Prime Gap Limit by Prime Triangles.

By V.L.M. Preemen

Introduction.

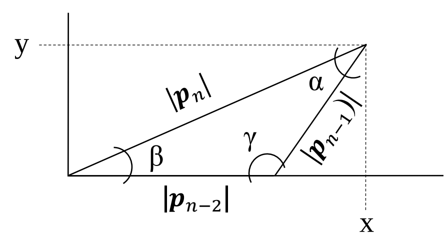
A method is presented to predict the location of the next prime based upon the 2 preceding prime numbers. The method is constructed with: triangles created from 3 succeeding prime numbers. The error difference found in the prediction correlates with the offset to a balanced prime number. A relation is found, the error difference converges to 0 giving a relation between the prime gap and prime numbers.

$$\lim_{n\to\infty}\left[-\frac{3{g_n}^2}{p_n}+\frac{12{g_n}^3}{{p_n}^2}-\frac{57{g_n}^4}{{p_n}^3}+\frac{300{g_n}^5}{{p_n}^4}-\frac{1686{g_n}^6}{{p_n}^5}+\mathcal{O}\left(\frac{1}{p^6}\right)\right]=0$$

1. "Fibonacci" Prime Vector triangles.

Based on an internet topic on [1] the question was if there are patterns based on triangles with the side's length based on primes. Triangles where constructed as a vector summation of prime numbers:

$$\boldsymbol{p}_n=\boldsymbol{p}_{n-2}+\boldsymbol{p}_{n-1} \tag{1}$$



The coordinate x, y is determined by applying the law of cosines.

$$\alpha=cos^{-1}\left(\frac{{p_{n-1}}^2+{p_n}^2-{p_{n-2}}^2}{2\cdot p_{n-1}\cdot\ p_n}\right)$$

$$\beta=cos^{-1}\left(\frac{{p_{n-2}}^2+{p_n}^2-{p_{n-1}}^2}{2\cdot p_{n-2}\cdot\ p_n}\right)$$

$$\gamma=cos^{-1}\left(\frac{{p_{n-2}}^2+{p_{n-1}}^2-{p_n}^2}{2\cdot p_{n-2}\cdot\ p_{n-1}}\right) \tag{2}$$

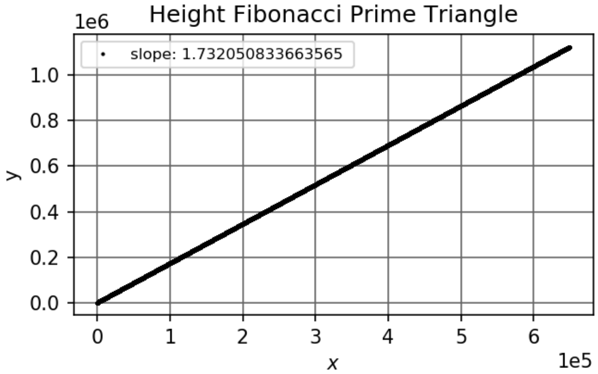
$$x=p_{n-2}+p_{n-1}\cdot cos(\pi-\gamma)$$

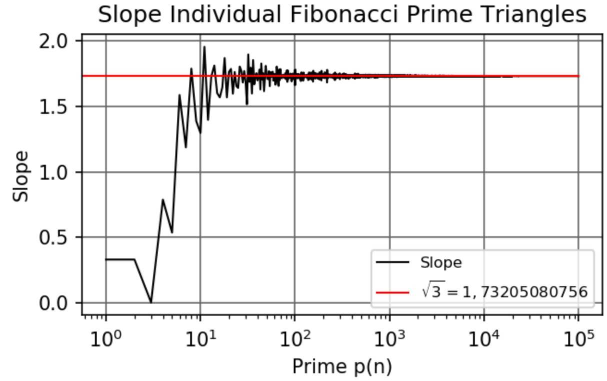
$$y=p_n\cdot cos\left(\frac{\pi}{2}-\beta\right)$$

The length of vector p_n can be calculated with [2]:

$$p_n=p_{n-2}\cdot\cos(\beta)+\sqrt{{p_{n-2}}^2\cdot cos^2(\beta)-{p_{n-2}}^2+{p_{n-1}}^2} \tag{3}$$

The constructed triangles are from here on called: "Fibonacci" Prime Vector triangles while just like the Fibonacci series the sum is taken from the two predecessors. Next analysis is performed from the first 100000 primes the coordinates: x and y are determined.





