THREE PROOFS FOR LEGENDRE'S CONJECTURE

Edigles Guedes

February 18, 2015

"They prevented me in the day of my calamity: but the LORD was my stay." − 2 Samuel 22:19.

Abstract. We write three proves for Legendre's conjecture: given an integer, $n > 0$, there is always one prime, p, such that $n^2 < p < (n+1)^2$, using the prime-counting function, the Bertrand's postulate and the Hardy-Wright's estimate.[1]

1. Introduction

The Legendre's conjecture, named after Adrien-Marie Legendre (1752-1833), says that: There is always one prime number between a square number and the next. Algebraically speaking, as originally proposed by G. H. Hardy and E. M. Wright, in his book *An Introduction to the Theory of Numbers*:

Theorem 1. *(Legendre's conjecture)* [1, p. 23] *There is always one prime between* n^2 *and* $(n+1)^2$.

Put yet another way, $\pi((n+1)^2) - \pi(n^2) > 0,$ where $\pi(n)$ denotes the classical prime-counting function.

In Landau's problems list, this conjecture was considered unproved, in 1912.

Chen Jingrun (1933-1996) demonstrated a weaker version of the conjecture: There is either a prime $n^2 < p < (n+1)^2$ or a semiprime $n^2 < pq < (n+1)^2,$ where q is one prime unequal to p. [1, p. 594]

H. Laurent [2, p. 427] noted this peculiar relation to prime numbers:

$$\frac{e^{\frac{2\pi i\Gamma(k)}{k}}-1}{e^{-\frac{2\pi i}{k}}-1} = \begin{cases} 1, \text{if } k \text{ is prime}, \\ 0, \text{if } k \text{ is composite}, \end{cases}$$

provided $k \in \mathbb{N}_{\geq 5}$, where $\Gamma(k)$ denotes the gamma function.

In 2013, I and the Dr. Raja Rama Gandhi [2, Theorem 2, pp. 5-7] demonstrated the Legendre's conjecture based in the partial summation for prime-counting function:

$$\pi(n) = 2 + \sum_{k=5}^{n} \frac{e^{\frac{2\pi i\Gamma(k)}{k}}-1}{e^{-\frac{2\pi i}{k}}-1},$$

for $n \in \mathbb{N}_{\geqslant 5}$, here $\Gamma(n)$ is the gamma function.

In section 3, we use the same strategy to demonstrate, again, the Legendre's conjecture; but, this time, we will utilize other partial summations for prime-counting function.

In section 4, we use a new strategy to prove the Legendre's conjecture, yet based on lower bound for prime-counting function, which we found in the book of Hardy and Wright [1], the Theorem 20, page 21.

1. 2010 *Mathematics Subject Classification.* Primary 11A41, Secondary 11A25.

Key words and phrases. Legendre's conjecture, Bertrand's postulate, Hardy-Wright's estimate, prime-counting function.

2. Preliminaries

In present, we will need of the two partial summation for prime-counting function:

Theorem 2. *For $n \in \mathbb{N}_{\geqslant 5}$, then*

$$\pi(n) = 2 + \sum_{k=5}^{n} \frac{1 - \cos\left(\frac{2\pi}{k}\right) - \cos\left(\frac{2\pi\Gamma(k)}{k}\right) + \cos\left(\frac{2\pi\Gamma(k)}{k} + \frac{2\pi}{k}\right)}{2 - 2\cos\left(\frac{2\pi}{k}\right)} \tag{1}$$

and

$$\pi(n) = 2 + \sum_{k=5}^{n} \frac{\left(1 - e^{-\frac{2\pi i\Gamma(k)}{k}}\right)\left(1 - e^{-\frac{2\pi i}{k}}\right)}{2 - 2\cos\left(\frac{2\pi}{k}\right)}, \tag{2}$$

where $\Gamma(n)$ denotes the gamma function.

