A Collection Of Cuboid Parametric Formulas

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1. Nicholas Saunderson (1740) and Leonhard Euler (1772)

This system was found independently by Saunderson, an English mathematician who was blind from birth, and Euler in Switzerland, who was also blind in his later years. When Euler still had one good eye, Frederick the Great nicknamed him “Cyclops”.

$$\begin{aligned} x &= 8t(t^4-1) \\ y &= 2t(3t^2-1)(t^2-3) \\ z &= (t^2-1)(t^2+4t+1)(t^2-4t+1) \\ \sqrt{x^2+y^2} &= 2t(5t^4-6t^2+5) \\ \sqrt{x^2+z^2} &= (t^2-1)(t^4+18t^2+1) \\ \sqrt{y^2+z^2} &= (t^2+1)^3 \end{aligned}$$

Note that if $t = h/k$ then

$$\begin{aligned} x\cdot k^6 &= 8hk(h^4-k^4) \;=\; 4pqr \\ y\cdot k^6 &= 2hk(3h^2-k^2)(h^2-3k^2) \;=\; q(r^2-4p^2) \\ z\cdot k^6 &= (h^2-k^2)(h^2+4hk+k^2)(h^2-4hk+k^2) \;=\; p(r^2-4q^2) \end{aligned}$$

where $(p,q,r) = (h^2-k^2, 2hk, h^2+k^2)$. Since $p^2+q^2=r^2$ we have

$$(x^2+y^2+z^2)\cdot k^{12} \;=\; r^2(p^4+18p^2q^2+q^4)\ .$$

In 1914 H. C. Pocklington proved that $p^4+18p^2q^2+q^4$ is never a perfect square when p and q are nonzero integers. Hence no Saunderson–Euler cuboid (x,y,z) can be a perfect cuboid. In 1977 E. Z. Chein proved that the dual (xy, xz, yz) also cannot be perfect.

It is easy to find the parametric formulas for the duals of the Saunderson–Euler cuboids. We have

$$\begin{aligned} xy &= 16t^2(t^4-1)(3t^2-1)(t^2-3) \\ xz &= 8t(t^2-1)(t^4-1)(t^4+4t+1)(t^4-4t+1) \\ yz &= 2t(t^2-1)(3t^2-1)(t^2-3)(t^2+4t+1)(t^2-4t+1) \end{aligned}$$

and since $\gcd(xy, xz, yz) = 2t(t^2-1)$ the dual (xy, xz, yz) reduces to

$$\begin{aligned}
Z &= 8t(t^2+1)(3t^2-1)(t^2-3) \\
Y &= 4(t^4-1)(t^2+4t+1)(t^2-4t+1) \\
X &= (3t^2-1)(t^2-3)(t^2+4t+1)(t^2-4t+1) \\
\sqrt{Z^2+Y^2} &= 4(t^2+1)^4 \\
\sqrt{Z^2+X^2} &= (t^2-3)(3t^6+53t^4-15t^2-1) \\
\sqrt{Y^2+X^2} &= (t^2+4t+1)(t^2-4t+1)(5t^4-6t^2+5)\ .
\end{aligned}$$

This parametric system has degree eight whereas the previous system for Saunderson–Euler cuboids had degree six. Note that the degree of any system is the maximum degree of the polynomials in t. All parametric systems for cuboids have even degree, but the Saunderson–Euler system is the only one known of degree six.

From a practical standpoint, it is much easier to compute the dual (xy, xz, yz) of a body cuboid directly from the original (x, y, z) so no more dual body cuboid formulas will be listed below.

2. Rignaux (1947)

$$\begin{aligned}
x &= 32t(t^2-2t-7)(t^3+5t^2+3t+7) \\
y &= (t-1)^2(t+7)(t^2-2t-7)(t^3+5t^2+19t+7) \\
z &= 8t(t^2-1)(t+3)(t+7)(t^2+7) \\
\sqrt{x^2+y^2} &= (t^2-2t-7)(t^6+10t^5+31t^4+44t^3+335t^2+42t+49) \\
\sqrt{x^2+z^2} &= 8t(t^6+10t^5+35t^4+28t^3+34t^2+154t+245) \\
\sqrt{y^2+z^2} &= (t-1)(t+7)(t^6+2t^5+31t^4+124t^3+207t^2+98t+49)
\end{aligned}$$

$$\begin{aligned}
x &= 8(t^2-4)(5t-4)(3t^2+2)(5t^3-11t^2+8t+2) \\
y &= 7t(t^2-4)(5t^2-8t+6)95t^3-32t^2+22t+16) \\
z &= 28(2t-3)((3t^2+2)(5t^2-6)(t^2-4t+2) \\
\sqrt{x^2+y^2} &= (t^2-4)(625t^6-2120t^5+3660t^4-2976t^3+612t^2+352t+128) \\
\sqrt{x^2+z^2} &= 4(3t^2+2)(50t^6-150t^5+17t^4+164t^3+292t^2-608t+260) \\
\sqrt{y^2+z^2} &= 7(25t^8-200t^7+584t^6-352t^5-548t^4+96t^3+976t^2-768t+288)
\end{aligned}$$

