A TOPOLOGICAL APPROACH TO THE TWIN PRIMES AND DE POLIGNAC CONJECTURES

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Abstract. We introduce a topology in the set of natural numbers via a subbase of open sets. With this topology, we obtain an irreducible (hyperconnected) space with no generic points. This fact allows proving that the cofinite intersections of subbasic open sets are always empty. That implies the validity of the Twin Primes Conjecture. On the other hand, the existence of strictly increasing chains of subbasic open sets shows that the Polignac Conjecture is false.

Key words: Irreducible (hyperconnected) space, gaps of prime numbers, primes in arithmetic progressions, generic point

1. Introduction

Let P be the set of prime numbers greater than 2 and $\mathbb{N}$ the set of stricly positive natural numbers.

We begin by introducing in P the topology for which the subsets $H_m = \{p \in P : p + 2m \in P\}$ (for every $m \in \mathbb{N}$) are a subbase of closed sets. We will call O_m the complementary open sets of the H_m and X the topological space we obtain.

We will also consider, now on the set $S = \{n \in \mathbb{N} : H_n \neq \varnothing\}$, the topology generated by the subbase of closed sets $H_p^* = \{m \in S : p + 2m \in P\}$ for all $p \in P$. We will call O_p^* to the complementary open sets of H_p^* and X^* to the topological space that is obtained.

X and X^* turn out to be both irreducible spaces (hyperconnected). In addition, X is T_1 (the maximum separation that supports an irreducible space).

The conjecture of the twin primes is equivalent to the fact that the closed H_1 contains an infinity of points and the conjecture of de Polignac is equivalent to that the same happens to the sets $H_m \cap O_{m-1} \cap \ldots \cap O_1$ for all the natural numbers m.

The article focuses on sets $\bigcap_{q \in I} O_q^*$ where $I \subset X$ due to the fact that there will be a finite H_m if and only if there exists a cofinite set $C \subset X$ such that

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$\bigcap_{q\in I} O_q^* \neq \varnothing$. Indeed:
$\bigcap_{q\in I} O_q^* \neq \varnothing \Leftrightarrow$ exists $m \in \mathbb{N}$ ($H_m \neq \varnothing$) such that for every $p \in C$ then $m \in O_p^* \Leftrightarrow$ exists $m \in \mathbb{N}$ such that for every $p \in C$ then $p \in O_m \Leftrightarrow$ exists m such that $C \subset O_m \Leftrightarrow$ exists m such that $H_m \subset X - C$ which is finite.
We start by proving that $\bigcap_{q\in I} O_q^*$ is open if and only if I is a finite set. In this case $\bigcap_{q\in I} O_q^* \neq \varnothing$ because X^* is irreductible. Then (in section 3) we show that the set $\bigcap_{q\in X} O_q^*$ (which turns out to be the set of the generic points of X^*) is empty. This tells us that the sets of the form $\bigcap_{q\in C} O_q^*$, where C is cofinite, cannot be dense.
Indeed: $\varnothing = \bigcap_{q\in X} O_q^* = \bigcap_{q\in C} O_q^* \cap \bigcap_{q\in X-C} O_q^*$ where $\bigcap_{q\in X-C} O_q^*$ is open because $X - C$ is finite (it is not empty because X^* is irreducible).
In section 4 is introduced the concept of extremal subespace. It is shown below that such subspaces cannot exist. Next we prove that if $\bigcap_{q\in O_n} O_q^*$, with O_n cofinite, was not empty, because there are no generic points, there should be an extreme set and so, so this subspace must be empty. As discussed above, this is equivalent to the fact that, for each $m \in \mathbb{N}$ such that $H_m \neq \varnothing$, H_m has an infinity of elements and, in particular, this is the case of H_1 which proves the conjecture of the twin primes.
Finally, it is shown that $H_m \cap O_{m-1} \cap ... \cap O_1 = \varnothing$ for infinite values of m and, therefore, the de Polignac's conjecture is false in an infinite amount of cases.

2. First properties of topological spaces X and X^*

Let P be the set of the odd prime numbers and $\mathbb{N}$ the set of the strictly positive integers.

For every $m \in \mathbb{N}$ we consider the subset defined as

$$O_m = \{p \in P : p + 2m \notin P\}$$

We will denote H_m the complementary set

$$X - O_m = \{p \in P : p + 2m \in P\}$$

We will call X to the topological space thus obtained. In order to clarify the notation, we will sometimes write $O(m)$ and $H(m)$ instead of O_m and H_m respectively.

We introduce in X the topology τ generated by the set of all the O_m as a subbase ([3]). This means that the open sets of this topology are all the reunions of finite intersections of the sets O_m, namely all sets of the form

$$\bigcup (O(i_1) \cap ... \cap O(i_n))$$

We start by proving some properties of the topological space (X, τ)

Proposition 2.1. *The space X is irreducible.*

Proof. It is enough to see that the intersection of a finite number of O_n,s is non-empty.

If we have $O(m_1) \cap ... \cap O(m_r) = \varnothing$ then $H(m_1) \cup ... \cup H(m_r) = X$. Let $\mu = \max\{m_1, ...m_r\}$. We know that there are arbitrarily large intervals of natural numbers that do not contain prime numbers (as an example of length $n-1$ of these intervals we can take $[n!+2, n!+n] \cap \mathbb{N}$). More specifically, for each natural number m there is a prime number q such that $s(q) - q > m$ where $s(q)$ is the prime number that follows q.

In particular, there must be a q such that $s(q) - q > 2\mu$ and, for this q, $\min\{j : q \in H_j\} > \mu \geq m_i$ for every $i = 1, ..., r$ or, in other words, $q \notin H(m_1) \cup H(m_r)$ which is a contradiction. $\square$

Proposition 2.2. *The space X is* T_1 *(Fréchet)*

Proof. Let $p \in X$. Let's see that $\{p\}$ is a closed set. given a j such that $1 \leq j < p$, because $GCD(j, p) = 1$, there is a $\lambda > 1$ such that $j + \lambda p \in X$. In fact, the Dirichlet theorem relative to prime numbers in arithmetic progressions ([1]; chapter 7) ensures the existence of an infinite amount of these λ.

