New Search Formulas For Computing Cuboids

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1. Introduction

It all started in a 1719 issue of the Hamburg periodical *Deliciæ Mathematicæ* in which Paul Halcke noted the curious identities

$$44^2 + 117^2 = 125^2 \ , \quad 44^2 + 240^2 = 244^2 \ , \quad 117^2 + 240^2 = 267^2$$

which of course prove that a rectangular box with edges 44, 117, and 240 must have surface diagonals 125, 244, and 267. A triple (x, y, z) satisfying $x^2 + y^2 = \alpha^2$, $x^2 + z^2 = \beta^2$, and $y^2 + z^2 = \gamma^2$ with $x, y, z, \alpha, \beta, \gamma$ all positive integers is called a *rational cuboid* or a *Pythagorean box* or an *Euler brick*. If (x, y, z) is a rational cuboid then so is (kx, ky, kz) for any integer $k \geq 1$. If $\gcd(x, y, z) = 1$ the cuboid is said to be *primitive*; it is easy to make any cuboid primitive: merely divide each edge by the gcd of the three edges. We will usually consider two cuboids which reduce to the same primitive to be equivalent, and thus we will regard $(44, 117, 240)$ and $(88, 234, 480)$ as essentially the same cuboid.

Primitive rational cuboids are somewhat rare; there are only five with all edges less than 1000, namely $(240, 117, 44)$, $(275, 252, 240)$, $(693, 480, 140)$, $(720, 132, 85)$, and $(792, 231, 160)$. There are 19 primitive cuboids with edges less than 10^4, and 65, 242, 704, 1884, 4631, and 10932 with edges less than 10^5 through 10^{10} respectively. Exact counts for higher powers of ten are unknown to me, but the pattern suggests the counts for 10^{11} and 10^{12} should be roughly 25800 and 60900.

A rational cuboid is said to be *perfect* if its body diagonal $\delta = \sqrt{x^2 + y^2 + z^2}$ is an integer. Halcke's cuboid is not perfect since $44^2 + 117^2 + 240^2 = 73225 = 5^2 \cdot 29 \cdot 101$ is not a perfect square, and indeed none of the 10932 cuboids with edges less than 10^{10} are perfect. Rational cuboids are sometimes called *body cuboids* because only their body diagonals need not be integers. Despite numerous computer searches over many years in many countries, no perfect rational cuboid has ever been found; on the other hand, no one has been able to prove that a perfect cuboid cannot exist. All computer searches are patterned on the same simple idea: find as many different cuboids as you can and hope you stumble across a perfect one. Back in 1920 Leonard Eugene Dickson **[12]** called the existence question a "difficult problem which has been recently attacked but not solved" and in 1994 Richard Guy **[15]** called it "this notorious unsolved problem."

This paper introduces some new search methods based on Pythagorean triangles. All my formulas immediately generalize to Gaussian cuboids, cuboids whose edges and diagonals are Gaussian integers $x + iy$, and to quadratic integer cuboids, cuboids whose edges and diagonals are of the form $a + b\sqrt{m}$ where a, b, m are ordinary integers and m is squarefree.

2. New Search Formulas for Body Cuboids

Let a, b, c, d be positive integers, and let $(p, q, r) = (a^2+b^2, a^2-b^2, 2ab)$ and $(s, t, u) = (c^2+d^2, c^2-d^2, 2cd)$; these are both Pythagorean triangles since $p^2 = q^2 + r^2$ and $s^2 = t^2 + u^2$. The four cases

Case	x	y	z	$\sqrt{x^2+y^2}$	$\sqrt{x^2+z^2}$	y^2+z^2
(1)	qu	ru	rt	pu	rs	$q^2u^2+r^2t^2$
(2)	qu	qt	rt	qs	pt	$q^2u^2+r^2t^2$
(3)	qt	rt	ru	pt	rs	$q^2t^2+r^2u^2$
(4)	qt	qu	ru	qs	pu	$q^2t^2+r^2u^2$

produce an efficient search algorithm: whenever either $q^2u^2 + r^2t^2$ or $q^2t^2 + r^2u^2$ is a perfect square, (x, y, z) is a body cuboid. For example if $(a, b, c, d) = (8, 5, 6, 5)$ then $(q, r, t, u) = (39, 80, 11, 60)$ with $q^2u^2 + r^2t^2 = 2500^2$ and so $(x, y, z) = (2340, 4800, 880)$ is a body cuboid using Case (1); in fact $(x, y, z) = 20 \cdot (117, 240, 44)$ is Halcke's cuboid.

