

Proof of the $3n+1$ problem for $n \geq 1$

by

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Abstract

I establish the existence of a unique binary pattern inherent to the $3n+1$ step, and then use this binary pattern to prove the $3n+1$ problem for all positive integers.

Introduction

Observe that every positive odd integer n , defined as $n = \sum_{i=0}^x 4^i$, $x \in \mathbf{Z}^+$

requires one $3n+1$ step and then $2(x+1)$ consecutive $n/2$ steps to be reduced to 1.

The truth of this statement will become apparent when the $3n+1$ step with such an integer is observed in base 2.

Example 1: Let $n = \sum_{i=0}^2 4^i = 21 = 10101_2$, then

$$10101_2 * 10_2 \Rightarrow 101010_2 + 10101_2 \Rightarrow 111111_2 + 1_2 = 1000000_2 = 2^6, \text{ and}$$

$$1000000_2 / 10_2 \Rightarrow 100000_2 / 10_2 \Rightarrow 10000_2 / 10_2 \Rightarrow 1000_2 / 10_2 \Rightarrow 100_2 / 10_2 \Rightarrow 10_2 / 10_2 = 1.$$

Therefore, the base 2 representation of positive integers furnishes more insight into the $3n+1$ problem than their base 10 representation.

Proof

Let \mathbf{O}^+ be the set of positive odd integers, then

$$\mathbf{O}^+ = \{x = \mathbf{Z} \mid x=2y+1, y \geq 0, y \in \mathbf{Z}\}.$$

Theorem 1: Let \mathbf{P} designate the $3n+1$ problem. Then if \mathbf{P} is true for all positive odd integers, it is true for all positive integers.

$$\forall a \in \mathbf{O}^+ : \mathbf{P}(a) \Rightarrow \forall b \in \mathbf{Z}^+ : \mathbf{P}(b)$$

Proof: Case 1: Power of two

Let $n = 2^x$, $x \in \mathbf{Z}^+$. Then n requires x consecutive $n/2$ steps to be reduced to 1.

Case 2: Odd integer multiplied by a power of two

Let $y = 2^x n$, $n \in \mathbf{O}^+$ and $x \in \mathbf{Z}^+$. Then x consecutive $n/2$ steps are required to have $y = n$.

Since these cases are exhaustive, it shows that if the $3n+1$ problem is true for all $a \in \mathbf{O}^+$ it has to be true for all $b \in \mathbf{Z}^+$.

The iteration between the $3n+1$ step and the $n/2$ step modifies every integer n , $n \in \mathbf{O}^+$ in such a way that, at some point the integer becomes 2^x , $x \in \mathbf{Z}^+$.

However, the process of this transformation is obscured by the $n/2$ step. In order to make the process apparent, the $n/2$ step is omitted and the addition of 1 in the $3n+1$ step is modified to compensate for the omission of the $n/2$ step.

Example 2: Let $n = 9 = 1001_2$, then $3n+2^x$ produces this pattern:

$$\begin{aligned} 1001_2 * 11_2 &\Rightarrow 11011_2 + 1_2 = 11100_2 \\ 11100_2 * 11_2 &\Rightarrow 1010100_2 + 100_2 = 1011000_2 \\ 1011000_2 * 11_2 &\Rightarrow 100001000_2 + 1000_2 = 100010000_2 \\ 100010000_2 * 11_2 &\Rightarrow 1100110000_2 + 10000_2 = 1101000000_2 \\ 1101000000_2 * 11_2 &\Rightarrow 100111000000_2 + 1000000_2 = 101000000000_2 \\ 101000000000_2 * 11_2 &\Rightarrow 1111000000000_2 + 1000000000_2 = 10000000000000_2 = 2^{13}. \end{aligned}$$

In example 2, the least significant bit transcends the most significant bit after six $3n+2^x$ steps, transforming n into a power of two.