For all $j = 1, ..., p-1$ let $\nu_j = \min\{\lambda > 1 : j + \lambda p \in X\}$. We have:

$$p \in H\left(\frac{1}{2}(1 + (\nu_1 - 1)p)\right) \cap ... \cap H\left(\frac{1}{2}(p - 1 + (\nu_{p-1} - 1)p)\right)$$

Let $q \in H\left(\frac{1}{2}(1 + (\nu_1 - 1)p)\right) \cap ... \cap H\left(\frac{1}{2}(p - 1 + (\nu_{p-1} - 1)p)\right)$ and suppose that $q \equiv i \neq 0 \pmod{p}$ where $i < p$ and let say that $q = i + \mu p$, $i < p$.

Because $q \in H\left(\frac{1}{2}(p-i+(\nu_{p-i}-1)\,p)\right)$ we have that

$$q+p-i+(\nu_{p-i}-1)\,p = i+\mu p+p-i+(\nu_{p-i}-1)\,p = (\mu+\nu_{p-i})\,p \in X$$

which is absurd because $\mu+\nu_{p-i}>1$. We deduce that it must be $i=0$ i.e. $q \equiv 0 \pmod p$ which implies $q=p$. So that $\{p\} = \bigcap_{j=1}^{p-1} H\left(\frac{1}{2}(j+(\nu_j-1)\,p)\right)$ which is closed.

(We observe that T_1 is the largest separation that allows an irreducible space). □

Remark 1: We emphasize that we have proved that every point in X can be written as a finite intersection of the closed sets H_m that contain it.

Proposition 2.3. *For every $m \in \mathbb{N}$ we have that $\bigcap_{\lambda\in\mathbb{N}} O_{\lambda m}$ is the set of the odd primes that divide m. In particular, if p is a prime number then $\bigcap_{\lambda\in\mathbb{N}} O_{\lambda p} = \{p\}$*

Proof. Suppose that p is a prime divisor of m. If p belonged to some $H_{\lambda m}$ then $p+2\lambda m = p\left(1+2\lambda\frac{m}{p}\right) \in X$ which is absurd so that $p \in \bigcap_{\lambda\in\mathbb{N}} O_{\lambda m}$

Let's see now that if p is an odd prime number that does not divide m then $p \notin \bigcap_{\lambda\in\mathbb{N}} O_{\lambda m}$. If p does not divide m then $GCD(p,m)=1$ and so, $GCD(p,2m)=1$. Applying again the Dirichlet theorem mentioned in proposition 2.2, there must be a λ such that $p+2\lambda m \in X$, that is, $p \in H_{\lambda m}$. □

Now we introduce a topology τ^* in the set $S=\{m\in\mathbb{N} : H_m \neq \varnothing\}$ taking the sets

$$O_p^* = \{m \in S : p+2m \notin X\}$$

(where p is any prime number) as subbase of open sets. We will call X^* to the topological space thus obtained.

In a similar way to what we have previously done with the space X we will write $H_p^* = \{m \in S : p+2m \in X\}$ to denote the complementary sets. Sometimes we will write $O^*(p)$ and $H^*(p)$ instead of O_p^* and H_p^*, respectively. Let's start now the study of the space X^*.

Remark 2: Before starting the study of this space, we note that we now have a new language in order to enunciate the conjectures that are the object of this paper. Indeed, with this language, the de Polignac conjecture says that, for all m, the set $H_m \cap O_1 \cap ... \cap O_{m-1}$ has infinite elements and the twin primes conjecture simply says that the closed set H_1 is infinite.

Proposition 2.4. *X^* is an irreducible topological space.*

Proof. Again it is sufficient to prove that the intersection of a finite number of open sets of the form O_q^* is non empty.

Specifically, if $q_1, ...q_r$ are different primes then

$$q_1 \cdot ... \cdot q_r \in O^* (q_1) \cap ... \cap O^* (q_r)$$

because if for some $j = 1, ...r$ were $q_1 \cdot ... \cdot q_r \in H^* (q_j)$ then

$$q_j + 2q_1 \cdot ... \cdot q_r = q_j (1 + 2q_1 \cdot q_2 \cdot ... \cdot \hat{q}_j \cdot ... \cdot q_r) \in X$$

which is absurd. In fact, if ( ) means "ideal" then

$$(q_1 \cdot ... \cdot q_r) \cap \mathbb{N} = \{\lambda q_1 \cdot ... \cdot q_r : \lambda \in \mathbb{N}\} \subset O^* (q_1) \cap ... \cap O^* (q_r)$$

□

Proposition 2.5. *We have:*

a) If $H_p^* \subset H_q^*$ the $p = q$

b) The subspaces H_p^* are irreducible.

Proof. a) If $H_p^* \subset H_q^*$ then every m in H_p^* belongs to H_q^* or, in other words, if $p \in H_m$ then $q \in H_m$ or, put in other way, $\{m : p \in H_m\} \subset \{m : q \in H_m\}$ and that implies

$$\bigcap_{q \in H_m} H_m \subset \bigcap_{p \in H_m} H_m$$

From remark 1 after proposition 2.2, we deduce that, for every r,

$$\bigcap_{q \in H_r} H_r = \{r\}$$

and, so, the previous inclusion implies $\{q\} \subset \{p\}$ or, what is the same, $p = q$.

b) We must to prove that if T and T' are closed sets then $H_p^* \subset T \cup T'$ implies either $H_p^* \subset T$ or $H_p^* \subset T'$. We begin proving that if $H_p^* \subset \bigcup_{i=1}^{r} H^* (q_i)$ then there is i such that $p = q$.

