Parametric Solutions for a Nearly-Perfect Cuboid

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Abstract

We consider nearly-perfect cuboids (NPC), where the only irrational is one of the face diagonals. Obtained are three rational parametrizations for NPC with one parameter.

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1 Introduction

For three centuries, no one has been able to prove the existence or otherwise of a perfect cuboid (PC) – a rectangular parallelepiped having integer (rational): three sides, three face (surface) diagonals and a space (body) diagonal.

If among the seven:

$$a, \; b, \; c, \; d_{ab}, \; d_{bc}, \; d_{ac}, \; d_s$$

only one is irrational, this is called a nearly-perfect cuboid (NPC). The NPC are divided into three types:

1. The space diagonal d_s is irrational,
2. One of the sides a, b, c is irrational,
3. One of the face diagonals d_{ab}, d_{bc}, d_{ac} is irrational.

In this paper we consider the third problem and show how parametric formulae for such NPC can be constructed.

PC problem is equivalent to the existence of an integer (rational) solution of the Diophantine system:

$$\begin{aligned} a^2 + b^2 &= d_{ab}^2, \\ b^2 + c^2 &= d_{bc}^2, \\ a^2 + c^2 &= d_{ac}^2, \\ a^2 + b^2 + c^2 &= d_s^2. \end{aligned}$$

In [1] are found complete parametrizations for NPC (only one face diagonal d_{ab} is irrational) by using a pair of solutions of the congruent number equation. Given parametrizations are not rational and depend on two parameters.

In [2–3] are summarized the literature written about PC. In numerous papers, only a few are related to the rational parametrizations [4–9].

In this paper we use the same notations as used in [1].

2 Basic Algebra

Theorem 3 in [1] establishes PC equivalent equation:

$$\Big(\frac{1-\gamma^2}{2\gamma}\Big)^2+\Big(\frac{1-\beta^2}{2\beta}\Big)^2=\Big(\frac{1-\alpha^2}{2\alpha}\Big)^2, \tag{1}$$

where α, β, γ are nontrivial rational numbers, i.e. from set

$$\mathbb{Q}\setminus\{0;\pm 1\}.$$

After simplification:

$$\Big(\frac{1-\gamma^2}{2\gamma}\Big)^2=\frac{\big(1-(\alpha\beta)^2\big)\Big(1-\Big(\dfrac{\alpha}{\beta}\Big)^2\Big)}{4(\alpha\beta)\cdot\Big(\dfrac{\alpha}{\beta}\Big)}\,.$$

By using new notations:

$$\xi\equiv\alpha\beta,\quad \zeta\equiv\frac{\alpha}{\beta}\,; \tag{2}$$

PC equation takes the form

$$\Big(\frac{1-\gamma^2}{2\gamma}\Big)^2=\frac{(1-\xi^2)(1-\zeta^2)}{4\xi\zeta}\,. \tag{3}$$

It is clear that the product (ratio) $\xi\zeta$ is square $(=\Box)$ and

$$\xi,\, \zeta \in \mathbb{Q} \setminus \{0; \pm 1\}, \quad \xi \neq \zeta.$$

From (2) and (3) we obtain:

Theorem 1. *The existence of Perfect cuboid is equivalent to the existence of nontrivial different ξ and ζ rational numbers satisfying three conditions:*

$$\xi\zeta = \square, \tag{4}$$

$$(1-\xi^2)(1-\zeta^2) = \square, \tag{5}$$

$$(1-\xi^2)(1-\zeta^2) + 4\xi\zeta = \square. \tag{6}$$

Let's discuss the identity like condition (5):

$$(1-T^2)\big(1-(4T^3-3T)^2\big) = \big[(1-T^2)(1-4T^2)\big]^2. \tag{7}$$

We try to replace ξ and ζ with T and $4T^3-3T$, respectively.

