The Proofs of Legendre's Conjecture and Related Conjectures

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Abstracts

In this paper, we are going to prove Legendre's Conjecture: There is a prime number between n^2 and $(n+1)^2$ for every positive integer n. We will also prove several related conjectures. The method that we use is to analyze a binomial coefficient. It has been developed from the method of analyzing a central binomial coefficient that was used by Paul Erdős to prove Bertrand's postulate / Chebyshev's theorem.

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1. Introduction

Legendre's Conjecture was proposed by Andrien-Marie Legendre [1]. The conjecture is one of Legendre's problems (1912) on prime numbers. It states that there is a prime number between n^2 and $(n+1)^2$ for every positive integer n.

In this paper, we will prove Legendre's Conjecture by analyzing the binomial coefficient $\binom{\lambda n}{n}$ where λ is a positive integer. It is developed from the method that was used by Paul Erdős [2] to prove Bertrand's postulate / Chebyshev's theorem [3].

In Section 1, we will define the prime decomposition operator and clarify some terms and concepts. In Section 2, we will derive some lemmas. In Section 3, we will develop a theorem to be used in the proofs of the conjectures in the later sections. In Section 4, we will prove Legendre's conjecture, and in Section 5, we will prove Oppermann's conjecture [4], Brocard's conjecture [5], and Andrica's conjecture [6].

Definition: $\Gamma_{a\geq p>b}\{n\}$ denotes the prime number decomposition operator. It is the product of the prime numbers in the decomposition of a positive integer n or a positive integer expression. In this operator, p is a prime number, a and b are real numbers, and $n \geq a \geq p > b \geq 1$.
It has some properties:

It is always true that $\Gamma_{a\geq p\geq b}\{n\} \geq 1$. **— (1.1)**

If no prime number in $\Gamma_{a\geq p>b}\{n\}$, then $\Gamma_{a\geq p>b}\{n\} = 1$, or vice versa, if $\Gamma_{a\geq p>b}\{n\} = 1$, then no prime number in $\Gamma_{a\geq p>b}\{n\}$ as in $\Gamma_{12\geq p>4}\{12\} = 11^0 \cdot 7^0 \cdot 5^0 = 1$. **— (1.2)**

If there is at least one prime number in $\Gamma_{a\geq p>b}\{n\}$, then $\Gamma_{a\geq p>b}\{n\} > 1$, or vice versa, if $\Gamma_{a\geq p>b}\{n\} > 1$, then there is at least one prime number in $\Gamma_{a\geq p>b}\{n\}$ as in $\Gamma_{4\geq p>2}\{12\} = 3 > 1$. **— (1.3)**

Similar to Paul Erdős' paper [2], we define *R(p)* by the inequalities $p^{R(p)} \leq \lambda n < p^{R(p)+1}$, and determine the $p\text{-}adic\ valuation$ of $\binom{\lambda n}{n}$.

$$v_p\left(\binom{\lambda n}{n}\right) = v_p((\lambda n)!) - v_p(((\lambda-1)n)!) - v_p(n!) = \sum_{i=1}^{R(p)}\left(\left\lfloor\frac{\lambda n}{p^i}\right\rfloor - \left\lfloor\frac{(\lambda-1)n}{p^i}\right\rfloor - \left\lfloor\frac{n}{p^i}\right\rfloor\right) \leq R(p)$$

because for any real numbers a and b, the expression of $\lfloor a+b\rfloor - \lfloor a\rfloor - \lfloor b\rfloor$ is 0 or 1.

Thus, if p divides $\binom{\lambda n}{n}$, then $v_p\left(\binom{\lambda n}{n}\right) \leq R(p) \leq \log_p(\lambda n)$, or $p^{v_p\left(\binom{\lambda n}{n}\right)} \leq p^{R(p)} \leq \lambda n$ **— (1.4)**

And if $\lambda n \geq p > \lfloor\sqrt{\lambda n}\rfloor$, then $0 \leq v_p\left(\binom{\lambda n}{n}\right) \leq R(p) \leq 1$ **— (1.5)**

Let $\pi(n)$ be the number of distinct prime numbers less than or equal to n. For the first six sequential natural numbers, there are three prime numbers 2, 3, and 5. For counting any successive set of six sequential natural numbers, there are at most two prime numbers added, $p \equiv 1$ (MOD 6) and $p \equiv 5$ (MOD 6). Thus, $\pi(n) \le \left\lfloor \frac{n}{3} \right\rfloor + 2 \le \frac{n}{3} + 2$. Since some of $n \equiv 1$ (MOD 6) and $n \equiv 5$ (MOD 6) are not prime numbers, as the number counts increase, $\pi(n)$ reduces from $\left\lfloor \frac{n}{3} \right\rfloor + 2$.