If $H_p^* \subset \bigcup_{i=1}^{r} H^*(q_i)$ then $\bigcap_{i=1}^{r} O^*(q_i) \subset O^*(p)$ which means that for all m such that $\{q_1, ...q_r\} \subset O_m$ we must have that $p \in O_m$, but for all $\lambda \in \mathbb{N}$,

$$\{q_1, ...q_r\} \subset O(\lambda q_1 \cdot ... \cdot q_r)$$

so that, for all $\lambda \in \mathbb{N}$, $p \in O(\lambda q_1 \cdot ... \cdot q_r)$ that is $p \in \bigcap_{\lambda \in N} O(\lambda q_1 \cdot ... \cdot q_r)$. By proposition 2.3, the last intersection is equal to $\{q_1, ..., q_r\}$ and therefore $p = q_i$ for some $i = 1, ..., r$

Suppose that $H_p^* \nsubseteq T \cup T' =$

$$\left(\bigcap_i \left(H^*(p_{i_1}) \cup ... \cup H^*\left(p_{i_{r(i)}}\right)\right)\right) \cup \left(\bigcap_j \left(H^*(q_{j_1}) \cup ... \cup H^*\left(q_{j_{s(j)}}\right)\right)\right)$$

and that $H_p^* \nsubseteq T'$. Then there must be a j such that
$H_p^* \nsubseteq H^*(q_{j_1}) \cup ... \cup H^*\left(q_{j_{s(j)}}\right)$ but, for every i, we have

$$H_p^* \subset \left(H^*(p_{i_1}) \cup ... \cup H^*\left(p_{i_{r(i)}}\right)\right) \cup \left(H^*(q_{j_1}) \cup ... \cup H^*\left(q_{j_{s(j)}}\right)\right)$$

This implies that for each i exists a t, $(1 \leq t \leq r_i)$ such that $H_p^* = H^*(p_{i_t})$ since we can't have $H_p^* = H^*(q_{j_k})$. Therefore, for every i,
$H_p^* \subset H^*(p_{i_1}) \cup ... \cup H^*\left(p_{i_{r(i)}}\right)$ and, finally,

$$H_p^* \subset \bigcap_i \left(H^*(p_{i_1}) \cup ... \cup H^*\left(p_{i_{r(i)}}\right)\right) = T$$

□

Proposition 2.6. *$H_n \subset H_m$ if and only if $\overline{\{m\}} \subset \overline{\{n\}}$ where the upper bar means closure in the space X^**

Proof. $\Rightarrow$)
Suppose that $m \notin \overline{\{n\}}$. Then there will be $p_1, ..., p_r$ such that
$m \in O^*(p_1) \cap ... \cap O^*(p_r)$ and $n \in H^*(p_1) \cup ... \cup H^*(p_r)$. So, for each $i = 1, ..., r$, p_i belongs to O_m and there is a j such that p_j belongs to H_n. As $H_n \subset H_m$, this p_j belongs to H_m which is a contradiction with $p_i \in O_m$ for every $i = 1, ..., r$
$\Leftarrow$)
$\overline{\{m\}} \subset \overline{\{n\}}$ implies $m \in \overline{\{n\}}$ and, therefore, for every p such that $m \in O_p^*$ we must have that $n \in O_p^*$ or, what is the same, for every p such that $n \in H_p^*$ we must have that $m \in H_p^*$. That means that, for every p, $p \in H_n$ implies $p \in H_m$. In other words, $H_n \subset H_m$. □

Proposition 2.7. $\overline{\{m\}} = \bigcap_{p \in H_m} H_p^*$

Proof. $\subset$)
Obviously, for every $p \in H_m$, we have $m \in H_p^*$ and therefore $m \in \bigcap_{p \in H_m} H_p^*$ so that

$$\overline{\{m\}} \subset \overline{\bigcap_{p \in H_m} H_p^*} = \bigcap_{p \in H_m} H_p^*$$

$\supset$)
Be $n \in \bigcap_{p \in H_m} H_p^*$. If $n \notin \overline{\{m\}}$ then will exist $q_1, ..., q_s$ such that $n \in O^*(q_1) \cap ... \cap O^*(q_s)$ and $m \in H^*(q_1) \cup ... \cap \cup H^*(q_s)$. This tell us that there is an i such that $m \in H^*(q_i)$, that is such that $q_i \in H_m$. So, in particular,we have that $n \in H^*(q_i)$ because $n \in \bigcap_{p \in H_m} H_p^*$. That contradicts the fact that $n \in O^*(q_1) \cap ... \cap O^*(q_s)$. □

Proposition 2.8. *$H_m \cap H_n = \varnothing$ if and only if $m \in \bigcap_{p \in H_n} O_p^*$*

Proof. We have successively: $H_m \cap H_n = \varnothing \Leftrightarrow H_n \subset O_m \Leftrightarrow$ for every $p \in H_n$ we must have that $p \in O_m \Leftrightarrow$ for all $p \in H_n$ it must have that $m \in O_p^* \Leftrightarrow$ $m \in \bigcap_{p \in H_n} O_p^*$.

□

As an example of this situation, it is easy to see that $H_{16} \cap H_{17} = \varnothing$: suppose that $p \epsilon H_{16} \cap H_{17}$ and check that p cannot be congruent neither with 0 nor whit 1 nor whit 2 (mod 3).

Proposition 2.9. *The following conditions are equivalents:*

i) $I \subset \mathbb{N}$ is a finite set.

ii) $\bigcap_{i \in I} O^*(p_i)$ is an open non-empty set.

iii) The interior of $\bigcap_{i \in I} O^*(p_i)$ is non-empty.

Proof. $i) \Rightarrow ii)$ and $ii) \Rightarrow iii)$ are obvious (note that in $i) \Rightarrow ii)$, $\bigcap_{i \in I} O^*(p_i)$ is not empty because I is finite and, by proposition 1.4, X^* is an irreducible space). We're going to see $iii) \Rightarrow i)$.