Proof. See [3 and 4, Theorem 1, pp.1-4]. □

The weak Bertrand's postulate says:

Theorem 3. (weak Bertrand's postulate) *For every $n \geq 1$ there is always, at least, one prime number, p, such that $n < p \leq 2n$.*

Proof. See [5, pp. 208-209]. □

Setting another way, $\pi(2n) - \pi(n) > 0$, here $\pi(n)$ denotes the classical prime-counting function.

Theorem 4. *For $n \in \mathbb{N}_{\geqslant 1}$, then*

$$\pi(n) \geqslant \frac{\log n}{2\log 2}. \tag{3}$$

Proof. See [1, Theorem 20, page 21]. □

3. Two proofs for theorem 1

Now, we will prove the Theorem 1.

3.1. First Proof.

Proof. PART 1. For $n \in \mathbb{N}_{\geqslant 5}$. Replace $(n+1)^2$ and n^2, respectively, into n, in Eq. (1), and obtain

$$\pi((n+1)^2) = 2 + \sum_{k=5}^{(n+1)^2} \frac{1 - \cos\left(\frac{2\pi}{k}\right) - \cos\left(\frac{2\pi\Gamma(k)}{k}\right) + \cos\left(\frac{2\pi\Gamma(k)}{k} + \frac{2\pi}{k}\right)}{2 - 2\cos\left(\frac{2\pi}{k}\right)} \tag{4}$$

$$= \left[2 + \sum_{k=5}^{2n} \frac{1 - \cos\left(\frac{2\pi}{k}\right) - \cos\left(\frac{2\pi\Gamma(k)}{k}\right) + \cos\left(\frac{2\pi\Gamma(k)}{k} + \frac{2\pi}{k}\right)}{2 - 2\cos\left(\frac{2\pi}{k}\right)}\right]$$

$$+ \sum_{k=2n+1}^{(n+1)^2} \frac{1 - \cos\left(\frac{2\pi}{k}\right) - \cos\left(\frac{2\pi\Gamma(k)}{k}\right) + \cos\left(\frac{2\pi\Gamma(k)}{k} + \frac{2\pi}{k}\right)}{2 - 2\cos\left(\frac{2\pi}{k}\right)}$$

$$= \pi(2n) + \sum_{k=2n+1}^{(n+1)^2} \frac{1 - \cos\left(\frac{2\pi}{k}\right) - \cos\left(\frac{2\pi\Gamma(k)}{k}\right) + \cos\left(\frac{2\pi\Gamma(k)}{k} + \frac{2\pi}{k}\right)}{2 - 2\cos\left(\frac{2\pi}{k}\right)}$$

and

$$\pi(n^2) = 2 + \sum_{k=5}^{n^2} \frac{1-\cos\left(\frac{2\pi}{k}\right) - \cos\left(\frac{2\pi\Gamma(k)}{k}\right) + \cos\left(\frac{2\pi\Gamma(k)}{k} + \frac{2\pi}{k}\right)}{2-2\cos\left(\frac{2\pi}{k}\right)} \tag{5}$$

$$= \left[2 + \sum_{k=5}^{n} \frac{1-\cos\left(\frac{2\pi}{k}\right) - \cos\left(\frac{2\pi\Gamma(k)}{k}\right) + \cos\left(\frac{2\pi\Gamma(k)}{k} + \frac{2\pi}{k}\right)}{2-2\cos\left(\frac{2\pi}{k}\right)}\right]$$

$$+ \sum_{k=n+1}^{n^2} \frac{1-\cos\left(\frac{2\pi}{k}\right) - \cos\left(\frac{2\pi\Gamma(k)}{k}\right) + \cos\left(\frac{2\pi\Gamma(k)}{k} + \frac{2\pi}{k}\right)}{2-2\cos\left(\frac{2\pi}{k}\right)}$$

$$= \pi(n) + \sum_{k=n+1}^{n^2} \frac{1-\cos\left(\frac{2\pi}{k}\right) - \cos\left(\frac{2\pi\Gamma(k)}{k}\right) + \cos\left(\frac{2\pi\Gamma(k)}{k} + \frac{2\pi}{k}\right)}{2-2\cos\left(\frac{2\pi}{k}\right)}.$$