For $n \ge 24$, $\pi(n) \le \left\lfloor \frac{n}{3} \right\rfloor + 1 \le \frac{n}{3} + 1$ **— (1.6)**

From the prime number decomposition,

when $n > \lfloor \sqrt{\lambda n} \rfloor$, $\binom{\lambda n}{n} = \Gamma_{\lambda n \ge p > n}\{\frac{(\lambda n)!}{n! \cdot ((\lambda-1)n)!}\} \cdot \Gamma_{n \ge p > \lfloor \sqrt{\lambda n} \rfloor}\{\frac{(\lambda n)!}{n! \cdot ((\lambda-1)n)!}\} \cdot \Gamma_{\lfloor \sqrt{\lambda n} \rfloor \ge p}\{\frac{(\lambda n)!}{n! \cdot ((\lambda-1)n)!}\}$

when $n \le \lfloor \sqrt{\lambda n} \rfloor$, $\binom{\lambda n}{n} \le \Gamma_{\lambda n \ge p > n}\{\frac{(\lambda n)!}{n! \cdot ((\lambda-1)n)!}\} \cdot \Gamma_{\lfloor \sqrt{\lambda n} \rfloor \ge p}\{\frac{(\lambda n)!}{n! \cdot ((\lambda-1)n)!}\}$

$\Gamma_{\lambda n \ge p > n}\{\frac{(\lambda n)!}{n! \cdot ((\lambda-1)n)!}\} = \Gamma_{\lambda n \ge p > n}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\}$ since all prime numbers in $n!$ do not appear in the range of $\lambda n \ge p > n$.

Referring to **(1.5)**, $\Gamma_{n \ge p > \lfloor \sqrt{\lambda n} \rfloor}\{\frac{(\lambda n)!}{n! \cdot ((\lambda-1)n)!}\} < \prod_{n \ge p} p$. It has been proved [7] that for $n \ge 3$, $\prod_{n \ge p} p \le 2^{2n-3}$. Thus, $\Gamma_{n \ge p > \lfloor \sqrt{\lambda n} \rfloor}\{\frac{(\lambda n)!}{n! \cdot ((\lambda-1)n)!}\} \le \prod_{n \ge p} p \le 2^{2n-3}$.

Referred to **(1.4)** and **(1.6)**, $\Gamma_{\lfloor \sqrt{\lambda n} \rfloor \ge p}\{\frac{(\lambda n)!}{n! \cdot ((\lambda-1)n)!}\} \le (\lambda n)^{\frac{\sqrt{\lambda n}}{3}+1}$ when $\lfloor \sqrt{\lambda n} \rfloor \ge 24$.

Thus, for $n \ge 3$ and $\lfloor \sqrt{\lambda n} \rfloor \ge 24$, $\binom{\lambda n}{n} \le \Gamma_{\lambda n \ge p > n}\{\frac{(\lambda n)!}{n! \cdot ((\lambda-1)n)!}\} \cdot 2^{2n-3} \cdot (\lambda n)^{\frac{\sqrt{\lambda n}}{3}+1}$ **— (1.7)**

2. Lemmas

Lemma 1: If a real number $x \ge 3$, then $\frac{2(2x-1)}{x-1} > \left(\frac{x}{x-1}\right)^x$ **— (2.1)**

Proof:

Let $f_1(x) = \frac{2(2x-1)}{x-1}$, then $f_1'(x) = \frac{2(x-1)(2x-1)' - 2(2x-1)(x-1)'}{(x-1)^2} = \frac{-2}{(x-1)^2} < 0$.

Thus $f_1(x)$ is a strictly decreasing function for $x > 1$.

Since $f_1(3) = 5$ and $\lim_{x \to \infty} f_1(x) = 4$,

thus, $5 \ge f_1(x) = \frac{2(2x-1)}{x-1} \ge 4$ for $x \ge 3$. **— (2.1.1)**

Let $f_2(x) = \left(\frac{x}{x-1}\right)^x$, then $f_2'(x) = \left(\left(\frac{x}{x-1}\right)^x\right)' = \left(e^{x \cdot \ln\frac{x}{x-1}}\right)' = e^{x \cdot \ln\frac{x}{x-1}} \cdot \left(x \cdot \ln\frac{x}{x-1}\right)'$

$f_2{}'(x) = \left(\frac{x}{x-1}\right)^x \cdot \left(\ln\frac{x}{x-1} + x \cdot \left(\ln\frac{x}{x-1}\right)'\right) = \left(\frac{x}{x-1}\right)^x \cdot \left(\ln\frac{x}{x-1} + x \cdot \frac{x-1}{x} \cdot \frac{x-1-x}{(x-1)^2}\right)$

$f_2{}'(x) = \left(\frac{x}{x-1}\right)^x \cdot \left(\ln\frac{x}{x-1} - \frac{1}{x-1}\right)$ **— (2.1.2)**

In **(2.1.2)**, $\frac{1}{x-1} = \frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3} + \frac{1}{x^4} + \frac{1}{x^5} + \frac{1}{x^6} + \cdots$ **— (2.1.3)**

Using the formula: $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \cdots$, we have

$\ln\frac{x}{x-1} = \ln\frac{1}{1+\frac{-1}{x}} = -\ln\left(1 + \frac{-1}{x}\right) = \frac{1}{x} + \frac{1}{2x^2} + \frac{1}{3x^3} + \frac{1}{4x^4} + \frac{1}{5x^5} + \frac{1}{6x^6} + \cdots$ **— (2.1.4)**

Thus for $x \geq 3$, $\ln\frac{x}{x-1} - \frac{1}{x-1} < 0$ **— (2.1.5)**

Since $\left(\frac{x}{x-1}\right)^x$ is a positive number for $x \geq 3$, $f_2{}'(x) = \left(\frac{x}{x-1}\right)^x \cdot \left(\ln\frac{x}{x-1} - \frac{1}{x-1}\right) < 0$. **— (2.1.6)**

Thus $f_2(x)$ is a strictly deceasing function for $x \geq 3$.