If the interior of $\bigcap_{i\in I} O^*(p_i)$ is non-empty then there are $q_1, ...q_s$ such that $O^*(q_1) \cap ... \cap O^*(q_s) \subset \bigcap_{i\in I} O^*(p_i)$ or, which is equivalent, $\bigcup_{i\in I} H^*(p_i) \subset H^*(q_1) \cup ... \cup H^*(q_s)$ and, therefore, for every $i \in I$ we have that $H^*(p_i) \subset H^*(q_1) \cup ... \cup H^*(q_s)$. Using the proposition 2.5, we obtain that, for every $i \in I$, there is a j $(1 \le j \le s)$ such that $H^*(p_i) = H^*(q_j)$. So that, for every $i \in I$, there is a j $(1 \le j \le s)$ such that $p_i = q_j$ and so, $\{p_i : i \in I\} \subset \{q_1, ...q_s\}$ and I is finite. □

Proposition 2.10. *Are equivalent:*

i) For every $C \subset \mathbb{N}$ cofinite, $\bigcap_{q\in C} O_q^* = \varnothing$

ii) For every $m \in \mathbb{N}$, the closed set H_m has infinite points.

Proof. $i) \Rightarrow ii)$ If there is a $m \in \mathbb{N}$ such that H_m is finite then O_m is cofinite and, nevertheless, $\bigcap_{q\in O_m} O_q^* \neq \varnothing$ because $m \in \bigcap_{q\in O_m} O_q^*$.

Indeed, $n \in \bigcap_{q\in O_m} O_q^* \Leftrightarrow n \in O_q^*$ for every $q \in O_m \Leftrightarrow q \in O_n$ for every $q \in O_m \Leftrightarrow O_m \subset O_n$ and so, $m \in \bigcap_{q\in O_m} O_q^*$ because, obviously $O_m \subset O_m$.

$ii) \Rightarrow i)$ We have that $m \in O_q^* \Leftrightarrow q \in O_m$ and so, given any $I \subset X$, $m \in \bigcap_{q\in I} O_q^* \Leftrightarrow m \in O_q^*$ for every $q \in I$ which is equivalent to $q \in O_m$ for all $q \in I$ or, what is the same, $I \subset O_m$. Now, if there is a cofinite set C such that $\bigcap_{q\in C} O_q^* \neq \emptyset$ and $m \in \bigcap_{q\in C} O_q^*$, then $C \subset O_m$ and, therefore, $H_m \subset X - C$ which is finite. □

Remark 3: We observe that if we prove $\bigcap_{q\in C} O_q^* = \varnothing$ for every cofinite C, in particular, we will have proven that H_1 is an infinite set which, as we have already explained in remark 2 after proposition 2.3, is equivalent to the Twin Primes Conjecture.

3. Generic points

Remember that a generic punt is one whose singleton is dense.

Proposition 3.1. *$H_m = \emptyset$ if and only if $\overline{\{m\}} = X^*$ namely if and only if m is a generic point in X^*.*

Proof. We have that $H_m = \emptyset \Leftrightarrow O_m = X$ and this is equivalent to $p \in O_m$ for every $p \in X$ or, in other words, $m \in O_p^*$ for every $p \in X$. That is the same as saying that m belongs to all open sets $U \subset X^*$ which is equivalent to say $\overline{\{m\}} = X^*$ since $\{m\}$ is dense if and only if m belongs to the all open sets. □

We will call Z the set $\{m \in X^* : H_m = \emptyset\}$. Let's see that $Z = \emptyset$ namely X^* has no generic points.

Proposition 3.2. $Z = \bigcap_{q \in X} O_q^*$

Proof. We have $m \in Z \Leftrightarrow O_m = X \Leftrightarrow q \in O_m$ for every $q \in X \Leftrightarrow m \in O_q^*$ for every $q \in X \Leftrightarrow m \in \bigcap_{q \in X} O_q^*$. □

Theorem 3.3. *$Z = \varnothing$ namely X^* has no generic points or, in other words, $H_m \neq \varnothing$ for every $m \in \mathbb{N}$*

Proof. Let $B_m = \bigcap_{p \in H_m} H_p^* \cap \bigcap_{q \in O_m} O_q^*$ for each $m \in X^*$.

$$\bigcap_{p \in H_m} H_p^* = \{n : H_m \subset H_n\} = \overline{\{m\}}$$

and

$$\bigcap_{q \in O_m} O_q^* = \{n : O_m \subset O_n\} = \{n : H_n \subset H_m\}$$

so that $B_m = \{n : H_n = H_m\}$ and is not empty because, at least, $m \in B_m$, We have:

$$B_m \cap \bigcap_{q \in H_m} O_q^* = \left(\bigcap_{p \notin H_m} H_p^* \cap \bigcap_{q \in H_m} O_q^* \right) \cap \bigcap_{q \in O_m} O_q^* = \varnothing \cap \bigcap_{q \in O_m} O_q^* = \varnothing$$

On the other hand,

$$B_m \cap \bigcap_{q \in H_m} O_q^* = \bigcap_{q \in H_m} H_q^* \cap \left(\bigcap_{q \in H_m} O_q^* \cap \bigcap_{q \in O_m} O_q^* \right) =$$

$$= \bigcap_{q\in H_m} H_q^* \cap \bigcap_{q\in H_m\cup O_m} O_q^* = \bigcap_{q\in H_m} H_q^* \cap \bigcap_{q\in X^*} O_q^* = \overline{\{m\}} \cap Z$$

and therefore, for every $m \in X^*$, $\overline{\{m\}} \cap Z = \varnothing$ so that $Z = \varnothing$.

□

Corollary 3.4. *Every even number is the difference of two prime numbers.*

Proof. $Z = \varnothing$ means $H_m \neq \varnothing$ for every m and this implies that, for every even number $2m$, exists a prime number p such that $p + 2m$ is also a prime number q. So $q - p = 2m$. □

Lemma 3.5. *Let $A, B \subset \mathbb{N}$ two disjoint sets, then*

$$\min(A \cap B) = \min(A) \Leftrightarrow \min(A) \in B$$

Proof. $\Rightarrow$) is obvious.