Definition 1: Let LSB be the least significant bit of $s \in \mathbf{Z}^+$, then

$$\text{LSB} = \{2^r, r \geq 0, r \in \mathbf{Z} \mid 2^r = s/t, t \in \mathbf{O}^+\}.$$

Theorem 2: The $3n+\text{LSB}$ step and the $3n+1$ step are isomorphic.

Proof: Suppose $n_0 \in \mathbf{O}^+$. Let $n_1 = 3n_0 + 1$ and $n_2 = n_1 / \text{LSB}$, then

$$\frac{3n_1 + \text{LSB}}{3n_2 + 1} = \frac{3n_1 + \text{LSB}}{3\left(\frac{n_1}{\text{LSB}}\right) + 1} = \frac{3n_1 + \text{LSB}}{\frac{3n_1 + \text{LSB}}{\text{LSB}}} = \text{LSB}.$$

$$\therefore 3n + \text{LSB} \equiv 0 \pmod{3n + 1}.$$

Because a modular congruence exists between the $3n+\text{LSB}$ step and the $3n+1$ step, they are therefore isomorphic.

The pattern in example 2 is composed of two functions. The first function increases the most significant power of two or most significant bit of n , and the second function increases the least significant power of two or least significant bit of n .

Let $m(x)$ be the function for repeated multiplication of n by 3 in terms of x , $x \in \mathbf{Z}^+$. Then $m(x) = 3^{x+\delta} n$.

Let $\text{lsb}(x)$ be the function for repeated multiplication by 4 ($3(\text{LSB}) + \text{LSB}$) of the least significant bit of n in terms of x , $x \in \mathbf{Z}^+$. Then $\text{lsb}(x) = 4^{x+\delta}$.

Definition 2: Let $f(x)$ be the function for the $3n+\text{LSB}$ step for $n \in \mathbf{O}^+$ in terms of x , $x \in \mathbf{Z}^+$. Then

$$f(x) = m(x) + \text{lsb}(x) = 3^{x+\delta} n + 4^{x+\delta}.$$

Suppose that $\text{Tlsb}(x)$ is the function that gives the true position of the least significant bit of the $3n+\text{LSB}$ step for $n \in \mathbf{O}^+$ in terms of x , $x \in \mathbf{Z}^+$. Then

$$\delta = \sum_1^x \text{Tlsb}(x) - \text{lsb}(x).$$

Example 3: $\text{Tlsb}(x) > \text{lsb}(x)$

Assume that multiplying n_k by 3 produces $\dots 001111100\dots$ somewhere in the binary representation of the result; and that the rightmost 1 is $\text{LSB}=2^x$. Let $\text{lsb}(x)=\text{Tlsb}(x)$. Adding LSB to n_k yields $\dots 010000000\dots$, then

$$\delta = \sum_x^x \text{Tlsb}(x) - \text{lsb}(x) = \sum_x^x 2^{x+5} - 2^{x+2} = \sum_x^x x+5-x-2 = \sum_x^x 3 = 3.$$

Example 4: $\text{Tlsb}(x) < \text{lsb}(x)$

Assume that multiplying n_k by 3 and adding LSB produces $\dots 001111100\dots$ somewhere in the binary representation of the result; and that the rightmost 1 is $\text{LSB}=2^x$. Let $\text{lsb}(x)=\text{Tlsb}(x)$. Then repeated multiplication by 3 and addition of LSB will produce this pattern:

$\dots 001111100\dots$ times 3 plus 2^x
 $\dots 101111000\dots$ times 3 plus 2^{x+1}
 $\dots 001110000\dots$ times 3 plus 2^{x+2}
 $\dots 101100000\dots$ times 3 plus 2^{x+3}
 $\dots 001000000\dots$, then

$$\delta = \sum_x^{x+3} \text{Tlsb}(x) - \text{lsb}(x) = \sum_x^{x+3} 2^{x+1} - 2^{x+2} = \sum_x^{x+3} x+1-x-2 = \sum_x^{x+3} -1 = -4.$$

$$\therefore (\delta < 0) \vee (\delta = 0) \vee (\delta > 0)$$

Assume $x \in \mathbf{Z}^+$, then $m(x) < \text{lsb}(x)$ implies that a single power of two is larger than a sum of powers of two.