There is no mystery about a, b, c, d : the three faces of the cuboid $(44, 117, 240)$ reduce to the primitive Pythagorean triangles $(89, 39, 80)$, $(61, 11, 60)$, and $(125, 117, 44)$ which are generated by $(8, 5)$, $(6, 5)$, and $(11, 2)$ respectively. Any two of these three generators will produce Halcke's cuboid. We have the following more complete results.

a	b	c	d	$\sqrt{x^2+z^2}$	Case	(x, y, z)
8	5	6	5	$20 \cdot 1 \cdot 125$	(1)	$20 \cdot (117, 240, 44)$
					(2)	$1 \cdot (2340, 429, 880)$
11	2	6	5	$11 \cdot 3 \cdot 89$	(3)	$11 \cdot (117, 44, 240)$
					(4)	$3 \cdot (429, 2340, 880)$
11	2	8	5	$4 \cdot 39 \cdot 61$	(1)	$4 \cdot (2340, 880, 429)$
					(2)	$39 \cdot (240, 117, 44)$

The three faces of the cuboid $(429, 880, 2340)$ reduce to the same three primitive Pythagorean triangle as $(44, 117, 240)$. Euler noticed that if (x, y, z) is a body cuboid then so is (xy, xz, yz) since $(xy)^2 + (xz)^2 = (x\gamma)^2$, $(xy)^2 + (yz)^2 = (y\beta)^2$, and $(xz)^2 + (yz)^2 = (z\alpha)^2$; (xy, xz, yz) is called the *derived cuboid* of (x, y, z) though I prefer to call it the *dual* since if we derive twice we get $(xy \cdot xz, xy \cdot yz, xz \cdot yz) = xyz \cdot (x, y, z)$ and we are back to our original body cuboid. The dual of Halcke's cuboid is $(429, 880, 2340)$ and the reader can easily check that the cuboids in Cases (1) and (2) are always duals, as are the cuboids in Cases (3) and (4).

To avoid trivial repetitions when searching for body cuboids the Pythagorean triangles (p, q, r) and (s, t, u) should be primitive, and this imposes well known restrictions on a, b, c, d : $a > b$, $c > d$, $\gcd(a, b) = \gcd(c, d) = 1$, and $a + b$ and $c + d$ must both be odd; also by symmetry we may assume $a \geq c$ and if $a = c$ then $b > d$. With these restrictions, the four cases in my search method will eventually find every primitive rational cuboid exactly three times. In fact Cases (1) and (2) separately will find every primitive cuboid exactly once, while Cases (3) and (4) together will find every primitive cuboid exactly once.

Why should we bother with four cases when Case (1) alone will find all primitive rational cuboids? One reason is that we get the cuboids in Cases (2) and (4) for free;

another is that some cuboids found early by Cases (3) and (4) might not be found for quite some time by Cases (1) and (2), and vice versa. For example the Case (3) cuboid $(476960913, 1325858000, 29528699520)$ is generated by $(a, b, c, d) = (1000, 703, 524, 501)$; it is also a Case (2) cuboid generated by $(a, b, c, d) = (6179, 6080, 524, 501)$ and a Case (1) cuboid generated by $(6179, 6080, 1000, 703)$ so if we were using only the Case (1) or (2) formulas this cuboid would not have been found for a very long time.

Computing Body Cuboids

We are now ready (at least in theory) to compute all primitive rational cuboids. Let a, b, c, d be positive integers with the restrictions just stated. As above, set $p = a^2 + b^2$, $q = a^2 - b^2$, $r = 2ab$, $s = c^2 + d^2$, $t = c^2 - d^2$, $u = 2cd$ so that (p, q, r) and (s, t, u) are primitive Pythagorean triangles. A lengthy search on a dozen computers found the following counts for $a \leq 7000$; $N(A)$ is the number of different primitive body cuboids (x, y, z) generated by (a, b, c, d) with $a \leq A$.