Since $f_2(3) = 3.375$ and $\lim_{x\to\infty} f_2(x) = e \approx 2.718$,

thus for $x \geq 3$, $3.375 \geq f_2(x) = \left(\frac{x}{x-1}\right)^x \geq e \approx 2.718$ **— (2.1.7)**

Since for $x \geq 3$, $f_1(x)$ has a lower bound of 4 and $f_2(x)$ has an upper bound of 3.375,

$f_1(x) = \frac{2(2x-1)}{x-1} > f_2(x) = \left(\frac{x}{x-1}\right)^x$ is proven. **— (2.1.8)**

Lemma 2: For $n \geq 2$ and $\lambda \geq 3$, $\binom{\lambda n}{n} > \frac{\lambda^{\lambda n-\lambda+1}}{n(\lambda-1)^{(\lambda-1)n-\lambda+1}}$ **— (2.2)**

Proof:

When $\lambda \geq 3$ and $n = 2$,

$\binom{\lambda n}{n} = \binom{2\lambda}{2} = \frac{2\lambda(2\lambda-1)(2\lambda-2)!}{2(2\lambda-2)!} = \lambda(2\lambda-1)$ **— (2.2.1)**

$\frac{\lambda^{\lambda n-\lambda+1}}{n(\lambda-1)^{(\lambda-1)n-\lambda+1}} = \frac{\lambda^{2\lambda-\lambda+1}}{2(\lambda-1)^{2(\lambda-1)-\lambda+1}} = \frac{\lambda(\lambda-1)}{2} \cdot \left(\frac{\lambda}{\lambda-1}\right)^\lambda$ **— (2.2.2)**

In **(2.1)** when $x = \lambda \geq 3$, we have $\frac{2(2\lambda-1)}{\lambda-1} > \left(\frac{\lambda}{\lambda-1}\right)^\lambda$ **— (2.2.3)**

Since $\frac{\lambda(\lambda-1)}{2}$ is a positive number for $\lambda \geq 3$, referring to **(2.2.1)** and **(2.2.2)**, when $\frac{\lambda(\lambda-1)}{2}$ multiplies to both sides of **(2.2.3)**, we have

$\left(\frac{\lambda(\lambda-1)}{2}\right)\left(\frac{2(2\lambda-1)}{\lambda-1}\right) = \lambda(2\lambda-1) = \binom{\lambda n}{n} > \left(\frac{\lambda(\lambda-1)}{2}\right)\left(\frac{\lambda}{\lambda-1}\right)^\lambda = \frac{\lambda^{\lambda n-\lambda+1}}{n(\lambda-1)^{(\lambda-1)n-\lambda+1}}$

Thus, $\binom{\lambda n}{n} > \frac{\lambda^{\lambda n-\lambda+1}}{n(\lambda-1)^{(\lambda-1)n-\lambda+1}}$ when $\lambda \geq 3$ and $n = 2$. **— (2.2.4)**

By induction on n, when $\lambda \geq 3$, if $\binom{\lambda n}{n} > \frac{\lambda^{\lambda n-\lambda+1}}{n(\lambda-1)^{(\lambda-1)n-\lambda+1}}$ is true for n, then for n+1, we have

$$\binom{\lambda(n+1)}{n+1} = \binom{\lambda n+\lambda}{n+1} = \frac{(\lambda n+\lambda)(\lambda n+\lambda-1)\cdots(\lambda n+2)(\lambda n+1)}{(\lambda n+\lambda-n-1)(\lambda n+\lambda-n-2)\cdots(\lambda n-n+1)(n+1)} \cdot \binom{\lambda n}{n}$$

$$\binom{\lambda(n+1)}{n+1} > \frac{(\lambda n+\lambda)(\lambda n+\lambda-1)\cdots(\lambda n+2)(\lambda n+1)}{(\lambda n+\lambda-n-1)(\lambda n+\lambda-n-2)\cdots(\lambda n-n+1)(n+1)} \cdot \frac{\lambda^{\lambda n-\lambda+1}}{n(\lambda-1)^{(\lambda-1)n-\lambda+1}}$$

$$\binom{\lambda(n+1)}{n+1} > \frac{(\lambda n+\lambda)(\lambda n+\lambda-1)\cdots(\lambda n+2)}{(\lambda n+\lambda-n-1)(\lambda n+\lambda-n-2)\cdots(\lambda n-n+1)} \cdot \frac{\lambda n+1}{n} \cdot \frac{1}{(n+1)} \cdot \frac{\lambda^{\lambda n-\lambda+1}}{(\lambda-1)^{(\lambda-1)n-\lambda+1}}$$

Notice $\frac{\lambda n+1}{n} > \lambda$, and $\frac{(\lambda n+\lambda)(\lambda n+\lambda-1)\cdots(\lambda n+2)}{(\lambda n+\lambda-n-1)(\lambda n+\lambda-n-2)\cdots(\lambda n-n+1)} > (\frac{\lambda}{\lambda-1})^{(\lambda-1)}$

because $\frac{\lambda n+\lambda}{\lambda n+\lambda-n-1} = \frac{\lambda}{\lambda-1}$; $\frac{\lambda n+\lambda-1}{\lambda n+\lambda-n-2} > \frac{\lambda}{\lambda-1}$; $\cdots$ $\frac{\lambda n+2}{\lambda n-n+1} > \frac{\lambda}{\lambda-1}$.

