Proof of Legendre´s conjecture

Abstract

By dividing the numbers in the interval

between N^2 and (N+1)^2 into 2 groups of numbers,

we prove that not all the numbers between N^2 and (N+1)^2

can be composite.

This proves Legendre´s conjecture.

Main Proof

We will start the proof by introducing **Theorem 1** and **Theorem 1.1**

Theorem 1

The number of numbers between 1 and Z: W(Z,N)

which are all divisible by

the prime numbers from 2 to N can be calculated using the

formula:

$$W(Z,N) = O(Z,N) - E(Z,N)$$

where :

O(Z,N) is the number of numbers less than Z which are

divisible with square-free numbers, **with repetitions**

, which have an odd number of prime factors

and where all the prime factors are less than or equal to N

E(Z,N) is the number of numbers less than Z divisible

with the square-free numbers, **with repetitions**

,which has an even number of prime factors

and where all the prime factors are less than or equal to N.

Said more clearly :

$$W(Z,N) = O(Z,N) - E(Z,N)$$

Where

$$O(Z,N) = \frac{Z - r_1}{2} + \frac{Z - r_2}{3} \dots + \frac{Z - r_x}{p_n} \dots$$

$$\wedge$$

$$E(Z,N) = \frac{Z - r_{11}}{2 \cdot 3} + \frac{Z - r_{21}}{2 \cdot 5} \dots \frac{Z - r_{x1}}{p_n \cdot p_{n-1}}$$

Where the denominators for $E(Z,N)$

are all the squarefree numbers less than Z

which have an even number of prime factors and

where all the prime factors are less than or equal to N.

The denominators for $O(Z,N)$

are all the squarefree numbers less than Z

which have an odd number of primefactors and

where all the prime factors are less than or equal to N.

r_1 to r_x are the remainders.
For instance Z-r2 is the largest number less than or equal to Z

which is divisible with 3.

This formula is derived directly from the inclusion-exclusion

principle, so the idea behind it is already known.

See https://en.wikipedia.org/wiki/Prime-counting_function

under "Algorithms for evaluating π(x)".

A way to calculate the number of numbers that all are divisible by the prime numbers from 2 to N in the interval

from N^2 to N^2+2N can be derived directly from Theorem 1.

It is :

Theorem 1.1

$$W\left(N^2+2N,N\right)-W\left(N^2,N\right)=W(2N,N)+d_1((N))+\left(d_3(N)-d_2(N)\right)$$
$$=2N-\left(\pi(2N)-\pi(N)+1\right)+d_1((N))+\left(d_3(N)-d_2(N)\right)$$

where $d(n_1)$

is the number of *extra occurrences* of numbers divisible with the primes

which are less than or equal to N in the interval from N^2 to N^2+2N.

$d(n_3)$ is the number of *extra occurrences* of numbers divisible with

the squarefree numbers which have all primefactors less than or equal to N

and which have an odd number of prime factors >=3,

in the interval from N^2 to N^2+2N

$d(n_2)$ is the number of *extra occurrences* of numbers divisible with the squarefree numbers which have all primefactors less than or equal to N

and which have an even number of prime factors, in the interval from

N^2 to N^2+2N

An *extra occurence* occurs when the number of numbers

divisible with some squarefree number is one larger in

the interval from N^2 to N^2+2N

than in the interval from 1 to 2N.

For instance if there are d numbers divisible with 3 in the

interval from 1 to 2N and (d+1) numbers divisible with 3 in the interval

from k to 2N+k, this will count as 1 extra occurence.

This formula is derived from subtracting $W\left(N^2,N\right)$ from $W\left(N^2+2N,N\right)$

Clearly $W\left(2N,N\right)=2N-\left(\pi\left(2N\right)-\pi\left(N\right)+1\right)$

Now we have introduced **Theorem 1** and **Theorem 1.1**

and will now go further to the actual proof where **Theorem 1.1**

will be used.

The composite numbers in the interval between N^2 and (N+1)^2 can be divided into 2 groups of numbers :

X : Composite numbers, which do not have any factors larger than 2N.

Y : Composite numbers, which have factors (not neccesarily prime factors)

larger than 2N (and less than the number itself).

Now we will find an estimate for how many X-numbers and Y-numbers

there generally are in the interval between N^2 and (N+1)^2.

<u>Number of X-numbers</u>

The number of X-numbers in the interval between N^2 and (N+1)^2

is always less than $\pi(2N)-\pi(N)$.