A	1000	2000	3000	4000	5000	6000	7000
$N(A)$	4874	10356	15846	21244	26546	31636	36830
$N(A)/A$	4.874	5.178	5.282	5.311	5.309	5.273	5.261

Alas, none of these 36830 cuboids are perfect. The ratios $N(A)/A$ suggest that these may be converging to some constant (almost surely between 5.1 and 5.4) so we may safely conjecture that $N(10^4)$ is roughly 52600 and that $N(10^5)$ is very roughly 526000.

I have been asked if further restrictions on (a, b, c, d) are possible, which might make the search go faster. Any more restrictions would cause some primitive Pythagorean triangles not to be tested, and we might miss a body cuboid.

Note that with the present restrictions $q^2u^2 + r^2t^2 \equiv 0 \bmod 4$ and $q^2t^2 + r^2u^2 \equiv 1 \bmod 4$. Believe me, I have tried lots of other ideas! A perfect square must be congruent to ± 1 modulo 5, but using this as a preliminary test only made the search slightly slower. My software (Ubasic) has a very fast integer square root: for example $isqrt(147) = 12$ with residue $res(147) = 3$, so that N is a perfect square if and only if $res(N) = 0$.

3. Face and Edge Cuboids

The search for a perfect cuboid has led to the introduction of two other types of cuboids. A *face cuboid* has integer body diagonal, but one of its face diagonals is irrational; the smallest example is $(153, 104, 672)$ with $153^2 + 104^2 = 185^2$, $104^2 + 672^2 = 680^2$, $153^2 + 104^2 + 672^2 = 697^2$, but $153^2 + 672^2 = 474993$ is not a perfect square. An *edge cuboid* has all diagonals integers, but one edge irrational; the smallest example is $(x, y, z) = (63, 60, \sqrt{-3344})$ where the edge z is both irrational and imaginary. It has been customary to allow z^2 to be negative since John Leech **[25]** first searched for edge cuboids in 1977. The smallest edge cuboid with real edges is $(x, y, z) = (576, 520, \sqrt{618849})$.

Obviously a face cuboid is perfect if its bad face diagonal is an integer, and an edge cuboid is perfect if its bad edge is an integer.

Pythagorean style formulas also exist for face and edge cuboids: nine cases for edge and thirty-five (!) for face cuboids.

Pythagorean Face Cuboid Search Formulas

Let a, b, c, d, p, q, r, s, t, u be the same as for body cuboids, and let $\alpha^2 = x^2 + y^2$, $\beta^2 = x^2 + z^2$, $\gamma^2 = y^2 + z^2$, $\delta^2 = x^2 + y^2 + z^2$.

Case	R^2	x	y	z	α	γ	δ
(1)	$p^2u^2 + r^2t^2$	qu	ru	rt	pu	rs	R
(2)	$q^2s^2 + r^2t^2$	qu	qt	rt	qs	pt	R
(3)	$p^2t^2 + r^2u^2$	qt	rt	ru	pt	rs	R
(4)	$q^2s^2 + r^2u^2$	qt	qu	ru	qs	pu	R
(5)	$q^2t^2 - r^2u^2$	R	ru	rt	qt	rs	pt
(6)	$q^2u^2 - r^2t^2$	R	rt	qt	qu	pt	qs
(7)	$r^2t^2 - q^2u^2$	R	qu	ru	rt	pu	rs
(8)	$r^2u^2 - q^2t^2$	R	qt	qu	ru	qs	pu

When R is an integer, (x, y, z) is a face cuboid with irrational diagonal $\beta = \sqrt{x^2 + z^2}$. In fact Case (8) cannot occur: $r^2u^2 - q^2t^2 \equiv 3 \bmod 4$ and so cannot be a perfect square. This case is included because $r^2u^2 - q^2t^2$ can be the square of a Gaussian integer, and will be used later. Hence for ordinary face cuboids we have only seven cases.

Why did I say there were 35 face cuboid formulas? In his 1977 paper **[25]** John Leech noted that face cuboids always occur in sets of five. If $F_1 = (x, y, z)$ is a face cuboid then so are $F_2 = (xz, xy, \gamma y)$, $F_3 = (\gamma z, xz, \delta y)$, $F_4 = (\delta y, xz, \alpha x)$, and $F_5 = (\alpha y, yz, xz)$. I call these five cuboids the *pentad* associated with (x, y, z). Combining these with the seven cases above produces 35 different formulas for face cuboids. You are cordially invited to write out the full table. For $a \leq 3500$ these 35 formulas produced 38795 different face cuboids, and none were perfect.