Thus $\binom{\lambda(n+1)}{n+1} > \frac{\lambda^{\lambda-1}}{(\lambda-1)^{(\lambda-1)}} \cdot \frac{\lambda}{1} \cdot \frac{1}{(n+1)} \cdot \frac{\lambda^{\lambda n-\lambda+1}}{(\lambda-1)^{(\lambda-1)n-\lambda+1}} = \frac{\lambda^{\lambda(n+1)-\lambda+1}}{(n+1)(\lambda-1)^{(\lambda-1)(n+1)-\lambda+1}}$ **— (2.2.5)**

From **(2.2.4)** and **(2.2.5)**, we have for $n \geq 2$ and $\lambda \geq 3$, $\binom{\lambda n}{n} > \frac{\lambda^{\lambda n-\lambda+1}}{n(\lambda-1)^{(\lambda-1)n-\lambda+1}}$

Thus, **Lemma 2** is proven.

3. A Prime Number between $(\lambda-1)n$ and λn

Proposition:

For $n \geq (\lambda-2) \geq 24$, there exists at least a prime number p such that $(\lambda-1)n < p \leq \lambda n$. **— (3.1)**

Proof:

When $n \geq (\lambda-2)$ in **(1.7)**, since $n + 1 = \sqrt{(n+2)n+1} > \sqrt{\lambda n}$, $\lfloor\sqrt{\lambda n}\rfloor$ is an integer at most one less than $(n + 1)$. Thus, $n \geq \lfloor\sqrt{\lambda n}\rfloor$ and **(1.7)** is valid for $n \geq (\lambda-2) \geq 24$.

From **(1.7)** and **(2.2)**, when $n \geq (\lambda-2) \geq 24$, we have

$\frac{\lambda^{\lambda n-\lambda+1}}{n(\lambda-1)^{(\lambda-1)n-\lambda+1}} < \binom{\lambda n}{n} \leq \Gamma_{\lambda n\geq p>n}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\} \cdot 2^{2n-3} \cdot (\lambda n)^{\frac{\sqrt{\lambda n}}{3}+1}$.

$\frac{\lambda^{\lambda n-\lambda+1}}{n(\lambda-1)^{(\lambda-1)n-\lambda+1}} < \Gamma_{\lambda n\geq p>n}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\} \cdot 2^{2n-3} \cdot (\lambda n)^{\frac{\sqrt{\lambda n}}{3}+1}$. Since $(\lambda n)^{\frac{\sqrt{\lambda n}}{3}+1} > 1$ and $2^{2n-3} > 1$,

$$\Gamma_{\lambda n\geq p>n}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\} > \frac{\lambda^{\lambda n-\lambda+1}}{(\lambda n)^{\frac{\sqrt{\lambda n}}{3}+1}\cdot 2^{2n-3}\cdot n(\lambda-1)^{(\lambda-1)n-\lambda+1}} = \frac{2\lambda^2\cdot\left(\left(\frac{\lambda-1}{4}\right)\cdot\left(\frac{\lambda}{\lambda-1}\right)^{\lambda}\right)^{(n-1)}}{(\lambda n)^{\frac{\sqrt{\lambda n}}{3}+2}}$$

Referring to (2.1.7), $\left(\frac{\lambda}{\lambda-1}\right)^{\lambda} > e$,

thus $\Gamma_{\lambda n\geq p>n}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\} > \frac{2\lambda^2\cdot\left(\left(\frac{\lambda-1}{4}\right)\cdot\left(\frac{\lambda}{\lambda-1}\right)^{\lambda}\right)^{(n-1)}}{(\lambda n)^{\frac{\sqrt{\lambda n}}{3}+2}} > \frac{2\lambda^2\cdot\left(\left(\frac{\lambda-1}{4}\right)\cdot e\right)^{(n-1)}}{(\lambda n)^{\frac{\sqrt{\lambda n}}{3}+2}} = f_3(n,\lambda)$ **— (3.2)**

Let $x = (y-2) \geq 24$, where both x and y are positive real numbers, and

$f_3(x,y) = \frac{2y^2\cdot\left(\left(\frac{y-1}{4}\right)\cdot e\right)^{(x-1)}}{(xy)^{\frac{\sqrt{xy}}{3}+2}} = \frac{2(x+2)^2\cdot\left(\left(\frac{x+1}{4}\right)\cdot e\right)^{(x-1)}}{(x\cdot(x+2))^{\frac{\sqrt{x\cdot(x+2)}}{3}+2}} > f_4(x) = \frac{2(x+2)^2\cdot\left(\left(\frac{x+1}{4}\right)\cdot e\right)^{(x-1)}}{(x\cdot(x+2))^{\frac{x+1}{3}+2}}$ **— (3.3)**

$f_4{}'(x) = f_4(x)\cdot\left(\frac{2}{x+2} + ln\left(\frac{x+1}{4}\right) + \frac{4}{3} - \frac{2}{x+1} - \frac{1}{3}ln(x\cdot(x+2)) - \frac{7}{3x} - \frac{5}{3(x+2)}\right) = f_4(x)\cdot f_5(x)$

where $f_5(x) = \frac{2}{x+2} + ln\left(\frac{x+1}{4}\right) + \frac{4}{3} - \frac{2}{x+1} - \frac{1}{3}ln(x\cdot(x+2)) - \frac{7}{3x} - \frac{5}{3(x+2)}$

$f_5{}'(x) = \frac{4x+6}{(x+1)^2\cdot(x+2)^2} + \frac{x^2+2x-2}{3x(x+1)(x+2)} + \frac{7}{3x^2} + \frac{5}{3(x+2)^2} > 0$

Thus, $f_5(x)$ is a strictly increasing function for $x \geq 24$.