Subtrating (4) to (5), it follows that

$$\pi((n+1)^2) - \pi(n^2) = \pi(2n) - \pi(n) \tag{6}$$

$$+ \sum_{k=2n+1}^{(n+1)^2} \frac{1-\cos\left(\frac{2\pi}{k}\right) - \cos\left(\frac{2\pi\Gamma(k)}{k}\right) + \cos\left(\frac{2\pi\Gamma(k)}{k} + \frac{2\pi}{k}\right)}{2-2\cos\left(\frac{2\pi}{k}\right)}$$

$$- \sum_{k=n+1}^{n^2} \frac{1-\cos\left(\frac{2\pi}{k}\right) - \cos\left(\frac{2\pi\Gamma(k)}{k}\right) + \cos\left(\frac{2\pi\Gamma(k)}{k} + \frac{2\pi}{k}\right)}{2-2\cos\left(\frac{2\pi}{k}\right)}$$

$$= \pi(2n) - \pi(n)$$

$$+ \sum_{k=2n+1}^{n^2} \frac{1-\cos\left(\frac{2\pi}{k}\right) - \cos\left(\frac{2\pi\Gamma(k)}{k}\right) + \cos\left(\frac{2\pi\Gamma(k)}{k} + \frac{2\pi}{k}\right)}{2-2\cos\left(\frac{2\pi}{k}\right)}$$

$$+ \sum_{k=n^2+1}^{(n+1)^2} \frac{1-\cos\left(\frac{2\pi}{k}\right) - \cos\left(\frac{2\pi\Gamma(k)}{k}\right) + \cos\left(\frac{2\pi\Gamma(k)}{k} + \frac{2\pi}{k}\right)}{2-2\cos\left(\frac{2\pi}{k}\right)}$$

$$- \sum_{k=n+1}^{2n} \frac{1-\cos\left(\frac{2\pi}{k}\right) - \cos\left(\frac{2\pi\Gamma(k)}{k}\right) + \cos\left(\frac{2\pi\Gamma(k)}{k} + \frac{2\pi}{k}\right)}{2-2\cos\left(\frac{2\pi}{k}\right)}$$

$$- \sum_{k=2n+1}^{n^2} \frac{1-\cos\left(\frac{2\pi}{k}\right) - \cos\left(\frac{2\pi\Gamma(k)}{k}\right) + \cos\left(\frac{2\pi\Gamma(k)}{k} + \frac{2\pi}{k}\right)}{2-2\cos\left(\frac{2\pi}{k}\right)}$$

$$= \pi(2n) - \pi(n)$$

$$+ \sum_{k=n^2+1}^{(n+1)^2} \frac{1-\cos\left(\frac{2\pi}{k}\right) - \cos\left(\frac{2\pi\Gamma(k)}{k}\right) + \cos\left(\frac{2\pi\Gamma(k)}{k} + \frac{2\pi}{k}\right)}{2-2\cos\left(\frac{2\pi}{k}\right)}$$

$$- \sum_{k=n+1}^{2n} \frac{1-\cos\left(\frac{2\pi}{k}\right) - \cos\left(\frac{2\pi\Gamma(k)}{k}\right) + \cos\left(\frac{2\pi\Gamma(k)}{k} + \frac{2\pi}{k}\right)}{2-2\cos\left(\frac{2\pi}{k}\right)}.$$

We easily noted the inequality

$$0=\min_{k\in\mathbb{N}_{\geqslant 5}}\left(\frac{1-\cos\left(\frac{2\pi}{k}\right)-\cos\left(\frac{2\pi\Gamma(k)}{k}\right)+\cos\left(\frac{2\pi\Gamma(k)}{k}+\frac{2\pi}{k}\right)}{2-2\cos\left(\frac{2\pi}{k}\right)}\right) \tag{7}$$