$\Leftarrow$) We always have $\min(A \cap B) \geq (A)$. If $\min(A \cap B) > \min(A)$ then $\min(A) \notin A \cap B$ and therefore, $\min(A) \notin B$. □

Lemma 3.6. *For every $m \in X^*$ there is an open set U such that* $\min(U) = m$

Proof. We will use induction on m.

If $m = 1$ we have, for example, $\min(O_7^*) = 1$. Suppose the lemma proved up to $m - 1$ and let $\min(U) = m - 1$. We know that for every $p \in X$, $\min\left(U \cap O_p^*\right) \geq \min(U) = m - 1$. We also know that $H_{m-1} \neq \varnothing$ (theorem 3.3) and for every $p \in H_{m-1}$ (namely $m - 1 \in H_p^*$) then $\min\left(U \cap O_p^*\right) > \min(U) = m - 1$ (lemma 3.4) i.e. $\min\left(U \cap O_p^*\right) \geq \min(U) = m$.

If $\min\left(U \cap O_p^*\right) > \min(U) = m$ for all $p \in H_{m-1}$, then $m \notin U \cap O_p^*$ to any $p \in H_{m-1}$. This means that for all $p \in H_{m-1}$, $m \in (X^* - U) \cup H_p^*$ or, in other words, for all $p \in H_{m-1}$, if $m \in U$ then $m \in H_p^*$ or, which is the same, $U \subset \bigcap_{p\in H_m} H_p^* = \overline{\{m-1\}}$. But this implies $\overline{U} = X^* \subset \overline{\{m-1\}}$ which tell us that $m - 1 \in Z$ wich contradicts the theorem 3.3. So, must exists $p \in H_{m-1}$ such that $\min\left(U \cap O_p^*\right) = \min(U) = m$. □

Proposition 3.7. *We have:*

a) X^* is T_0 if and only if $O_m \neq O_n$ for every m, n such that $m \neq n$.

b) X^* is T_1 if and only if $O_m \nsubseteq O_n$ for every m, n such that $m \neq n$.

Proof. We will show *b)*. The proof of a) is analogous.
X^* is not T_1 if and only if there are m, n with $m \neq n$ and such that $n \in O^*_{p_1} \cap ... \cap O^*_{p_r}$ for every $p_1, ...p_r$ such that $m \in O^*_{p_1} \cap ... \cap O^*_{p_r}$. That is equivalent to that exist m, n with $m \neq n$ and such that $\{p_1, ...p_r\} \subset O_n$ for every $p_1, ...p_r$ with $\{p_1, ...p_r\} \subset O_m$. Again, this amounts to that for each finite set F, $F \subset O_m$ implies $F \subset O_n$ and that is equivalent to $O_m \subset O_n$ □

Corollary 3.8. *X^* is a T_0 topological space (Kolmogorov) and, in particular, by proposition 3.6, $m \neq n \Rightarrow O_m \neq O_n$ (or $H_m \neq H_n$)*

Proof. Let $m < n$ be two points of X^*. By lemma 3.5, there is an open set U such that $\min(U) = m$. Then $m \in U$ but $n \notin U$ because $n < m = \min(U)$.
□

4. Extremality

Definition Let $I \subset X$. We will say that I is an *extremal set* or, simply, an *extremal* if $\bigcap_{q\in I} O^*_q \neq \varnothing$ and $O^*_p \cap \bigcap_{q\in I} O^*_q = \bigcap_{q\in I\cup\{p\}} O^*_p = \varnothing$ for every $p \in X - I$.

Proposition 4.1. *We have:*

a) If $\bigcap_{q\in I} O^*_q \neq \varnothing$ then $\bigcap_{q\in X-I} H^*_q \subset \overline{\bigcap_{q\in I} O^*_q}$

b) $I \subset X$ is an extremal set if and only if $\bigcap_{q\in I} O^*_q \neq \varnothing$ and

$$\bigcap_{q\in X-I} H^*_q = \overline{\bigcap_{q\in I} O^*_q}$$

Proof. $a)$ If $\bigcap_{q\in X-I} H^*_q = \varnothing$ there is nothing to prove. Otherwise, $m \in \bigcap_{q\in X-I} H^*_q \Leftrightarrow m \in H^*_q$ for every $q \in X - I \Leftrightarrow q \in H_m$ for every $q \in X - I \Leftrightarrow X - I \subset H_m \Leftrightarrow O_m \subset I^{(1)}$. Consider $q_1, ...q_r$ such that $m \in O^*(q_1) \cap ... \cap O^*(q_r)$. This is equivalent to $\{q_1, ...q_r\} \subset O_m$ and, by **(1)**, we have that $\{q_1, ...q_r\} \subset I$ so that $O^*(q_1) \cap ... \cap O^*(q_r) \cap \bigcap_{q\in I} O^*_q = \bigcap_{q\in I} O^*_q \neq \varnothing$.
Therefore, $m \in \overline{\bigcap_{q\in I} O^*_q}$

b) We know that in every topological space, if A is an open set and B is any set, then $A \cap \overline{B} \subset \overline{A \cap B}$ (see [2]; 1.7, prop. 5). In our case this tells us that for every $p \in X - I$, we have $O_p^* \cap \overline{\bigcap_{q\in I} O_q^*} \subset \overline{\bigcap_{q\in I\cup\{p\}} O_q^*}$ but this last set is empty because I is an extremal and, so, $\overline{\bigcap_{q\in I} O_q^*} \subset H_p^*$ for every $p \in X - I$, that is $\overline{\bigcap_{q\in I} O_q^*} \subset \bigcap_{p\in X-I} H_p^*$. Part *a)* completes the proof. Let's see the reciprocal of part *b)*.