3. Andrew Bremner (1988)

$$
\begin{aligned}
x &= 4(2t-1)(t^2-2t-1)(t^3+t^2-t+1)\\
y &= (t-1)^2(t+3)(t^2-2t-1)(t^3+t^2+3t-1)\\
z &= 2t(t^2-1)(t+3)(2t-1)(t^2-t+2)\\
\sqrt{x^2+y^2} &= (t^2-2t-1)(t^6+2t^5-t^4+19t^2-18t+5)\\
\sqrt{x^2+z^2} &= 2(2t-1)(t^6+2t^5-4t^3+5t^2+2t+2)\\
\sqrt{y^2+z^2} &= (t-1)(t+3)(t^6-2t^5+9t^4+t^2-2t+1)
\end{aligned}
$$

$$
\begin{aligned}
x &= 4(2t^2+2t-1)(3t^2-2t+1)(t^3+3t^2-t-1)\\
y &= (t+1)(t^2+6t-3)(3t^2-2t+1)(t^3-t^2-5t+1)\\
z &= 2t(t-1)(t^2+t+2)(t^2+6t-3)(2t^2+2t-1)\\
\sqrt{x^2+y^2} &= (3t^2-2t+1)(t^6+6t^5+23t^4+24t^3-5t^2-14t+5)\\
\sqrt{x^2+z^2} &= 2(2t^2+2t-1)(t^6+t6^5+16t^4-20t^3+5t^2-2t+2)\\
\sqrt{y^2+z^2} &= (t^2+6t-3)(5t^6+2t^5-7t^4+8t^3+13t^2-6t+1)
\end{aligned}
$$

$$
\begin{aligned}
x &= t(t^2-4t-8)(3t^2+8t+6)(t^3+4t^2-4)\\
y &= 2(t+1)(t+2)(t^2+t+2)(t^2-4t-8)(2t^2+2t-1)\\
z &= 4(2t^2+2t-1)(3t^2+8t+6)(t^3-4t-2)\\
\sqrt{x^2+y^2} &= (t^2-4t-8)(5t^6+28t^5+58t^4+44t^3+2t^2+8)\\
\sqrt{x^2+z^2} &= (3t^2+8t+6)(t^6+8t^4+28t^3+16t^2+8)\\
\sqrt{y^2+z^2} &= 2(2t^2+2t-1)(t^6+t^4+44t^3+116t^2+112t+40)
\end{aligned}
$$

$$
\begin{aligned}
x &= t^2(t-4)(t^2-2)(t^3-4t^2+8t-4)\\
y &= 2t(t-1)(t-2)(t-4)(2t-1)(t^2-t+2)\\
z &= 4(2t-1)(t^2-2)(t^3-4t^2+4t-2)\\
\sqrt{x^2+y^2} &= t(t-4)(t^6-4t^5+14t^4-36t^3+50t^2-32t+8)\\
\sqrt{x^2+z^2} &= (t^2-2)(t^6-8t^5+24t^4-36t^3+48t^2-32t+8)\\
\sqrt{y^2+z^2} &= 2(2t-1)(t^6-8t^5+25t^4-36t^3+28t^2-16t+8)
\end{aligned}
$$

$$
\begin{aligned}
x &= 8t(t-2)(t+2)(t-4)(3t+2)(t^2-t+2)\\
y &= (t-2)^2(3t+2)(t^2+4t-4)(t^3-6t^2-4t-8)\\
x &= 4t^2(t-4)(t^2+4t-4)(t^3-2t^2+4t+8)\\
\sqrt{x^2+y^2} &= (t-2)(3t+2)(t^6-4t^5+4t^4+144t^2-64t+64)\\
\sqrt{x^2+z^2} &= 4t(t-4)(t^6+2t^5+10t^4-16t^3+32t+32)\\
\sqrt{y^2+z^2} &= (t^2+4t-4)(5t^6-36t^5+76t^4-16t^2+64t+64)
\end{aligned}
$$