It is observed that:

- On large scale the "Fibonacci" Prime Vector triangle height seems to grow linear.
- The slope y = 1,7320508022x of the line is remarkable close to: sqrt(3), 1,73205080756888.
- Remarkable: 2+3=5 is the only observed vector set where angular component is zero (no triangle but line).

The following questions arise:

- Do all "Fibonacci" Prime Vector triangles exist, are there combinations what are not possible?
- Where does the slope come from?

2. Extreme "Fibonacci" Prime Vector triangles.

Vector set 2+3=5 is the only observed triangle with slope of 0. Do there exist more of these triangles with slope 0? To answer this question, the parity of the prime numbers is used. Prime number 2 is the only even prime so: 2(even)+3(odd)=5(odd), for all others parity does not match: all other primes are odd, there exists only one triangle with slope 0.

Do there exist any "Fibonacci" Prime Vector triangles what are not possible? If so, the following expression should not hold:

$$p_{n-2} + p_{n-1} > p_n \tag{4}$$

The sum (total length) of the preceding primes should be larger than the length of the vector p_n. The first question is: what is the biggest possible triangle? This should be a triangle where the preceding prime is a maximum. This occurs for twin prime numbers. Expression (4) will then become:

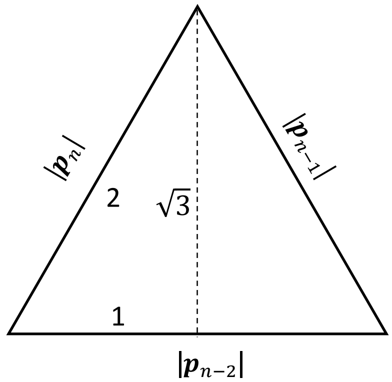
$$p_n < 2p_{n-1} - 2 \tag{5}$$

Expression (5) is known as: Bertrand–Chebyshev theorem [3]. This way to conclude that all "Fibonacci" Prime Vector triangles exist.

It is remarkable that all other triangles (not twin prime) also tend to the slope of: $\sqrt{3}$. This can be explained by [3]: gaps get arbitrarily smaller in proportion to the primes. This means that two following prime numbers are nearly equal to one other if $n \to \infty$. The gap g_n will become neglectable to the magnitude of the prime numbers.

$$p_n = p_{n-1} + g_n \tag{5}$$

For twin primes the "Fibonacci" Prime Vector triangles are the biggest. This is for large prime numbers almost an equilateral triangle (difference 2 can be neglected for large prime numbers). An equilateral triangle has the following ratio: 1, 2, $\sqrt{3}$. The maximum slope for twin prime triangles then is: $\sqrt{3}$ like observed in the analysis.



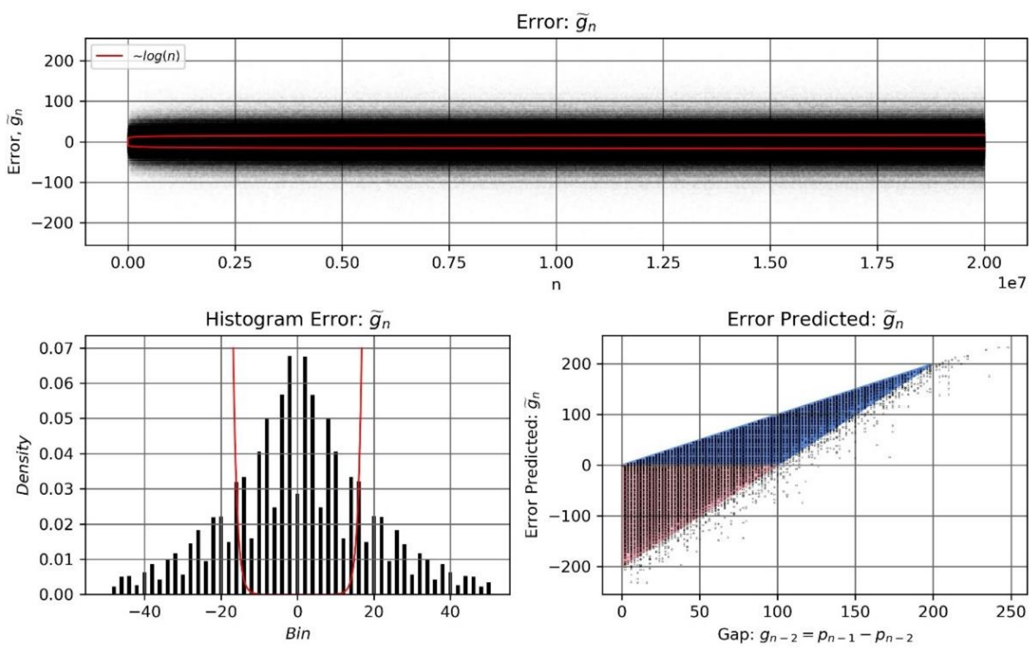
3. Prediction of next prime.