$\overline{\bigcap_{q\in I} O_q^*} = \bigcap_{p\in X-I} H_p^* \Rightarrow \bigcap_{q\in I} O_q^* \subset \bigcap_{p\in X-I} H_p^*$ which means that for every m such that $I \subset O_m$ we have that $X - I \subset H_m$ that is to say that for every m such that $I \subset O_m$ we have that $I = O_m$. If there was $p \in X - I$ such that $\bigcap_{q\in I\cup\{p\}} O_q^* \neq \varnothing$ then there would be n such that $I \subset I \cup \{p\} \subset O_n$ and by the previous observation we have that $I = I \cup \{p\} = O_n$ which implies $p \in I$. This is a contradiction. Therefore, $\bigcap_{q\in I\cup\{p\}} O_q^* = \varnothing$ for every $p \in X - I$ and I is an extremal set. □

Proposition 4.2. *I is extremal $\Leftrightarrow$ exists n such that $I = O_n$ and $\bigcap_{q\in O_n} O_q^* = \{m : O_m = O_n\}$.*

Proof. $\Rightarrow$) We have $n \in \bigcap_{q\in I} O_q^* \Leftrightarrow I \subset O_n$. In addition, if I is extremal then $\bigcap_{q\in I} O_q^* \subset H_p^*$ for every $p \in X - I$ which implies that, for every $p \in X - I$, if $I \subset O_n$, then $p \in H_n$. Therefore, if $I \subset O_n$ then $X - I \subset H_n$. That means that, if $I \subset O_n$ then $O_n \subset I$ and, so, $O_n = I$. In short, we must have $O_n = I$ for every $n \in \bigcap_{q\in I} O_q^*$ and so, $\bigcap_{q\in I} O_q^* = \{m : O_m = O_n\}$.

$\Leftarrow$)
$\bigcap_{q\in I} O_q^* \neq \varnothing$ because $I = O_n$ and therefore $n \in \bigcap_{q\in O_n} O_q^* = \bigcap_{q\in I} O_q^*$. If there was a $p \in X - I$ such that $\bigcap_{q\in I\cup\{p\}} O_q^* \neq \varnothing$ then there should be m that $I \cup \{p\} \subset O_m$ but $I \subsetneqq I \cup \{p\} \subset O_m$ and, in particular, $I \subset O_m$ which implies $m \in \bigcap_{q\in I} O_q^*$ and $I = O_n \nsubseteq O_m$ which is absurd.

Definition: If $I \subset X$ we will call $N(I)$ the set

$$N(I) = \left\{ p \in X : O_p^* \cap \bigcap_{q\in I} O_q^* = \varnothing \right\} = \left\{ p \in X : \bigcap_{q\in I\cup\{p\}} O_q^* = \varnothing \right\}$$

We have:

a) If $\bigcap_{q\in I} O_q^* = \varnothing$ then $N(I) = X$ and reciprocally. Indeed, $N(I) = X$ implies that $\bigcap_{q\in I} O_q^* \subset H_p^*$ for every $p \in X$, namely $\bigcap_{q\in I} O_q^* \subset \bigcap_{p\in X} H_p^*$. But $m \in \bigcap_{p\in X} H_p^* \Leftrightarrow p \in H_m$ for every $p \in X$ which is equivalent to $H_m = X$ and this is absurd due to the existence of gaps of prime numbers of arbitrary length.

b) If $\bigcap_{q\in I} O_q^* \neq \varnothing$ is dense, then $N(I) = \varnothing$. □

Proposition 4.3. *If* $\bigcap_{q\in X-N(I)} O_q^* \neq \varnothing$ *then* $\bigcap_{q\in N(I)} H_q^* = \overline{\bigcap_{q\in X-N(I)} O_q^*}$ *and so, by proposition* 4.1*,* $X - N(I)$ *is extremal.*

Proof. $\bigcap_{q\in X-N(I)} O_q^* \neq \varnothing$ implies $N(I) \neq \varnothing$ because, otherwise,

$$\bigcap_{q\in X-N(I)} O_q^* = \bigcap_{q\in X} O_q^* = Z = \varnothing$$

We have that $\bigcap_{q\in I} O_q^* \subset H_p^*$ for all the $p \in N(I)$ and so, $\bigcap_{q\in I} O_q^* \subset \bigcap_{p\in N(I)} H_p^*$ which implies $\overline{\bigcap_{q\in I} O_q^*} \subset \bigcap_{p\in N(I)} H_p^{*(1)}$ because $\bigcap_{p\in N(I)} H_p^*$ is a closed set.

Given that $\bigcap_{q\in X-N(I)} O_q^* \neq \varnothing$, by the proposition 4.1*a*), we have that

$$\bigcap_{q\in N(I)} H_q^* \subset \overline{\bigcap_{q\in X-N(I)} O_q^{*}}^{(2)}$$

On the other hand, $N(I) \subset X - I \Rightarrow I \subset X - N(I) \Rightarrow$

$$\Rightarrow \bigcap_{q\in X-N(I)} O_q^* \subset \bigcap_{q\in I} O_q^* \Rightarrow \overline{\bigcap_{q\in X-N(I)} O_q^*} \subset \overline{\bigcap_{q\in I} O_q^{*}}^{(3)}$$

Therefore, $\overline{\bigcap_{q\in I} O_q^*} \subset \bigcap_{p\in N(I)} H_p^* \subset \overline{\bigcap_{q\in X-N(I)} O_q^*} \subset \overline{\bigcap_{q\in I} O_q^*}$ where inclusions come, respectively, from **(1)**, **(2)** and **(3)**. We finally obtain that

$$\bigcap_{p\in N(I)} H_p^* = \overline{\bigcap_{q\in X-N(I)} O_q^*}$$

. □

Proposition 4.4. *If* $\bigcap_{q\in I} O_q^* \neq \varnothing$ *then* I *is extremal if and only if* $N(I) = X - I$.