$$\leqslant\frac{1-\cos\left(\frac{2\pi}{k}\right)-\cos\left(\frac{2\pi\Gamma(k)}{k}\right)+\cos\left(\frac{2\pi\Gamma(k)}{k}+\frac{2\pi}{k}\right)}{2-2\cos\left(\frac{2\pi}{k}\right)}$$

$$\leqslant\max_{k\in\mathbb{N}_{\geqslant 5}}\left(\frac{1-\cos\left(\frac{2\pi}{k}\right)-\cos\left(\frac{2\pi\Gamma(k)}{k}\right)+\cos\left(\frac{2\pi\Gamma(k)}{k}+\frac{2\pi}{k}\right)}{2-2\cos\left(\frac{2\pi}{k}\right)}\right)=1.$$

From (6) and (7), it follows that

$$\pi(2n)-\pi(n)$$

$$+\sum_{k=n^2+1}^{(n+1)^2}\min_{k\in\mathbb{N}_{\geqslant 5}}\left(\frac{1-\cos\left(\frac{2\pi}{k}\right)-\cos\left(\frac{2\pi\Gamma(k)}{k}\right)+\cos\left(\frac{2\pi\Gamma(k)}{k}+\frac{2\pi}{k}\right)}{2-2\cos\left(\frac{2\pi}{k}\right)}\right)$$

$$-\sum_{k=n+1}^{2n}\min_{k\in\mathbb{N}_{\geqslant 5}}\left(\frac{1-\cos\left(\frac{2\pi}{k}\right)-\cos\left(\frac{2\pi\Gamma(k)}{k}\right)+\cos\left(\frac{2\pi\Gamma(k)}{k}+\frac{2\pi}{k}\right)}{2-2\cos\left(\frac{2\pi}{k}\right)}\right)$$

$$\leqslant\pi((n+1)^2)-\pi(n^2)<\pi(2n)-\pi(n)$$

$$+\sum_{k=n^2+1}^{(n+1)^2}\max_{k\in\mathbb{N}_{\geqslant 5}}\left(\frac{1-\cos\left(\frac{2\pi}{k}\right)-\cos\left(\frac{2\pi\Gamma(k)}{k}\right)+\cos\left(\frac{2\pi\Gamma(k)}{k}+\frac{2\pi}{k}\right)}{2-2\cos\left(\frac{2\pi}{k}\right)}\right)$$

$$-\sum_{k=n+1}^{2n}\max_{k\in\mathbb{N}_{\geqslant 5}}\left(\frac{1-\cos\left(\frac{2\pi}{k}\right)-\cos\left(\frac{2\pi\Gamma(k)}{k}\right)+\cos\left(\frac{2\pi\Gamma(k)}{k}+\frac{2\pi}{k}\right)}{2-2\cos\left(\frac{2\pi}{k}\right)}\right).$$

Thereafter, we encounter

$$\pi(2n)-\pi(n)+\sum_{k=n^2+1}^{(n+1)^2}0-\sum_{k=n+1}^{2n}0\leqslant\pi((n+1)^2)-\pi(n^2)$$

$$<\pi(2n)-\pi(n)+\sum_{k=n^2+1}^{(n+1)^2}1-\sum_{k=n+1}^{2n}1,$$

consequently,

$$\pi(2n)-\pi(n)\leqslant\pi((n+1)^2)-\pi(n^2)<\pi(2n)-\pi(n)+2n+1-n,$$

that is,

$$\pi(2n)-\pi(n)\leqslant\pi((n+1)^2)-\pi(n^2)<\pi(2n)-\pi(n)+n+1. \tag{8}$$

From Theorem 3 (weak Bertrand's postulate) and Eq. (8), obviously, I have

$$0<\pi(2n)-\pi(n)\leqslant\pi((n+1)^2)-\pi(n^2)<\pi(2n)-\pi(n)+n+1.$$

In other words, $\pi((n+1)^2)-\pi(n^2)>0$, and this prove the first part, for $n\in\mathbb{N}_{\geqslant 5}$.