This is because there always are numbers divisible with the primes from N and 2N

in the interval from N^2 to (N+1)^2.

For instance there is a number divisible with 13 between 7^2 and 8^2.

That the X-numbers can not have a composite number between N and 2N

as a factor can be seen from the fact, that it would have to have another primefactor less than N, which would imply that the number would have a number greater than 2N and less than the number itself, as a divisor.

There could be 2 numbers divisible with 13 between 7^2 and 8^2.

However this would mean that at least one of these numbers would be divisible by 2*13.

Since 2*p_(n+1), where p_(n+1) is the prime following N , is always greater than 2N, and the X-number must not have any factors greater than 2N, we get that the number of X-numbers in the interval between N^2 and (N+1)^2 is always less than equal to $\pi(2N)-\pi(N)$.

The reason it must be less than $\pi(2N)-\pi(N)$ is that the X-number must be divisible with exactly 2 primes,

One of them must be larger than N/2 and less than N.

The other larger than N and less than 2N.

If the X-number was divisible a prime less than N/2, then it would have a factor greater than 2N and therefore not be a X-number.

So the maximum number of X-number is : $\pi(N)-\pi\left(\frac{N}{2}\right)$

Number of Y-numbers

The Y-numbers will always have all their prime factors, which is less than

N, be less than (N+1)/2.

This is because

(N+1)^2/2N = (N/N)*((N+1)/2)

By using **Theorem 1.1** we get that the number of Y-numbers between N^2 and (N+1)^2

divisible with primes less than N/2 can be written as :

$$Y = W\left(N^2 + 2N, \frac{N}{2}\right) - W\left(N^2, \frac{N}{2}\right)$$
$$= W\left(2N, \frac{N}{2}\right) + d_1\left(\left(\frac{N}{2}\right)\right) + \left(d_3\left(\frac{N}{2}\right) - d_2\left(\frac{N}{2}\right)\right)$$
$$= 2N - \left(\pi(2N) - \pi\left(\frac{N}{2}\right) + 1\right) + d_1\left(\left(\frac{N}{2}\right)\right) + \left(d_3\left(\frac{N}{2}\right) - d_2\left(\frac{N}{2}\right)\right)$$

Where $d_1\left(\frac{N}{2}\right) + d_3\left(\frac{N}{2}\right) - d_2\left(\frac{N}{2}\right)$ is the extra occurences for

the squarefree numbers, which have primes less than or equal to N/2 as their factors.

Notice that if we have 1 extra number divisible with 3 and have another

extra number divisible with 3*p1*p2,

then since we already have a maximum number of numbers divisible with 3,

the extra number divisible with 3*p1*p2 cannot add to the

amount of composite numbers.

We can do this reasoning for any prime less than N/2.

This means that for

$$Y = 2N - \left(\pi(2N) - \pi\left(\frac{N}{2}\right) + 1 \right) + d_1\left(\frac{N}{2}\right) + d_3\left(\frac{N}{2}\right) - d_2\left(\frac{N}{2}\right)$$

that $Y \leq 2N - \left(\pi(2N) - \pi\left(\frac{N}{2}\right) + 1 \right) + \pi\left(\frac{N}{2}\right) = 2N - \pi(2N) + 2 \cdot \pi\left(\frac{N}{2}\right) - 1$

Now we add the maximum number of X-numbers and

maximum number of Y-numbers together.

$$MaxX + MaxY = 2N - \pi(2N) + 2 \cdot \pi\left(\frac{N}{2}\right) - 1 + \left(\pi(N) - \pi\left(\frac{N}{2}\right) \right)$$

$$= 2N + \pi\left(\frac{N}{2}\right) + \pi(N) - 1 - \pi(2N)$$

It can be shown that $\pi(2N) > \pi\left(\frac{N}{2}\right) + \pi(N) - 1$

for all values N>36

Since $\pi(2N) \approx \frac{2N}{\ln(2N)} \approx 2 \cdot \frac{N}{\ln(N)}$

and $\pi(N)+\pi\left(\frac{N}{2}\right)+\approx\frac{N}{\ln(N)}+\frac{\frac{N}{2}}{\ln(\frac{N}{2})}\approx\frac{3}{2}\frac{N}{\ln(N)}$

Since 2 > 3/2, we get that the inequality is true for sufficiently large values of N.

By verification we find that this value is N=36.

So the number of composite numbers between N^2 and (N+1)^2

is always less than 2N. (Because Legendre´s conjecture has been verified for N>36)

So Legendre´s conjecture is true.

Q.E.D