From condition (4), it is possible if

$$4T^2 - 3 = \square\,. \tag{8}$$

The complete rational parametrization of (8) is

$$T = \frac{t^2+3}{4t}\,,$$

where t is an arbitrary (nonzero) rational number.

We obtain the rational parametrization of conditions (4) and (5) by formulae:

$$\xi = \frac{t^2+3}{4t}, \quad \zeta = \frac{t^2+3}{4t}\left(\frac{t^2-3}{2t}\right)^2. \tag{9}$$

Of course (9) represents the incomplete parametrization of (4) and (5), because identity (7) does not describe all rational solutions of (5).

3 The First Parametrization

PC equation (1) is obtained from the system [1]:

$$\begin{aligned}
\frac{d_s}{a} &= \frac{1+\alpha^2}{2\alpha}, \quad \frac{d_{bc}}{a} = \frac{1-\alpha^2}{2\alpha}, \\
\frac{d_{ac}}{a} &= \frac{1+\beta^2}{2\beta}, \quad \frac{c}{a} = \frac{1-\beta^2}{2\beta}, \\
\frac{d_{ab}}{a} &= \frac{1+\gamma^2}{2\gamma}, \quad \frac{b}{a} = \frac{1-\gamma^2}{2\gamma}.
\end{aligned} \tag{10}$$

From (2) follows

$$\alpha^2 = \xi\zeta, \quad \beta^2 = \frac{\xi}{\zeta}, \tag{11}$$

and if (4) and (5) are satisfied, then the ratios

$$\frac{d_s}{a}, \quad \frac{d_{bc}}{a}, \quad \frac{d_{ac}}{a}, \quad \frac{c}{a}, \quad \frac{b}{a}$$

are rationals. So,

Theorem 2. *Nearly-perfect cuboid (only one face diagonal is irrational) is obtained by nontrivial different ξ and ζ rational numbers satisfying two conditions:*

$$\xi\zeta = \square,$$
$$(1-\xi^2)(1-\zeta^2) = \square.$$

From (9) and (11):

$$\alpha = \frac{t^4-9}{8t^2}, \quad \beta = \frac{2t}{t^2-3}. \tag{12}$$

By substituting (12) in (10) we get the first parametrization for NPC:

I parametrization

$$\begin{aligned} a &= 16t^2(t^4-9), \\ b &= (t^4-10t^2+9)(t^4+2t^2+9), \\ c &= 4t(t^2+3)(t^4-10t^2+9); \end{aligned}$$

$$\begin{aligned} d_{ac} &= 4t(t^2+3)(t^4-2t^2+9), \\ d_{bc} &= (t^4-1)(t^4-81), \\ d_s &= t^8+46t^4+81. \end{aligned}$$

4 The Second Parametrization

In [1] three PC equations are obtained so, naturally we expect other parametrizations.

Consider the PC equation from Theorem 1 in [1]:

$$\left(\frac{2\alpha}{1+\alpha^2}\right)^2+\left(\frac{2\gamma}{1+\gamma^2}\right)^2=\left(\frac{2\beta}{1+\beta^2}\right)^2. \tag{13}$$

After simplification:

$$\left(\frac{2\gamma}{1+\gamma^2}\right)^2=\frac{4\beta^2\big(1-(\alpha\beta)^2\big)\left(1-\left(\frac{\alpha}{\beta}\right)^2\right)}{(1+\alpha^2)^2(1+\beta^2)^2}\,.$$

PC equation (13) is obtained from the following system [1]:

$$\begin{aligned}\frac{a}{d_s}&=\frac{2\alpha}{1+\alpha^2}\,, & \frac{d_{bc}}{d_s}&=\frac{1-\alpha^2}{1+\alpha^2}\,,\\ \frac{b}{d_s}&=\frac{1-\beta^2}{1+\beta^2}\,, & \frac{d_{ac}}{d_s}&=\frac{2\beta}{1+\beta^2}\,,\\ \frac{c}{d_s}&=\frac{2\gamma}{1+\gamma^2}\,, & \frac{d_{ab}}{d_s}&=\frac{1-\gamma^2}{1+\gamma^2}\,.\end{aligned} \tag{14}$$