PART 2. We calculate explicitly from $n=1$ at $n=4$. For $n=1$, thus $\pi(2^2)-\pi(1^2)=2-0=2>0$; for $n=2$, thus $\pi(3^2)-\pi(2^2)=4-2=2>0$; for $n=3$, thus $\pi(4^2)-\pi(3^2)=6-4=2>0$; for $n=4$, thus $\pi(5^2)-\pi(4^2)=9-6=3>0$. This completes the proof. □

3.2. Second Proof.

Proof. PART 1. For $n\in\mathbb{N}_{\geqslant 5}$. Replace $(n+1)^2$ and n^2, respectively, into n, in Eq. (2), and obtain

$$\pi((n+1)^2)=2+\sum_{k=5}^{(n+1)^2}\frac{\left(1-e^{-\frac{2\pi i\Gamma(k)}{k}}\right)\left(1-e^{-\frac{2\pi i}{k}}\right)}{2-2\cos(\frac{2\pi}{k})} \tag{9}$$

$$=\left[2+\sum_{k=5}^{2n}\frac{\left(1-e^{-\frac{2\pi i\Gamma(k)}{k}}\right)\left(1-e^{-\frac{2\pi i}{k}}\right)}{2-2\cos(\frac{2\pi}{k})}\right]+\sum_{k=2n+1}^{(n+1)^2}\frac{\left(1-e^{-\frac{2\pi i\Gamma(k)}{k}}\right)\left(1-e^{-\frac{2\pi i}{k}}\right)}{2-2\cos(\frac{2\pi}{k})}$$

$$=\pi(2n)+\sum_{k=2n+1}^{(n+1)^2}\frac{\left(1-e^{-\frac{2\pi i\Gamma(k)}{k}}\right)\left(1-e^{-\frac{2\pi i}{k}}\right)}{2-2\cos(\frac{2\pi}{k})}$$

and

$$\pi(n^2)=2+\sum_{k=5}^{n^2}\frac{\left(1-e^{-\frac{2\pi i\Gamma(k)}{k}}\right)\left(1-e^{-\frac{2\pi i}{k}}\right)}{2-2\cos(\frac{2\pi}{k})} \tag{10}$$

$$=\left[2+\sum_{k=5}^{n}\frac{\left(1-e^{-\frac{2\pi i\Gamma(k)}{k}}\right)\left(1-e^{-\frac{2\pi i}{k}}\right)}{2-2\cos(\frac{2\pi}{k})}\right]+\sum_{k=n+1}^{n^2}\frac{\left(1-e^{-\frac{2\pi i\Gamma(k)}{k}}\right)\left(1-e^{-\frac{2\pi i}{k}}\right)}{2-2\cos(\frac{2\pi}{k})}$$

$$=\pi(n)+\sum_{k=n+1}^{n^2}\frac{\left(1-e^{-\frac{2\pi i\Gamma(k)}{k}}\right)\left(1-e^{-\frac{2\pi i}{k}}\right)}{2-2\cos(\frac{2\pi}{k})}.$$

Subtrating (9) to (10), it follows that

$$\pi((n+1)^2)-\pi(n^2)=\pi(2n)-\pi(n) \tag{11}$$

$$+\sum_{k=2n+1}^{(n+1)^2}\frac{\left(1-e^{-\frac{2\pi i\Gamma(k)}{k}}\right)\left(1-e^{-\frac{2\pi i}{k}}\right)}{2-2\cos(\frac{2\pi}{k})}-\sum_{k=n+1}^{n^2}\frac{\left(1-e^{-\frac{2\pi i\Gamma(k)}{k}}\right)\left(1-e^{-\frac{2\pi i}{k}}\right)}{2-2\cos(\frac{2\pi}{k})}$$

$$=\pi(2n)-\pi(n)+\sum_{k=2n+1}^{n^2}\frac{\left(1-e^{-\frac{2\pi i\Gamma(k)}{k}}\right)\left(1-e^{-\frac{2\pi i}{k}}\right)}{2-2\cos(\frac{2\pi}{k})}+\sum_{k=n^2+1}^{(n+1)^2}\frac{\left(1-e^{-\frac{2\pi i\Gamma(k)}{k}}\right)\left(1-e^{-\frac{2\pi i}{k}}\right)}{2-2\cos(\frac{2\pi}{k})}$$