Using example 2 as an illustration:

$$m(x) - \text{lsb}(x) = 9(3^{x+2}) - 4^{x+2} = 0 \text{ for } x \approx 5.6377.$$

The integer after the root necessitates that $m(x) < \text{lsb}(x)$. In other words, it requires six $3n + \text{LSB}$ steps for 9 to converge to 2^{13} .

Theorem 3: For all positive odd integers n , there exists a positive integer x such that $m(x) < \text{lsb}(x)$.

$$\forall n (n \in \mathbf{O}^+) \exists x \in \mathbf{Z}^+ (m(x) < \text{lsb}(x))$$

Proof: Case 1:

$$\delta \leq -1, \delta \in \mathbf{Z}$$

$$\text{Assume } n \in \mathbf{O}^+ \text{ and let } m(x) - \text{lsb}(x) = 3^{x-\delta}n - 4^{x-\delta} = 0.$$

$$\text{Then } x = \frac{\log(1/n)}{\log(3/4)} + \delta.$$

$$\therefore \exists! x \in \mathbf{R}^+ (3^{x-\delta}n - 4^{x-\delta} = 0) \Rightarrow \exists x \in \mathbf{Z}^+ (m(x) < \text{lsb}(x))$$

Case 2:

$$\delta = 0$$

$$\text{Assume } n \in \mathbf{O}^+ \text{ and let } m(x) - \text{lsb}(x) = 3^x n - 4^x = 0.$$

$$\text{Then } x = \frac{\log(1/n)}{\log(3/4)}.$$

$$\therefore \exists! x \in \mathbf{R}^+ (3^x n - 4^x = 0) \Rightarrow \exists x \in \mathbf{Z}^+ (m(x) < \text{lsb}(x))$$

Case 3:

$$\delta \geq 1, \delta \in \mathbf{Z}$$

$$\text{Assume } n \in \mathbf{O}^+ \text{ and let } m(x) - \text{lsb}(x) = 3^{x+\delta}n - 4^{x+\delta} = 0.$$

$$\text{Then } x = \frac{\log(1/n)}{\log(3/4)} - \delta.$$

$$\therefore \exists! x \in \mathbf{R}^+ (3^{x+\delta}n - 4^{x+\delta} = 0) \Rightarrow \exists x \in \mathbf{Z}^+ (m(x) < \text{lsb}(x))$$

Because these cases are exhaustive, it shows that

$$\forall n (n \in \mathbf{O}^+) \exists x \in \mathbf{Z}^+ (m(x) < \text{lsb}(x)).$$

For all $n \in \mathbf{O}^+$ there exists an $x \in \mathbf{Z}^+$ such that $m(x) < \text{lsb}(x)$ (Theorem 3), therefore $f(x)$ converges to 2^y , $y \in \mathbf{Z}^+$. And since the $3n+\text{LSB}$ step and the $3n+1$ step are isomorphic (Theorem 2), it can be concluded that if $a_0 = n$, $n \in \mathbf{O}^+$, then

$$a_{i+1} = \begin{cases} a_i/2 & \text{for even } a_i \\ 3a_i+1 & \text{for odd } a_i \end{cases}, \text{ converges to } 1.$$

Because the $3n+1$ problem is true for all positive odd integers, then by Theorem 1 the truth extends to all positive integers. Therefore, if $a_0 = n$, $n \in \mathbf{Z}^+$, then

$$a_{i+1} = \begin{cases} a_i/2 & \text{for even } a_i \\ 3a_i+1 & \text{for odd } a_i \end{cases}, \text{ converges to } 1.$$

Q.E.D