By substituting (12) in (14), we obtain the second parametrization for NPC:

II parametrization

$$\begin{aligned} a &= 16t^2(t^4-9)(t^4-2t^2+9), \\ b &= (t^4-10t^2+9)(t^8+46t^4+81), \\ c &= 4t(t^2-3)(t^4-10t^2+9)(t^4+2t^2+9); \end{aligned}$$

$$\begin{aligned} d_{ac} &= 4t(t^2-3)(t^8+46t^4+81), \\ d_{bc} &= (t^4-2t^2+9)(t^8-82t^4+81), \\ d_s &= (t^4-2t^2+9)(t^8+46t^4+81). \end{aligned}$$

5 The Third Parametrization

From Theorem 2 in [1] PC equation is

$$\left(\frac{2\gamma}{1-\gamma^2}\right)^2 + \left(\frac{2\beta}{1-\beta^2}\right)^2 = \left(\frac{2\alpha}{1-\alpha^2}\right)^2. \tag{15}$$

After simplification:

$$\left(\frac{2\gamma}{1-\gamma^2}\right)^2 = \frac{4\alpha^2\big(1-(\alpha\beta)^2\big)\left(1-\left(\frac{\beta}{\alpha}\right)^2\right)}{(1-\alpha^2)^2(1-\beta^2)^2}\,.$$

In this case insert

$$\alpha\beta = \frac{t^2+3}{4t}\,, \quad \frac{\beta}{\alpha} = \frac{t^2+3}{4t}\left(\frac{t^2-3}{2t}\right)^2$$

into the generating system [1] for PC equation (15):

$$\frac{d_s}{a} = \frac{1+\alpha^2}{1-\alpha^2}, \quad \frac{d_{bc}}{a} = \frac{2\alpha}{1-\alpha^2},$$
$$\frac{d_{ac}}{a} = \frac{1+\beta^2}{1-\beta^2}, \quad \frac{c}{a} = \frac{2\beta}{1-\beta^2},$$
$$\frac{d_{ab}}{a} = \frac{1+\gamma^2}{1-\gamma^2}, \quad \frac{b}{a} = \frac{2\gamma}{1-\gamma^2}.$$

We receive the third parametrization for NPC:

III parametrization

$$a = (t^4-1)(t^4-81),$$
$$b = 4t(t^2-3)(t^4+2t^2+9),$$
$$c = 16t^2(t^4-9);$$

$$d_{ac} = t^8+46t^4+81,$$
$$d_{bc} = 4t(t^2-3)(t^4+10t^2+9),$$
$$d_s = (t^4-2t^2+9)(t^4+10t^2+9).$$

6 The Basis of a Computer Search

The goal of this paper is to find a perfect cuboid. The obtained parametrizations can form the basis for a computer search. The first, second and third parametrizations give for diagonal d_{ab} conditions of rationality, respectively:

$$\Big[16t^2(t^4-9)\Big]^2+\Big[(t^4-10t^2+9)(t^4+2t^2+9)\Big]^2=\Box,$$

$$\Big[16t^2(t^4-9)(t^4-2t^2+9)\Big]^2+$$
$$+\Big[(t^4-10t^2+9)(t^8+46t^4+81)\Big]^2=\Box,$$

$$\Big[(t^4-1)(t^4-81)\Big]^2+\Big[4t(t^2-3)(t^4+2t^2+9)\Big]^2=\Box.$$

If, by using a computer, it is possible to find the nontrivial ($\neq 0,\pm1,\pm3$) rational t, for which one of the given expressions is square, then PC problem has a solution. Otherwise from Theorem 1 it must be proved that two nontrivial different rational numbers which satisfied the (4)–(6) conditions do not exist.

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