Pythagorean Edge Cuboid Search Formulas

There are twelve cases to consider. Let $Z = \delta^2 - x^2 - y^2$ be the square of the irrational edge. The first four cases are

Case	x	y	δ	γ	β
(1)	rs	pu	ps	pt	qs
(2)	qs	pt	ps	pu	rs
(3)	qs	pu	ps	pt	rs
(4)	rs	pt	ps	pu	qs

and when $\alpha = \sqrt{x^2 + y^2}$ is an integer $(x, y, \sqrt{Z})$ is an edge cuboid. The other cases are

Case	x	y	α	β	δ
(5)	rt	qt	pt	qu	qs
(6)	ru	qu	pu	qt	qs
(7)	qt	rt	pt	ru	rs
(8)	qu	ru	pu	rt	rs
(9)	qu	qt	qs	rt	pt
(10)	ru	rt	rs	qt	pt
(11)	qt	qu	qs	ru	pu
(12)	rt	ru	rs	qu	pu

and when $\gamma = \sqrt{\delta^2 - x^2}$ is an integer $(x, y, \sqrt{Z})$ is an edge cuboid. In fact Cases (2), (7), and (11) cannot occur because of modulo 4 congruences; however they can produce Gaussian edge cuboids. Hence we have just nine formulas for ordinary edge cuboids. For $a \leq 3500$ my computers found 7542 real edge cuboids and 5505 imaginary edge cuboids. None were perfect.

4. Gaussian Cuboids

The inspiration for this section was the title "A Perfect Cuboid in Gaussian Integers" of a 1994 article by W. T. A. Colman **[10]**, but I was rather disappointed that his example was $(4, 3, 5i)$ with $4^2+3^2 = 5^2$, $4^2+(5i)^2 = (3i)^2$, $3^2+(5i)^2 = (4i)^2$, and $4^2+3^2+(5i)^2 = 0$. Colman observed that $(2pq, p^2-q^2, (p^2+q^2)i)$ was a more general solution, but even then we have $(2pq)^2 + (p^2-q^2)^2 + [(p^2+q^2)i]^2 = 0$. Colman concluded his paper by asking whether a perfect Gaussian cuboid existed with a nonzero body diagonal. In other words, is there a cuboid in which all three edges and all four diagonals are nonzero Gaussian integers?

Associates

Each Gaussian body cuboid $(a+bi, c+di, e+fi)$ has as many as 31 *associates*: certainly $(a-bi, c-di, e-fi)$, $(b+ai, d+ci, f+ei)$, and $(b-ai, d-ci, f-ei)$ are also body cuboids. And if (x, y, z) is a Gaussian body cuboid so are the eight cuboids $(\pm x, \pm y, \pm z)$ and this brings the total to 32. Of course if some of the real or imaginary parts are zero or if some are equal then the total will be smaller. Henceforth we will regard any associate of a Gaussian body cuboid to be essentially the same as the original.

Computing Gaussian Cuboids

If we replace the generators a, b, c, d with Gaussian integers $a_1 + ia_2$, $b_1 + ib_2$, $c_1 + ic_2$, $d_1 + id_2$ then $p, q, r, s, t, u, x, y, z$ all become Gaussian integers and the search formulas from Sections 2 and 3 produce Gaussian cuboids. Searches on several computers found almost half a million non-associated Gaussian body cuboids, and I honestly expected a few of these to be perfect, but none were. Similarly my computers found over a hundred thousand non-associated Gaussian face cuboids. Again, none were perfect.

I did not bother to compute any Gaussian edge cuboids. Every Gaussian body cuboid produces three different Gaussian edge cuboids, and conversely every Gaussian edge cuboid produces a single Gaussian body cuboid. Let (x, y, z) be a Gaussian body cuboid with $\Delta = x^2 + y^2 + z^2$ not the square of a Gaussian integer; then $(\sqrt{\Delta}, ix, iy)$, $(\sqrt{\Delta}, ix, iz)$, and $(\sqrt{\Delta}, iy, iz)$ are distinct Gaussian edge cuboids. On the other hand, if $(\sqrt{X}, y, z)$ is a Gaussian edge cuboid with $d^2 = X + y^2 + z^2$ then (d, iy, iz) is a Gaussian body cuboid.