Proof. We have $X - I \subset N(I) \Leftrightarrow \bigcap_{q\in I} O_q^* \cap O_p^* = \varnothing$ for every $p \in X - I \Leftrightarrow I$ is extremal. But if $\bigcap_{q\in I} O_q^* \neq \varnothing$ then $X - I \subset N(I)$ which is equivalent to $X - I = N(I)$ because we always have $N(I) \subset X - I$. $\square$

Proposition 4.5. $N(I) = \bigcap_{I\subset O_m} H_m$

Proof. We have $I \subset O_m \Leftrightarrow m \in \bigcap_{q\in I} O_q^*$ and therefore $p \in \bigcap_{I\subset O_m} H_m \Leftrightarrow p \in H_m$ for every $m \in \bigcap_{q\in I} O_q^* \Leftrightarrow m \in H_p^*$ for every $m \in \bigcap_{q\in I} O_q^* \Leftrightarrow O_p^* \cap \bigcap_{q\in I} O_q^* = \varnothing \Leftrightarrow p \in N(I)$. $\square$

Proposition 4.6. *If* $\bigcap_{q\in I} O_q^* \neq \varnothing$ *then* I *is an extremal* $\Leftrightarrow I = \bigcup_{I\subset O_m} O_m$

Proof. By proposition 4.4, I is an extremal if and only if $N(I) = X - I$ and, by proposition 4.5, $N(I) = \bigcap_{I\subset O_m} H_m$. $\square$

Proposition 4.7. *If* $I \subset J$ *then* $N(I) \subset N(J)$

Proof. $I \subset J$ implies $\bigcap_{q\in J} O_q^* \subset \bigcap_{q\in I} O_q^*$. If $p \in N(I)$ then $O_p^* \cap \bigcap_{q\in I} O_q^* = \varnothing$ and so $O_p^* \cap \bigcap_{q\in J} O_q^* = \varnothing$ which means that $p \in N(J)$. $\square$

Proposition 4.8. *If* $\bigcap_{q\in I} O_q^* \neq \varnothing$ *and* $\bigcap_{q\in X-N(I)} O_q^* \neq \varnothing$ *then* $X - N(I)$ *is extremal.*

Proof. If $X - N(I)$ is not extremal then must exists $p \in X - N(X - N(I)) = N(I)$ such that $\bigcap_{q\in (X-N(I))\cup\{p\}} O_q^* \neq \varnothing$ but $p \in N(I)$ implies $\bigcap_{q\in I\cup\{p\}} O_q^* = \varnothing$. However, $I \subset X - N(I) \Rightarrow I \cup \{p\} \subset (X - N(I)) \cup \{p\}$ and so, $\varnothing \neq \bigcap_{q\in (X-N(I))\cup\{p\}} O_q^* \subset \bigcap_{q\in I\cup\{p\}} O_q^*$ which is absurd. $\square$

Corollary 4.9. *If* $\bigcap_{q\in I} O_q^* \neq \varnothing$ *then* $\bigcap_{q\in X-N(I)} O_q^* = \varnothing$

Proof. Suppose $\bigcap_{q\in X-N(I)} O_q^* \neq \varnothing$. We have $I \neq X$ because $\bigcap_{q\in I} O_q^* \neq \varnothing$. Be J any set such that $J \supsetneq I$ (such J exists because $I \neq X$). Then $N(I) \subset N(J) \Rightarrow X - N(J) \subset X - N(I)$ and so, $\varnothing \neq \bigcap_{q\in X-N(I)} O_q^* \subset \bigcap_{q\in X-N(J)} O_q^*$. Therefore, $\varnothing \neq \bigcap_{q\in X-N(I)} O_q^* \subset \bigcap_{q\in X-N(J)} O_q^*$ and, by proposition 4.8, $X - N(J)$ is extremal. Now, $X - N(J) \subset X - N(I)$ and since $X - N(J)$ is extremal and $J = O_\nu = X - N(J) \subset X - N(I) = O_\mu = I$, we must have that $X - N(J) = X - N(I)$ and so, $J = I$ which is a contradiction. □

Theorem 4.10. *There are not extremal sets.*

Proof. Suppose I is an extremal. Then $\bigcap_{i\in I} O_q^* \neq \varnothing$ and $I = X - N(I)$ so that $\bigcap_{q\in X-N(I)} O_q^* \neq \varnothing$ which is impossible by corollary 4.9. □

Theorem 4.11. *For every* $n \in \mathbb{N}$ *(such that* $H_n \neq \varnothing$*),* H_n *is an infinite set. In particular,* H_1 *is infinite and the Twin Primes Conjecture is true.*

Proof. If not, O_n would be cofinite and so, $\bigcap_{q\in O_n} O_q^*$ (which is non empty because $n \in \bigcap_{q\in O_n} O_q^*$) is not dense. Indeed, $\bigcap_{q\in O_n} O_q^* \cap \bigcap_{q\in H_n} O_q^* = Z = \varnothing$ (see proposition 3.3) where $\bigcap_{q\in H_n} O_q^*$ is open because H_n is finite. From theorem 4.10, O_n is not extremal and thus, $O_n \subsetneqq X - N(O_n)$ and, therefore, there is a p such that $p \in X - N(O_n)$ and $p \notin O_n$. So we have that $\bigcap_{q\in O_n\cup\{p\}} O_q^* \neq \varnothing$ is not dense and $O_n \subsetneqq O_n \cup \{p\}$

Iterating this process we will arrive to $\bigcap_{q\in X-\{r\}} O_q^* \neq \varnothing$ and $X - \{r\} = X - N(X - \{r\})$ because $N(X - \{r\}) = \{r\}$. This implies that $X - \{r\}$ is an extremal set in contradiction with proposition 4.10. □

Theorem 4.12. *The de Polignac Conjecture is false.*

Proof. Let's take any O_ν. As $\bigcap_{q\in O_\nu} O_q^* \neq \varnothing$, by proposition 4.11, $\bigcap_{q\in O_\nu} O_q^*$ is dense and, therefore, if $p \in H_\nu$, we must have $\bigcap_{q\in O_\nu \cup\{p\}} O_q^* \neq \varnothing$.

