}, m], m > \exists M$. ([10]).$\sum_{n=1}^{x}\frac{1}{n}\mu(n)$ changes sign in $[m^{(1-\epsilon')}, m], m > \exists M$([10]). □

Remark:$\sum_{n=1}^{x}\frac{1}{n}\mu(n)$ and $\sum_{n=1}^{x}\mu(n)(x \leq m)$ change sign $\log m \times \frac{\gamma_1}{\pi} \approx 4.5 \log m$ times. This value does not depend on ϵ', for $m = 1000$, that is near 31 times. By ([10]), this value gives upper bound, so the almost destance of change sign point is given.

Theorem 1.

$$\sum_{n=1}^{m}\mu(n)\ (t.c.\ \sum_{n=1}^{m}\frac{m}{n}\mu(n))$$

t.c. means "too close" in the meaning of this theorem 1.

Proof. m_0 is previous (from m) point satisfies $\sum_{n\leq m_0}\mu(n) = 0$. m' is the after (from m) point satrisfies $\sum_{n\leq m'}\mu(n) = 0$. By lemma 1, $\sum_{n\leq m}\frac{1}{n}\mu(n)$ and $\sum_{n\leq m}\mu(n)$ hold many change sign points. The case $\sum_{n\leq m_0}\mu(n) = 0$, I get $\sum_{n\leq m_0}\frac{1}{n}\mu(n)$ $(t.c.\ 0)$. (By later example, $m_0 = 920$, $\sum_{n=1}^{920}\mu(n) = 0$ case, $\sum_{n=1}^{920}\frac{920}{n}\mu(n) \risingdotseq 2.00191$ $(t.c.\ 0)$.) $\sum_{n\leq m}\frac{1}{n}\mu(n)$ changes sign frequently. I get $\frac{K}{m_0} > |\sum_{n\leq m_0}\frac{1}{n}\mu(n)|$. The last term is $\frac{1}{m_0}\mu(m_0)$. I get $0 = \sum_{n\leq m_0}\mu(n)$ $(t.c.\ \sum_{n\leq m_0}\frac{m_0}{n}\mu(n))$. From m_0, $\sum_{n\leq m}\frac{m}{n}\mu(n)$ changes value $\sum_{n\leq m'-1}\frac{1}{n}\mu(n)+\mu(m')$, $\sum_{n\leq m}\mu(n)$ changes value $\mu(m'), m' \leq m$. (The mass of change is less than next $\sum_{n\leq m'}\mu(n) = 0$ and $\sum_{n\leq m'}\frac{m'}{n}\mu(n)$'s defference.) These two formulas too close. ("too close" means $\sum_{n\leq m_0}\frac{m_0}{n}\mu(n)$'s absolute value is less than $\sum_{n\leq m}\mu(n)$ or $\sum_{n\leq m}\frac{m}{n}\mu(n), m_0 \leq m \leq m'$.) From m_0 to m', $\sum_{n\leq m}\frac{1}{n}\mu(n)$ are too small or take same sign. □

example:m=1000

$$\sum_{n=1}^{1000}\mu(n) = 2$$

$$\sum_{n=1}^{1000}\frac{1000}{n}\mu(n) \risingdotseq 4.411$$

$m_0 = 920$

$$\sum_{n=1}^{920} \mu(n) = 0$$

$$\sum_{n=1}^{920} \frac{920}{n} \mu(n) \risingdotseq 2.00191$$

$$\sum_{m=1}^{920} \sum_{n=1}^{m-1} \frac{1}{n} \mu(n) \risingdotseq 2.00191$$

$m' = 1002$

$$\sum_{n=1}^{1002} \mu(n) = 0$$

$$\sum_{n=1}^{1002} \frac{1002}{n} \mu(n) \risingdotseq 2.41969$$

example:m=10000

$$\sum_{n=1}^{10000} \mu(n) = -23$$

$$\sum_{n=1}^{10000} \frac{10000}{n} \mu(n) \risingdotseq -20.827$$

$m_0 = 9256$

$$\sum_{n=1}^{9256} \mu(n) = 0$$

$$\sum_{n=1}^{9256} \frac{9256}{n} \mu(n) \risingdotseq 3.62119$$

$$\sum_{m=1}^{9256} \sum_{n=1}^{m-1} \frac{1}{n} \mu(n) \risingdotseq 3.62119$$

$m' = 11117$

$$\sum_{n=1}^{11117} \mu(n) = 0$$

$$\sum_{n=1}^{11117} \frac{11117}{n} \mu(n) \risingdotseq 0.414323$$

These results means $\sum_{n=1}^{m}\mu(n) < \sum_{n=1}^{m}\frac{m}{n}\mu(n)$.
example:m=100000

$$\sum_{n=1}^{100000}\mu(n) = -48$$

$$\sum_{n=1}^{100000}\frac{100000}{n}\mu(n) \fallingdotseq -48.7228$$

example:m=1000000

$$\sum_{n=1}^{1000000}\mu(n) = 212$$

$$\sum_{n=1}^{1000000}\frac{1000000}{n}\mu(n) \fallingdotseq 200.605$$

example:m=10000000

$$\sum_{n=1}^{10000000}\mu(n) = 1037$$

$$\sum_{n=1}^{10000000}\frac{10000000}{n}\mu(n) \fallingdotseq 1015.24$$

These results means $\sum_{n=1}^{m}\mu(n) > \sum_{n=1}^{m}\frac{m}{n}\mu(n)$. The common result is $\sum_{n=1}^{m}\mu(n) \approx \sum_{n=1}^{m}\frac{m}{n}\mu(n)$

Theorem 2.

$$|\sum_{n\le m}\mu(n)| < Km^{\frac{1}{2}+\epsilon}$$

I get R.H.

Proof. First,

$$|\sum_{m'=1}^{m}\sum_{n=1}^{m'-1}\frac{1}{n}\mu(n)| << |\sum_{n\le m}\frac{m}{n}\mu(n)|$$

By theorem 1. The case $\sum_{n=1}^{m}\frac{m}{n}\mu(n) > 0$, I prove

$$\sum_{n=1}^{m}\frac{1}{n}\mu(n) << Km^{-\frac{1}{2}+\epsilon}$$

$\sum_{n\le m_0}\mu(n) = 0, m_0 < m$

$$\int_{m_0}^{m} Kx^{-\frac{1}{2}+\epsilon}dx = 2Km^{\frac{1}{2}+\epsilon} - 2Km_0^{\frac{1}{2}+\epsilon}$$

$2Km^{\frac{1}{2}+\epsilon} - 2Km_0^{\frac{1}{2}+\epsilon}$ is "as ratio" too big. By this result, I get

$$\sum_{n=1}^{m} \frac{1}{n}\mu(n) << Km^{-\frac{1}{2}+\epsilon}$$

$$|\sum_{n=1}^{m} \frac{m}{n}\mu(n)| < Km^{\frac{1}{2}+\epsilon}$$

$\sum_{m'=1}^{m} \sum_{n=1}^{m'-1} \frac{1}{n}\mu(n)$ is negative term. I get

$$|-\sum_{m'=1}^{m} \sum_{n=1}^{m'-1} \frac{1}{n}\mu(n) + \sum_{n=1}^{m} \frac{m}{n}\mu(n)| < Km^{\frac{1}{2}+\epsilon}$$

$$|\sum_{n\leq m} \mu(n)| < Km^{\frac{1}{2}+\epsilon}$$

□

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