When $x = 24$, $f_5(x) = \frac{2}{24+2} + ln\left(\frac{24+1}{4}\right) + \frac{4}{3} - \frac{2}{24+1} - \frac{1}{3}ln(24) - \frac{1}{3}ln(24+2) - \frac{7}{72} - \frac{5}{78} > 0$,

thus, for $x \geq 24$, $f_5(x) > 0$.

Then, $f_4{}'(x) = f_4(x)\cdot f_5(x) > 0$ and $f_4(x)$ is a strictly increasing function for $x \geq 24$.

When $x = 24$, $f_4(x) = \frac{2\cdot(26)^2\cdot\left(\frac{25}{4}\right)^{23}\cdot e^{23}}{(24\cdot 26)^{\frac{24+1}{3}+2}} > \frac{2.6606\text{E}+31}{7.6484\text{E}+28} > 1$, thus for $x \geq 24$, $f_4(x) > 1$.

Thus, when $x = (y-2) \geq 24$, $f_3(x,y) = \frac{2y^2\cdot\left(\left(\frac{y-1}{4}\right)\cdot e\right)^{(x-1)}}{(xy)^{\frac{\sqrt{xy}}{3}+2}} > f_4(x) > 1$. **— (3.4)**

$\frac{\partial f_3(x,y)}{\partial x} = f_3(x,y)\cdot\left(ln\left(\frac{y-1}{4}\right) + 1 - \frac{\sqrt{y}}{6\sqrt{x}}\cdot ln(yx) - \frac{\sqrt{y}}{3\sqrt{x}} - \frac{2}{x}\right) = f_3(x,y)\cdot f_6(x,y)$ **— (3.5)**

where $f_6(x,y) = ln\left(\frac{y-1}{4}\right) + 1 - \frac{\sqrt{y}}{6\sqrt{x}}\cdot ln(yx) - \frac{\sqrt{y}}{3\sqrt{x}} - \frac{2}{x}$

$\frac{\partial f_6(x,y)}{\partial x} = \frac{\sqrt{y}}{12x\sqrt{x}}\cdot ln(y) + \frac{\sqrt{y}}{12x\sqrt{x}}\cdot ln(x) + \frac{\sqrt{y}}{6x\sqrt{x}} + \frac{\sqrt{y}}{6x\sqrt{x}} + \frac{2}{x^2} > 0$

Thus, $f_6(x,y)$ is a strictly increasing function for $x \geq y-2 \geq 24$.

When $x = (y-2) \geq 24$, $f_6(x,y) \geq ln\left(\frac{26-1}{4}\right) + 1 - \frac{\sqrt{26}}{6\sqrt{24}}\cdot ln(24\cdot 26) - \frac{\sqrt{26}}{3\sqrt{24}} - \frac{2}{24} > 0$.

Thus, $\frac{\partial f_3(x,y)}{\partial x} = f_3(x,y)\cdot f_6(x,y) > 0$. $f_3(x,y)$ is a strictly increasing function for $x \geq y-2 \geq 24$.

When $x = y - 2 \ge 24$, $f_3(x,y) \ge \frac{2\cdot 26^2\cdot\left(\left(\frac{26-1}{4}\right)\cdot e\right)^{(24-1)}}{(24\cdot 26)^{\frac{\sqrt{24\cdot 26}}{3}+2}} > \frac{2.6606\text{E}+31}{7.3274\text{E}+28} > 1.$

Thus, for $x \ge y - 2 \ge 24$, $f_3(x,y) = \frac{2y^2\cdot\left(\left(\frac{y-1}{4}\right)\cdot e\right)^{(x-1)}}{(xy)^{\frac{\sqrt{xy}}{3}+2}} > 1.$ **— (3.6)**

Let $n = \lfloor x\rfloor$ and $\lambda = \lfloor y\rfloor$, referring to **(3.2)**, when $n \ge (\lambda - 2) \ge 24$,

we have $\Gamma_{\lambda n\ge p>n}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\} > \frac{2\lambda^2\cdot\left(\left(\frac{\lambda-1}{4}\right)\cdot\left(\frac{\lambda}{\lambda-1}\right)^{\lambda}\right)^{(n-1)}}{(\lambda n)^{\frac{\sqrt{\lambda n}}{3}+2}} > 1$ **— (3.7)**

In $\Gamma_{\lambda n\ge p>n}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\}$, $p \ge n+1 = \sqrt{n^2+2n+1} > \sqrt{(n+2)n} \ge \lfloor\sqrt{\lambda n}\rfloor$. Referring to **(1.5)**, we have $0 \le v_p\left(\Gamma_{\lambda n\ge p>n}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\}\right) \le R(p) \le 1$.

$\Gamma_{\lambda n\ge p>n}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\} =$

$= \Gamma_{\lambda n\ge p>(\lambda-1)n}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\}\cdot\prod_{i=1}^{i=\lambda-2}\left(\Gamma_{\frac{(\lambda-1)n}{i}\ge p>\frac{\lambda n}{i+1}}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\}\cdot\Gamma_{\frac{\lambda n}{i+1}\ge p>\frac{(\lambda-1)n}{i+1}}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\}\right)$

In $\prod_{i=1}^{\lambda-2}\left(\Gamma_{\frac{(\lambda-1)n}{i}\ge p>\frac{\lambda n}{i+1}}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\}\right)$, every distinct prime number p in this range in the numerator $(\lambda n)!$ has the form of $(i)!\cdot p^i$. It also has the same form of $(i)!\cdot p^i$ in the denominator $((\lambda-1)n)!$. Thus, referring to **(1.2)**, $\prod_{i=1}^{\lambda-2}\left(\Gamma_{\frac{(\lambda-1)n}{i}\ge p>\frac{\lambda n}{i+1}}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\}\right) = 1.$