$$-\sum_{k=n+1}^{2n}\frac{\left(1-e^{-\frac{2\pi i\Gamma(k)}{k}}\right)\left(1-e^{-\frac{2\pi i}{k}}\right)}{2-2\cos(\frac{2\pi}{k})}-\sum_{k=2n+1}^{n^2}\frac{\left(1-e^{-\frac{2\pi i\Gamma(k)}{k}}\right)\left(1-e^{-\frac{2\pi i}{k}}\right)}{2-2\cos(\frac{2\pi}{k})}$$

$$=\pi(2n)-\pi(n)+\sum_{k=n^2+1}^{(n+1)^2}\frac{\left(1-e^{-\frac{2\pi i\Gamma(k)}{k}}\right)\left(1-e^{-\frac{2\pi i}{k}}\right)}{2-2\cos(\frac{2\pi}{k})}-\sum_{k=n+1}^{2n}\frac{\left(1-e^{-\frac{2\pi i\Gamma(k)}{k}}\right)\left(1-e^{-\frac{2\pi i}{k}}\right)}{2-2\cos(\frac{2\pi}{k})}.$$

We easily noted the inequality

$$0=\min_{k\in\mathbb{N}_{\geqslant 5}}\left(\frac{\left(1-e^{-\frac{2\pi i\Gamma(k)}{k}}\right)\left(1-e^{-\frac{2\pi i}{k}}\right)}{2-2\cos(\frac{2\pi}{k})}\right)\leqslant\frac{\left(1-e^{-\frac{2\pi i\Gamma(k)}{k}}\right)\left(1-e^{-\frac{2\pi i}{k}}\right)}{2-2\cos(\frac{2\pi}{k})} \tag{12}$$

$$\leqslant\max_{k\in\mathbb{N}_{\geqslant 5}}\left(\frac{\left(1-e^{-\frac{2\pi i\Gamma(k)}{k}}\right)\left(1-e^{-\frac{2\pi i}{k}}\right)}{2-2\cos(\frac{2\pi}{k})}\right)=1.$$

From (11) and (12), it follows that

$$\pi(2n)-\pi(n)+\sum_{k=n^2+1}^{(n+1)^2}\min_{k\in\mathbb{N}_{\geqslant 5}}\left(\frac{\left(1-e^{-\frac{2\pi i\Gamma(k)}{k}}\right)\left(1-e^{-\frac{2\pi i}{k}}\right)}{2-2\cos\left(\frac{2\pi}{k}\right)}\right)$$

$$-\sum_{k=n+1}^{2n}\min_{k\in\mathbb{N}_{\geqslant 5}}\left(\frac{\left(1-e^{-\frac{2\pi i\Gamma(k)}{k}}\right)\left(1-e^{-\frac{2\pi i}{k}}\right)}{2-2\cos\left(\frac{2\pi}{k}\right)}\right)\leqslant\pi((n+1)^2)-\pi(n^2)<\pi(2n)-\pi(n)$$

$$+\sum_{k=n^2+1}^{(n+1)^2}\max_{k\in\mathbb{N}_{\geqslant 5}}\left(\frac{\left(1-e^{-\frac{2\pi i\Gamma(k)}{k}}\right)\left(1-e^{-\frac{2\pi i}{k}}\right)}{2-2\cos\left(\frac{2\pi}{k}\right)}\right)-\sum_{k=n+1}^{2n}\max_{k\in\mathbb{N}_{\geqslant 5}}\left(\frac{\left(1-e^{-\frac{2\pi i\Gamma(k)}{k}}\right)\left(1-e^{-\frac{2\pi i}{k}}\right)}{2-2\cos\left(\frac{2\pi}{k}\right)}\right).$$