5. Quadratic Integer Cuboids

We seek a perfect cuboid whose three edges and four diagonals are nonzero quadratic integers of the form $j+k\sqrt{m}$ where j, k, m are integers and m is squarefree. An anonymous referee (at another journal) observed that if (x, y, z) is a rational body cuboid then it is perfect in $Z[\sqrt{D}]$ where $D = x^2+y^2+z^2$ and saw no point in looking any further. I regard such examples as trivial and his lack of curiosity rather surprising, to put it politely.

If we put $a = a_1 + a_2\sqrt{m},\ b = b_1 + b_2\sqrt{m},\ c = c_1 + c_2\sqrt{m},\ d = d_1 + d_2\sqrt{m}$ then p, q, r, s, t, u are all quadratic integers, and the search formulas in Section 2 will find quadratic integer body cuboids. A simple program checked $a_1 \le 12$, $|a_2| \le a_1$, $1 \le b_1 \le a_1$, $|b_2| \le b_1$, $1 \le c_1 \le a_1$, $|c_2| \le c_1$, $1 \le d_1 \le a_1$, $|d_2| \le d_1$ for $m = 2, 3, 5, 6, 7, 10, 11, 13, 14, \cdots$. This search took about an hour for each m and even though hundreds of thousands of quadratic integer body cuboids were found on the first day, none were perfect. I quickly became convinced that this would be another bridge to nowhere.

I was wrong. On the second day (June 3, 2011) the computer suddenly displayed several perfect quadratic cuboids for $m = 31$. The first was generated by

$$(a, b, c, d) = (5 - \sqrt{31}, 3 + \sqrt{31}, 1 + \sqrt{31}, 4)$$

and its edges were $(x, y, z) = (736 + 224\sqrt{31}, 3840, 736 - 224\sqrt{31})$. All four diagonals were quadratic integers. I then noticed that all the edge coefficients were divisible by 32, so in reality the computer had found the smaller perfect cuboid $(23 + 7\sqrt{31}, 120, 23 - 7\sqrt{31})$ and I added a line of code to divide all coefficients by $\gcd(x_1, x_2, y_1, y_2, z_1, z_2)$ whenever a perfect cuboid (x, y, z) was found. I also noticed that we may assume $\gcd(a_1, a_2, b_1, b_1) = 1 = \gcd(c_1, c_2, d_1, d_2)$ which eliminates a great many equivalent cuboids, and makes the program run faster. A bit of explanation is in order here. Rational and Gaussian cuboids are always reduced to primitives by dividing by the gcd of the sides. For quadratic cuboids this is usually not possible because you cannot always compute the gcd of two quadratic integers: it has long been known that the Euclidean algorithm works in $Z[\sqrt{m}]$ only for the values $m = -11, -7, -3, -2, -1, 2, 3, 5, 6, 11, 13, 17, 19, 21, 29, 33, 37, 41, 57$, and 73. See **[13]**, **[16]**, and **[17]**.

For $m = 31$ my computer found dozens of perfect quadratic integer cuboids with $\gcd(x_1, x_2, y_1, y_2, z_1, z_2) = 1$; the first two were $(23 + 7\sqrt{31}, 120, 23 - 7\sqrt{31})$ and

$$(6180 - 1020\sqrt{31}, 3029 - 556\sqrt{31}, 660 - 165\sqrt{31})\ .$$

These two cuboids are not really different since

$$\frac{6180 - 1020\sqrt{31}}{120} = \frac{3029 - 556\sqrt{31}}{23 - 7\sqrt{31}} = \frac{660 - 165\sqrt{31}}{23 + 7\sqrt{31}} = \frac{103}{2} + \frac{17}{2}\sqrt{31}\ .$$

A simple computer program checked that *all* the perfect cuboids found for $m = 31$ were of the form $(23+7\sqrt{31}, 120, 23-7\sqrt{31})\cdot(e+f\sqrt{31})/2$ so in this sense they were all equivalent.