Therefore, when $n \ge \lambda - 2 \ge 24$,

$\Gamma_{\lambda n\ge p>n}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\} = \Gamma_{\lambda n\ge p>(\lambda-1)n}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\}\cdot\prod_{i=1}^{i=\lambda-2}\left(\Gamma_{\frac{\lambda n}{i+1}\ge p>\frac{(\lambda-1)n}{i+1}}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\}\right) > 1.$

— (3.8)

From **(1.1)**, $\Gamma_{\lambda n\ge p>(\lambda-1)n}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\} \ge 1$ and $\prod_{i=1}^{i=\lambda-2}\left(\Gamma_{\frac{\lambda n}{i+1}\ge p>\frac{(\lambda-1)n}{i+1}}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\}\right) \ge 1$, and in **(3.8)** at last one of these two parts is greater than 1.

When $n \ge \lambda - 2 \ge 24$, if $\Gamma_{\lambda n\ge p>(\lambda-1)n}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\} > 1$, then referring to **(1.3)**, there exists at least a prime number p such that $(\lambda-1)n < p \le \lambda n$. **— (3.9)**

If $\prod_{i=1}^{i=\lambda-2}\left(\Gamma_{\frac{\lambda n}{i+1}\ge p>\frac{(\lambda-1)n}{i+1}}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\}\right) = 1$, then $\Gamma_{\lambda n\ge p>(\lambda-1)n}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\} > 1$. **— (3.10)**

If $\prod_{i=1}^{i=\lambda-2}\left(\Gamma_{\frac{\lambda n}{i+1}\ge p>\frac{(\lambda-1)n}{i+1}}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\}\right) > 1$, then at least one factor $\Gamma_{\frac{\lambda n}{i+1}\ge p>\frac{(\lambda-1)n}{i+1}}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\} > 1$.

When a factor $\Gamma_{\frac{\lambda n}{i+1}\ge p>\frac{(\lambda-1)n}{i+1}}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\} > 1$, let $y_{i+1} = \frac{n}{i+1}$, then $y_{i+1} \ge \frac{\lambda-2}{i+1} \ge \frac{24}{i+1}$. We have

$\Gamma_{\lambda y_{i+1} \geq p > (\lambda-1) y_{i+1}} \{\frac{(\lambda n)!}{((\lambda-1)n)!}\} > 1$. Thus, when $y_{i+1} \geq \frac{\lambda-2}{i+1} \geq \frac{24}{i+1}$, there exists at least a prime number p such that $(\lambda - 1) \cdot y_{i+1} < p \leq \lambda \cdot y_{i+1}$

Since $n > y_{i+1} > \frac{\lambda-2}{i+1} > \frac{24}{i+1}$, there exists at least a prime number p such that $(\lambda - 1)n < p \leq \lambda n$.

Thus, If $\prod_{i=1}^{i=\lambda-2} \left(\Gamma_{\frac{\lambda n}{i+1} \geq p > \frac{(\lambda-1)n}{i+1}} \{\frac{(\lambda n)!}{((\lambda-1)n)!}\} \right) > 1$, then $\Gamma_{\lambda n \geq p > (\lambda-1)n} \{\frac{(\lambda n)!}{((\lambda-1)n)!}\} > 1$. **— (3.11)**

Referring to **(3.7)**, **(3.9)**, **(3.10)**, and **(3.11)**, then $\Gamma_{\lambda n \geq p > (\lambda-1)n} \{\frac{(\lambda n)!}{((\lambda-1)n)!}\} > 1$ when $n \geq \lambda - 2 \geq 24$.

Referring to **(1.3)**, there exists at least a prime number p such that $(\lambda - 1)n < p \leq \lambda n$.

Thus, **(3.1)**, the **Proposition**, is proven. It becomes a theorem: **Theorem (3.1)**.

4. The Proof of Legendre's Conjecture

Legendre's Conjecture states that there is a prime number between n^2 and $(n + 1)^2$ for every positive integer n. **— (4.1)**

Proof:

Referring to **Theorem (3.1)**, for integers $j \geq k - 2 \geq 24$, there exists at least a prime number p such that $j(k - 1) < p \leq jk$. **— (4.2)**

When $k = j + 1 \geq 26$, then $j = k - 1 \geq 25$

Applying $k = j + 1$ into **(4.2)**, then $j^2 < p \leq j(j + 1) < (j + 1)^2$

Let $n = j \geq 25$, then we have $n^2 < p < (n + 1)^2$. **— (4.3)**

For $1 \leq n \leq 24$, we have a table, **Table 1**, that shows Legendre's conjecture valid. **— (4.4)**

Table 1: For $1 \leq n \leq 24$, there is a prime number between n^2 and $(n + 1)^2$.

n	1	2	3	4	5	6	7	8	9	10	11	12
n^2	1	4	9	16	25	36	49	64	81	100	121	144
p	3	5	11	19	29	41	53	67	83	103	127	149
$(n+1)^2$	4	9	16	25	36	49	64	81	100	121	144	169
n	13	14	15	16	17	18	19	20	21	22	23	24
n^2	169	196	225	256	289	324	361	400	441	484	529	576
p	173	199	229	263	307	331	373	409	449	491	541	587
$(n+1)^2$	196	225	256	289	324	361	400	441	484	529	576	625

Combining **(4.3)** and **(4.4)**, we have proven Legendre's conjecture.