If $n_1 \in \bigcap_{q\in O_\nu \cup\{p\}} O_q^*$ then $O_\nu \subsetneqq O_{n_1}$. Iterating this process we get a strict chain:

$$O_\nu \subsetneqq O_{n_1} \subsetneqq O_{n_2} \subsetneqq O_{n_3} \subsetneqq ...$$

which shows that the space X is not noetherian ([4]) and, in particular, by proposition 3.7, X^* is not a T_1 space. Let's consider the complementary chain

$$H_\nu \supsetneqq H_{n_1} \supsetneqq H_{n_2} \supsetneqq H_{n_3} \supsetneqq ...$$

All the n_j are different because the H_{n_j} are and so, there must be a k such that $n_k > \nu$. We have $H_\nu \supsetneqq H_{n_k}$ which implies $H_{n_k} \cap O_\nu \subset H_\nu \cap O_\nu = \varnothing$ so that

$$H_{n_k} \cap O_1 \cap O_2 \cap ... \cap O_\nu \subset H_{n_k} \cap O_\nu = \varnothing$$

and, therefore

$$H_{n_k} \cap O_1 \cap O_2 \cap ... \cap O_\nu \cap O_{\nu+1} \cap ... \cap O_{n_k-1} = \varnothing$$

(note that $n_k > \nu \Rightarrow n_{k-1} \geq \nu$ which means that the subscript n_{k-1} is either ν or "comes" after ν).
The thesis is derived from remark 2 after the proposition 2.3. □

Corollary 4.13. *There are no gaps of prime numbers of length* $2n$ *for infinite values of* n.

Proof. It suffices to take $\nu = n_k$ and repeat the previous reasoning indefinitely. □

Note however that, since H_m is always an infinite set if $H_m \neq \varnothing$, for each $m \in \mathbb{N}$ such that $H_m \neq \varnothing$ there are infinite couples of prime numbers that differ in $2m$ units (although, perhaps, they are not couples of consecutive prime numbers).

Theorem 4.14. *The subspaces* $\bigcap_{q\in I} O_q^*$ *can be classified as follows according to the size of the set* I*:*

a) If I *is finite then* $\bigcap_{q\in I} O_q^*$ *is open and reciprocally. What's more,* $\bigcap_{q\in I} O_q^*$ *is dense because* X^* *is an irreducible space.*

b) F I *is infinite but not cofinite and* $\bigcap_{q\in I} O_q^* \neq \varnothing$ *then* $\bigcap_{q\in I} O_q^*$ *is dense (but not open).*

c) If I *is cofinite then* $\bigcap_{q\in I} O_q^* = \varnothing$

On the other hand, there are sets I *that are infinite (but no cofinite) and fullfil that* $\bigcap_{q\in I} O_q^* = \varnothing$*.*

Proof. a), b) and c) have been demonstrated along the article. Let us see that there exist sets I such that they are infinite (not cofinite) and such that $\bigcap_{q\epsilon I} O_q^* = \varnothing$.

For any prime number p we call $\sigma(p)$ the prime number that follows p. If $p \in H_k$ we will write $\sigma_k(p)$ to indicate the prime number of H_k that follows p. In order to clarify this, since $H_2 = \{3, 7, 13, 19, ...\}$, we have $\sigma(3) = 5$ but $\sigma_2(3) = 7$. Let $h_1 = \min(H_1) = 3$. We will define by recurrence h_k for each $k \in \mathbb{N}$. Considere $u_k = \min\{m \epsilon H_k : m > h_{k-1}\}$. Then we define:

$$h_k = \begin{cases} u_k \text{ if } u_k \neq \sigma(h_{k-1}) \\ \\ \sigma_k(u_k) \text{ if } u_k = \sigma(h_{k-1}) \end{cases}$$

Let's do $I = \{h_k : k \in \mathbb{N}\}$. Let's see that this I satisfies the conditions

1. I is infinite
2. I is not cofinite
3. $\bigcap_{q\in I} O_q^* = \varnothing$

1. If $u^k \neq \sigma(h_{k-1})$ then $h_k = \min\{m \in H_k : m > h_{k-1}\}$ which is greater than h_{k-1} by definition. This also happens if $u_k = \sigma(h_{k-1})$ (even more reason why) because, then,

$$h_k = \sigma_k(u_k) > u_k = \min\{m \in H_k : m > h_{k-1}\} > h_{k-1}$$

This tells us that the sequence $(h_k)_{k\in\mathbb{N}}$ is strictly increasing and therefore I is infinite.

2. For any k we have that $h_k \neq \sigma(h_{k-1})$. Indeed:

$h_k = u_k = \min\{m \in H_k : m > h_{k-1}\}$ if $u_k \neq \sigma(h_{k-1})$ and $h_k = \sigma_k(u_k) = \sigma_k(\sigma(h_{k-1}))$ if $u_k = \sigma(h_{k-1})$ This tells us that I is not cofinite because $\{\sigma(h_k) : k \in \mathbb{N}\} \subset X - I$.

3. $\bigcap\limits_{q \in I} O_q^* \neq \varnothing \Leftrightarrow$ there is n such that $I \subset O_n \Leftrightarrow$ there is n such that $I \cap H_n = \varnothing$. But this does not happen for any n given that, by the construction of I, we have $I \cap H_n \neq \varnothing$ to all n. □

Proposition 4.15. *X^* is not a sober space*

Proof. If X^* were sober, all closed irreducible set must be the closure of a point (exactly one, to be precise). By proposition 2.5 (b) the closet subspaces H_p^* are irreducible and therefore we should have that $H_p^* = \overline{\{m\}}$ for some m. By proposition 2.7, $\overline{\{m\}} = \bigcap\limits_{q \in H_m} H_q^*$ and we have $H_p^* = \bigcap\limits_{q \in H_m} H_q^*$. This implies $H_p^* \subset H_q^*$ for every $q \in H_m$. Now, by proposition 2.5 (a) we obtain $q = p$ for every $q \in H_m$. We deduce that $H_m = \{p\}$ in contradiction to theorem 4.11. □

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