Searches for squarefree $|m| \le 10000$ were carried out and found the first six of the following seven perfect quadratic integer cuboids.

	x	y	z
(1)	$23 + 7\sqrt{31}$	120	$23 - 7\sqrt{31}$
(2)	$224 - 28\sqrt{65}$	$128 - 16\sqrt{65}$	$195 - 24\sqrt{65}$
(3)	$8\sqrt{91}$	$15\sqrt{91}$	99
(4)	190	$8\sqrt{-91}$	$15\sqrt{-91}$
(5)	8160	$2056\sqrt{-510}$	73759
(6)	$15\sqrt{-627}$	$8\sqrt{-627}$	442
(7)	$15\sqrt{627}$	$8\sqrt{627}$	119

	$\sqrt{x^2+y^2}$	$\sqrt{x^2+z^2}$	$\sqrt{y^2+z^2}$	$\sqrt{x^2+y^2+z^2}$
(1)	$7+23\sqrt{31}$	64	$7-23\sqrt{31}$	136
(2)	$260-32\sqrt{65}$	$296-37\sqrt{65}$	$232-29\sqrt{65}$	$325-40\sqrt{65}$
(3)	$17\sqrt{91}$	125	174	190
(4)	174	125	$17\sqrt{-91}$	99
(5)	$2024\sqrt{-510}$	74209	57311	57889
(6)	$17\sqrt{-627}$	233	394	119
(7)	$17\sqrt{627}$	394	233	442

Of course many other perfect cuboids were also found, but each of them were multiples of one of the first six of these seven primitive perfect cuboids.

The symmetries in cuboid (1) are intriguing and suggest a specialized search might find similar perfect cuboids that are not multiples of (1). Elementary algebra shows that the identities

$$(a+b\sqrt{m})^2+(a-b\sqrt{m})^2=(2e)^2$$
$$(a\pm b\sqrt{m})^2+c^2=(b\pm a\sqrt{m})^2$$
$$(a+b\sqrt{m})^2+(a-b\sqrt{m})^2+c^2=d^2$$

are equivalent to $a^2+b^2m=2e^2$, $(a^2-b^2)(m-1)=c^2$, and $(a^2+b^2)(m+1)=d^2$. Clearly the squarefree number m must be positive, and we may assume $\gcd(a,b)=1$. A simple computer search for $m<10000$ and $0<b<a\leq 10000$ found only the one solution

$$(m,a,b,c,d,2e)=(31,23,7,120,136,64)$$

after ten days on a single computer, and this is just cuboid (1).

The elusive cuboid (7) is closely related to cuboid (6). Obviously if $(p\sqrt{m},q\sqrt{m},s)$ is a perfect cuboid with $p^2+q^2=r^2$, $p^2m+s^2=t^2$, $q^2m+s^2=u^2$, $r^2m+s^2=v^2$ then $(p\sqrt{-m},q\sqrt{-m},v)$ is also a perfect cuboid. The perfect cuboids (3) and (4) are also related in this way. Naturally I was concerned that my computers had not found cuboid (7), and a longer search of $a_1\leq 20$ produced nothing for $m=627$. The reason for this is that the Pythagorean triangle $(15\sqrt{627},119,394)$ is not of the form $(2abk,(a^2-b^2)k,(a^2+b^2)k)$ where $k,a,b\in Z[\sqrt{627}]$ because neither $15\sqrt{627}$ nor 119 can be equal to $2abk$.

6. Afterthoughts

Are there other nontrivial primitive perfect cuboids in the quadratic integers? It was never my intention to give a complete list — I was overjoyed to find even one! Perhaps some reader of this paper will find others. The fact that the classical Pythagorean representation $(2abk,(a^2-b^2)k,(a^2+b^2)k)$ does not always seem to work in $Z[\sqrt{m}]$ may mean that other nontrivial primitive perfect cuboids are hiding somewhere in these rings.

I once believed that eventually someone (why not me?) would find a perfect rational cuboid and cover himself in glory, and I really did expect my little army of computers to find at least a few nontrivial perfect Gaussian cuboids, but my skepticism increased as the months dragged by and nothing happened. However, the somewhat naive search for quadratic integer cuboids established a simple pattern: for each squarefree m either a perfect cuboid popped up in a few seconds soon followed by dozens of multiples, or no perfects at all were found. Primitive body cuboids become much sparser as their edges grow larger, and their body diagonals have an ever smaller chance of being integers: in number theory such statistical arguments prove nothing at all, and yet they make a large perfect rational cuboid seem an even more remarkable object, assuming the creature does exist. Maybe your computer will find one. Hope Springs Eternal.

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