Extension of Legendre's conjecture

There are at least two prime numbers, p_n and p_m , between j^2 and $(j+1)^2$ for every positive integer j such that $j^2 < p_n \le j(j+1)$ and $j(j+1) < p_m < (j+1)^2$ where p_n is the n^{th} prime number, p_m is the m^{th} prime number, and $m \ge n + 1$. **— (4.5)**

Proof:

Referring to **Theorem (3.1)**, for integers $j \ge k-2 \ge 24$, there exists at least a prime number p such that $j(k-1) < p \le jk$.

When $k-1 = j \ge 25$, then $j(k-1) = j^2 < p_n \le jk = j(j+1)$. Thus, there is at least a prime number p_n such that $j^2 < p_n \le j(j+1)$ when $j = k-1 \ge 25$.

When $j = k-2 \ge 25$, then $k = j + 2$. Thus, $j(k-1) = j(j+1) < p_m \le jk = j\,(j+2) < (j+1)^2$. Thus, there is at least another prime number p_m such that $j(j+1) < p_m < (j+1)^2$ when $j = k-2 \ge 25$.

Thus, when $j \ge 25$, there are at least two prime numbers p_n and p_m between j^2 and $(j+1)^2$ such that $j^2 < p_n \le j(j+1) < p_m < (j+1)^2$ where $m \ge n + 1$ for $p_m > p_n$. **— (4.6)**

For $1 \le j \le 24$, we have a table, **Table 2**, that shows **(4.5)** valid. **— (4.7)**

Table 2: For $1 \le j \le 24$, there are 2 prime numbers such that $j^2 < p_n \le j(j+1) < p_m < (j+1)^2$.

j	1	2	3	4	5	6	7	8	9	10	11	12
j^2	1	4	9	16	25	36	49	64	81	100	121	144
p_n	2	5	11	19	29	41	53	67	83	103	127	149
$j(j+1)$	2	6	12	20	30	42	56	72	90	110	132	156
p_m	3	7	13	23	31	43	59	73	97	113	137	163
$(j+1)^2$	4	9	16	25	36	49	64	81	100	121	144	169
j	13	14	15	16	17	18	19	20	21	22	23	24
j^2	169	196	225	256	289	324	361	400	441	484	529	576
p_n	173	199	229	263	393	331	373	409	449	491	541	587
$j(j+1)$	182	210	240	272	306	342	380	420	462	506	552	600
p_m	191	211	251	277	311	349	389	431	467	521	557	613
$(j+1)^2$	196	225	256	289	324	361	400	441	484	529	576	625

Combining **(4.6)** and **(4.7)**, we have proven **(4.5)**. It becomes a theorem: **Theorem (4.5)**.

5. The Proofs of Three Related Conjectures

Oppermann's conjecture was proposed by Ludvig Oppermann [4] in March 1877. It states that for every integer $x > 1$, there is at least one prime number between $x(x-1)$ and x^2, and at least another prime between x^2 and $x(x+1)$. **— (5.1)**

Proof:

Theorem (4.5) states there are at least two prime numbers, p_n and p_m, between j^2 and $(j+1)^2$ for every positive integer j such that $j^2 < p_n \leq j(j+1)$ and $j(j+1) < p_m < (j+1)^2$ where $m \geq n+1$.

$j(j+1)$ is a composite number except $j = 1$. Since $j^2 < p_n \leq j(j+1)$ is valid for every positive integer j, when we replace j with $j+1$, we have $(j+1)^2 < p_v < (j+1)(j+2)$.

Thus, we have $j(j+1) < p_m < (j+1)^2 < p_v < (j+1)(j+2)$. **— (5.2)**

When $x > 1$, then $(x-1) \geq 1$. Substitute j with $(x-1)$ in **(5.2)**, we have

$x(x-1) < p_m < x^2 < p_v < x(x+1)$ **— (5.3)**

Thus, we have proven Oppermann's conjecture.

Brocard's conjecture is named after Henri Brocard [5]. It states that there are at least 4 prime numbers between $(p_n)^2$ and $(p_{n+1})^2$, where p_n is the n^{th} prime number, for every $n > 1$. **— (5.4)**

Proof:

Theorem (4.5) states there are at least two prime numbers, p_n and p_m, between j^2 and $(j+1)^2$ for every positive integer j such that $j^2 < p_n \leq j(j+1)$ and $j(j+1) < p_m < (j+1)^2$ where $m \geq n+1$. When $j > 1$, $j(j+1)$ is a composite number. Then **Theorem (4.5)** can be written as $j^2 < p_n < j(j+1)$ and $j(j+1) < p_m < (j+1)^2$.

In the series of prime numbers: p_1=2, p_2=3, p_3=5, p_4=7, p_5=11... all prime numbers except p_1 are odd numbers. Their gaps are two or more. Thus when $n > 1$, $(p_{n+1} - p_n) \geq 2$.

Thus, we have $p_n < (p_n + 1) < p_{n+1}$ when $n > 1$. **— (5.5)**

Applying **Theorem (4.5)** to **(5.5)**, when $n > 1$, we have at least two prime numbers p_{m1}, p_{m2} in between $(p_n)^2$ and $(p_n+1)^2$ such that $(p_n)^2 < p_{m1} < p_n(p_n+1) < p_{m2} < (p_n+1)^2$, and at least two more prime numbers p_{m3}, p_{m4} in between $(p_n+1)^2$ and $(p_{n+1})^2$ such that $(p_n+1)^2 < p_{m3} < p_{n+1}(p_n+1) < p_{m4} < (p_{n+1})^2$.