The slope for "Fibonacci" Prime Vector triangles is determined: $\sqrt{3}$ this corresponds to an angle of: $\beta = \frac{\pi}{3}$. With help of expression (3) we can predict the next prime. Expression (3) will become:

$$\tilde{p}_n = \frac{1}{2}p_{n-2} + \sqrt{-\frac{3}{4}p_{n-2}^2 + p_{n-1}^2} \tag{6}$$

With the error between the prediction and the expected prime number:

$$\varepsilon(n) = \tilde{p}_n - p_n \tag{7}$$



When the predicted prime gap is zero these forms balanced prime set, Wiki. The lower right graph shows the predicted error as function of actual Prime gap from its predecessors. There occurs an observed symmetry between positive errors and negative errors:

- The symmetry in the errors is unexpected. Red triangle (negative error) in graph has observed the same number then the blue triangle (positive).
- Twin primes only contribute negative errors.
- The error range for bigger gaps is smaller.

4. Error convergence.

Two functions are defined: ε_1 and ε_2. Function ε_1 is based upon prime triangles created from 3 following prime numbers. Function ε_2 is based upon the error regarding a balanced prime number.

$$\varepsilon_1(n) = \frac{1}{2}p_{n-2} - p_n + \sqrt{-\frac{3}{4}p_{n-2}^2 + p_{n-1}^2}$$

$$\varepsilon_2(n) = 2p_{n-1} - p_{n-2} - p_n$$

$$\Delta\varepsilon(n) = \varepsilon_1(n) - \varepsilon_2(n)$$

$$\Delta\varepsilon(n) = \frac{3}{2}p - 2(p+g) + \sqrt{-\frac{3}{4}p^2 + (p+g)^2}$$

$$\Delta\varepsilon(n) = -\frac{3g^2}{p} + \frac{12g^3}{p^2} - \frac{57g^4}{p^3} + \frac{300g^5}{p^4} - \frac{1686g^6}{p^5} + \mathcal{O}\left(\frac{1}{p^6}\right) \tag{8}$$

The difference between both functions $\Delta\varepsilon$ converges to 0 for $n \to \infty$ [4]. Expression (12) is found by converting 11 to a power series [5]. The first term of expression (12) can also be found numerical:

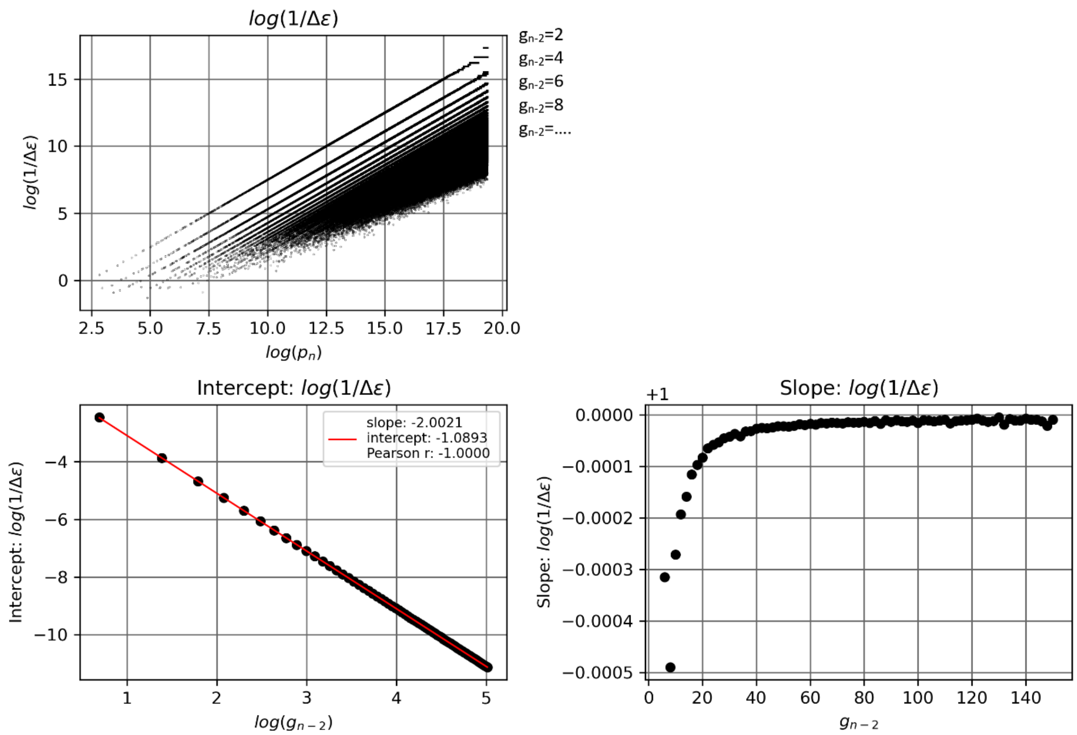
$$\log(1/\Delta\varepsilon) = slope \cdot \log(p_n) + intercept$$

$$intercept \approx -2.0021 \cdot \log(g_{n-2}) - 1.0893$$

$$slope \to 1$$

$$\Delta\varepsilon(n) \approx -\frac{a g_{n-2}^b}{p_n} \approx -\frac{3g_n^2}{p_n} \tag{9}$$

More information is found here: [6] and [7].



Summary.

All triangles based on the length of three following prime numbers exist.

The height of these triangles will eventually grow with a maximum slope of: $\sqrt{3}$. This slope enables to predict the position of the next prime based on the two preceding prime numbers.

The error difference between the prime triangles and balanced primes converges to 0. This analytical relation is identical as observed in the data. This prime gap limit could be related to: Oppermann's and Legendre's conjecture.

$$\lim_{n\to\infty}\left[-\frac{3{g_n}^2}{p_n}+\frac{12{g_n}^3}{{p_n}^2}-\frac{57{g_n}^4}{{p_n}^3}+\frac{300{g_n}^5}{{p_n}^4}-\frac{1686{g_n}^6}{{p_n}^5}+\mathcal{O}\left(\frac{1}{p^6}\right)\right]=0$$

The convergence of this error difference gives us an expression how fast the prime gaps converge.

Any feedback is welcome:

vpreemen@hotmail.com
mailto:vpreemen@hotmail.com?subject=Primenumbers

References.

[1] Original post on group: Unsolved problems.
https://groups.io/g/UnsolvedProblems/topic/triangels_with_all_sides/75781241?p=,,,20,0,0,0::recentpostdate%2Fsticky,,,20,2,0,75781241

[2] Wolfram Alpha: Cosine Rule.
https://www.wolframalpha.com/input/?i=solve+b%5E2%3Da%5E2%2Bc%5E2-2*a*c*cos%28beta%29+for+c

[3] Bertrand–Chebyshev theorem.
https://en.wikipedia.org/wiki/Bertrand%27s_postulate
https://en.wikipedia.org/wiki/Prime_gap#Upper_bounds

[4] Wolfram Alpha: Prime gap error convergence.
https://www.wolframalpha.com/input/?i=limit+y%3D0.5*x%2Bsqrt%28-0.75*x%5E2%2B%28x%2B1%29%5E2%29-2*%28x%2B1%29%2Bx++for+x+to+inf

[5] Wolfram Alpha: Series error convergence.
https://www.wolframalpha.com/input/?i=series+1.5p-2%28p%2Bg%29%2Bsqrt%28-0.75p%5E2%2B%28p%2Bg%29%5E2%29+for+p%3Dinfinity

[6] SE: Prime Triangles.
https://math.stackexchange.com/q/3798631/650339

[7] SE: Error convergence
https://math.stackexchange.com/q/3822679/650339

Extra Information.

[a] Animations on YouTube.
https://youtu.be/YOsASuAv54Y

[b] Animations on YouTube.
https://youtu.be/clKzRNfuk80