Therefrom, we find

$$\pi(2n)-\pi(n)+\sum_{k=n^2+1}^{(n+1)^2}0-\sum_{k=n+1}^{2n}0\leqslant\pi((n+1)^2)-\pi(n^2)$$

$$<\pi(2n)-\pi(n)+\sum_{k=n^2+1}^{(n+1)^2}1-\sum_{k=n+1}^{2n}1,$$

consequently,

$$\pi(2n)-\pi(n)\leq\pi((n+1)^2)-\pi(n^2)<\pi(2n)-\pi(n)+2n+1-n,$$

that is,

$$\pi(2n)-\pi(n)\leqslant\pi((n+1)^2)-\pi(n^2)<\pi(2n)-\pi(n)+n+1. \tag{13}$$

From Theorem 3 (weak Bertrand's postulate) and Eq. (13), obviously, I have

$$0<\pi(2n)-\pi(n)\leqslant\pi((n+1)^2)-\pi(n^2)<\pi(2n)-\pi(n)+n+1.$$

In other words, $\pi((n+1)^2)-\pi(n^2)>0$, and this prove the first part, for $n\in\mathbb{N}_{\geqslant 5}$.

PART 2. We calculate explicitly from $n=1$ at $n=4$. For $n=1$, thus $\pi(2^2)-\pi(1^2)=2-0=2>0$; for $n=2$, thus $\pi(3^2)-\pi(2^2)=4-2=2>0$; for $n=3$, thus $\pi(4^2)-\pi(3^2)=6-4=2>0$; for $n=4$, thus $\pi(5^2)-\pi(4^2)=9-6=3>0$. This completes the proof. □

4. A inedited short proof for theorem 1

Now, we present an unpublished proof for Legendre's conjecture.

Proof. For $n\in\mathbb{N}_{\geqslant 1}$, we set $(n+1)^2$ and n^2, respectively, into n, in Eq. (3), and obtain

$$\pi((n+1)^2)>\frac{2\log(n+1)}{2\log 2}=\frac{\log(n+1)}{\log 2} \tag{14}$$

and

$$\pi(n^2)\geqslant\frac{2\log n}{2\log 2}=\frac{\log n}{\log 2}. \tag{15}$$

Subtracting (14) to (15), we meet

$$\pi((n+1)^2)-\pi(n^2)>\frac{\log(n+1)-\log n}{\log 2}>0,$$

since $\log(n+1)>\log n$. Hence, $\pi((n+1)^2)-\pi(n^2)>0$. This gives us the desired result. □

References

[1] Hardy, G. H. and Wright, E. M., *An introduction to the theory of numbers*. Edited and revised by D. H. Heath-Brown and J. H. Silverman. With a foreword by Andrew Wiles. 6th ed. Oxford: Oxford University Press. xxi, 621 p. (2008). MSC(2000): 11-01-11Axx-11Mxx, *Reviewer*: Karl-Bernhard Gundlach (Malburg). Zbl 1159.11001

[2] Dickson, Leonard Eugene, *History of the theory of numbers*. Vol. I: Divisibility and primality. Reprint of the 1919 original published by Carnegie Institution, Washington, DC. Mineola, NY: Dover Publications, xii, 486 p. (2005). MSC2000: 11-00-01A75. Zbl 1214.11001.

[3] Guedes, Edigles. *An elementary proof of Legendre's conjecture*, available in viXra:1307.0142.

[4] Gandhi, Raja Rama and Guedes, Edigles, *An elementary proof of Legendre's conjecture*, Asia Journal of Mathematics and Physics, Volume 2013, Article ID amp0041, p. 1-7, ISSN 2308-3131, http://scienceasia.asia.

[5] Ramanujan, Srinivasa, *Collected papers of Srinivasa Ramanujan*. Edited by G. H. Hardy, P. V. Seshu Aiyar and B. M. Wilson. With a new preface and commentary by Bruce C.Berndt. Third printing of the 1926 original. Providence, RI: AMS Chelsea Publishing, xxxviii, 426 p. (2000). MSC 2000: 11-03-01A75-33-03. Zbl 1110.11001.

E-mail adress: edigles.guedes@gmail.com