Thus, there are at least 4 prime numbers between $(p_n)^2$ and $(p_{n+1})^2$ for $n > 1$ such that

$(p_n)^2 < p_{m1} < p_n(p_n+1) < p_{m2} < (p_n+1)^2 < p_{m3} < p_{n+1}(p_n+1) < p_{m4} < (p_{n+1})^2$ **— (5.6)**

Thus, Brocard's conjecture is proven.

Andrica's conjecture is named after Dorin Andrica [6]. It is a conjecture regarding the gaps between prime numbers. The conjecture states that the inequality $\sqrt{p_{n+1}} - \sqrt{p_n} < 1$ holds for all n where p_n is the n^{th} prime number. If $g_n = p_{n+1} - p_n$ denotes the n^{th} prime gap, then Andrica's conjecture can also be rewritten as $g_n < 2\sqrt{p_n} + 1$. **— (5.7)**

Proof:

From **Theorem (4.5)**, for every positive integer j, there are at least two prime numbers p_n and p_m between j^2 and $(j+1)^2$ such that $j^2 < p_n \leq j(j+1) < p_m < (j+1)^2$ where $m \geq n+1$.
Since $m \geq n+1$, we have $p_m \geq p_{n+1}$.

Thus, we have $j^2 < p_n$. **— (5.8)**

And $p_{n+1} \leq p_m < (j+1)^2$. **— (5.9)**

Since j, p_n , p_{n+1} and $(j+1)$ are positive integers,
$j < \sqrt{p_n}$ **— (5.10)**

And $\sqrt{p_{n+1}} < j + 1$ **— (5.11)**

Applying **(5.10)** to **(5.11)**, we have $\sqrt{p_{n+1}} < \sqrt{p_n} + 1$. **— (5.12)**
Thus, $\sqrt{p_{n+1}} - \sqrt{p_n} < 1$ holds for all n since in **Theorem (4.5)**, j holds for all positive integers.

Using the prime gap to prove the conjecture, from **(5.8)** and **(5.9)**, we have
$g_n = p_{n+1} - p_n < (j+1)^2 - j^2 = 2j + 1$. From **(5.10)**, $j < \sqrt{p_n}$.
Thus, $g_n = p_{n+1} - p_n < 2\sqrt{p_n} + 1$. **— (5.13)**

Thus, Andrica's conjecture is proven.

6. References

[1] *Wikipedia,* https://en.wikipedia.org/wiki/Legendre%27s_conjecture

[2] P. Erdős, *Beweis eines Satzes von Tschebyschef*, Acta Sci. Math. (Szeged) **5** (1930-1932), 194-198

[3] M. Aigner and G. M. Ziegler, *Proofs from THE BOOK (4th ed.), Chapter 2*, Springer, 2010.

[4] *Wikipedia,* https://en.wikipedia.org/wiki/Oppermann%27s_conjecture

[5] *Wikipedia,* https://en.wikipedia.org/wiki/Brocard%27s_conjecture

[6] *Wikipedia,* https://en.wikipedia.org/wiki/Andrica%27s_conjecture

[7] Wikipedia, https://en.wikipedia.org/wiki/Proof_of_Bertrand%27s_postulate, Lemma 4.

7. Appendix

Q & A

Q: Why does it need to define the prime number decomposition operator?
A: The prime number decomposition operator $\Gamma_{a\geq p>b}\{n\}$ is needed because it has the properties that if $\Gamma_{a\geq p>b}\{n\}$ = 1, then there is no prime number less than or equal to a but greater than b; and that if $\Gamma_{a\geq p>b}\{n\}$ > 1, then there exists at least one prime number less than or equal to a but greater than b. In this operator, n is an integer, p is a prime number, a and b are real numbers, and $n \geq a \geq p > b \geq 1$. Thus, one can determine immediately whether a prime number exists in a certain range with this operator.

Q: What is the logic in the proof of $\Gamma_{\lambda n\geq p>n}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\} > 1$ for $n \geq (\lambda - 2) \geq 24$?
A: First, to prove $\Gamma_{\lambda n\geq p>n}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\} > f_3(n,\lambda)$ for $n \geq (\lambda - 2) \geq 24$. Then to replace $f_3(n,\lambda)$ with $f_3(x,y)$ to make a continuous function.
Second, to prove that if $x = (y - 2) \geq 24$, then $f_3(x,y) > 1$.
Third, to prove if $x > (y - 2) \geq 24$, then $f_3(x,y) > 1$ by showing $\frac{\partial f_3(x,y)}{\partial x} > 0$. Thus $f_3(x,y) > 1$ for $x \geq (y - 2) \geq 24$.
Forth, let $n = \lfloor x \rfloor$ and $\lambda = \lfloor y \rfloor$ in $f_3(x,y)$, thus, to prove $f_3(n,\lambda) > 1$ for $n \geq (\lambda - 2) \geq 24$ also $\Gamma_{\lambda n\geq p>n}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\} > f_3(n,\lambda) > 1$.
The send and third steps are similar to integer conduction: major premise, to prove that $f_3(n,\lambda) > 1$ for $n = (\lambda - 2) \geq 24$; minor premise, to prove that if $f_3(n,\lambda) > 1$, then $f_3((n+1),\lambda) > 1$; conclusion, thus $f_3(n,\lambda) > 1$ for $n \geq (\lambda - 2) \geq 24$.