4. Allan J. MacLeod (1991)

$$\begin{aligned}
x &= 8t^2(t-1)^2(2t-1)(2t^2-2t+1)^2 \\
y &= 2t(t-1)(16t^8-64t^7+112t^6-112t^5+84t^4-56t^3+28t^2-8t+1) \\
Z &= -(2t-1)^2(768t^{16}-6144t^{15}+23040t^{14}-53760t^{13}+86656t^{12} \\
&\quad -100608t^{11}+84480t^{10}-49280t^9+16656t^8+960t^7 \\
&\quad -5376t^6+3936t^5-1784t^4+560t^3-120t^2+16-1) \\
\sqrt{x^2+y^2} &= 2t(t-1)(16t^8-64t^7+144t^6-208t^5+18t^4-104t^3+36t^2-8t+1) \\
\sqrt{x^2+Z} &= (2t-1)(16t^8-64t^7+112t^6-112t^5+84t^4-56t^3+28t^2-8t+1) \\
\sqrt{y^2+Z} &= (2t^2-2t+1)(16t^8-64t^7+80t^6-16t^5-52t^4+56t^3-28t^2+8t-1) \\
\sqrt{x^2+y^2+Z} &= (2t^2-2t+1)(16t^8-64t^7+112t^6-112t^5+84t^4-56t^3+28t^2-8t+1)
\end{aligned}$$

This is the only example I have seen of a parametric system for edge cuboids. Note that $Z = z^2$ where the edge z need not be rational — indeed, if z happened to be rational for some rational t then (x, y, z) would be a perfect rational cuboid ! !

Note also that if

$$A = 16t^8 - 64t^7 + 112t^6 - 112t^5 + 84t^4 - 56t^3 + 28t^2 - 8t + 1$$

then $y = 2t(t-1)A$, $\sqrt{x^2+Z} = (2t-1)A$, and $\sqrt{x^2+y^2+Z} = (2t^2-2t+1)A$.

MacLeod investigated the roots of the 16th degree polynomial in the formula for Z and determined that Z is positive for $-0.5291 < t < 0.2571$ and $0.7429 < t < 1.5291$. Otherwise $Z = z^2$ is negative and z would be imaginary.

5. Norihito Narumiya and Hironori Shiga (2001)

$$\begin{aligned}
x &= 2(t^2-5)(t^2-5t+5)(t^2-4t+5)^2 \\
y &= 4t(t-2)9(2t-5)(t^2-5t+5)(t^2-4t+5) \\
z &= t(t-1)(t-2)(t-3)(t-5)(2t-5)(3t-5) \\
\sqrt{x^2+y^2} &= 2(t^2-5t+5)(t^2-4t+5)(t^4-4t^3+8k^2-20k+25) \\
\sqrt{x^2+z^2} &= 2t^8-26t^7+141t^6-446t^5+1066t^4-2230t^3+3525t^2-3250t+1250 \\
\sqrt{y^2+z^2} &= t(t-2)(2t-5)(5t^4-48t^3+166t^2-240t+125)
\end{aligned}$$

$$\begin{aligned}
x &= 8t(2t^2+1)(48t^8-16t^6+4t^2-3) \\
y &= (4t^4-12t^2+1)(48t^8-16t^6+4t^2-3) \\
z &= 4(64t^{12}-80t^{10}+80t^8-120t^6+20t^4-5t^2+1) \\
\sqrt{x^2+y^2} &= (4t^2+20t^2+1)(48t^8-16t^6+4t^2-3) \\
\sqrt{x^2+z^2} &= 4(64t^{12}+208t^{10}-112t^8+56t^6-28t^4+13t^2+1) \\
\sqrt{y^2+z^2} &= 320t^{12}-640t^{10}+560t^8-64t^6+140t^4-40t^2+5
\end{aligned}$$

6. Terry Raines (2004)

$$\begin{aligned}
x &= 8t(t^2-1)(t^2-25)(t+3)(3t+5)\\
y &= (t-1)(t+3)(t-5)(3t+5)(t^2+2t+5)(t^2+10t+5)\\
z &= 4(5-t^2)(t^2+10t+5)(t^2+2t+5)^2\\
\sqrt{x^2+y^2} &= (t-1)(t+3)(t-5)(3t+5)(t^4+12t^3+62t^2+60t+25)\\
\sqrt{x^2+z^2} &= 4(t^8+14t^7+72t^6+26t^5-322t^4+130t^3+1800t^2+1750t+625)\\
\sqrt{y^2+z^2} &= (t^2+2t+5)(t^2+10t+5)(5t^4+4t^3-26t^2+20t+125)
\end{aligned}$$

$$\begin{aligned}
x &= t(t+2)(2t+1)(t^2+2t+3)(3t^2+2t+1)\\
y &= 4t(t+2)(2t+1)(t^2-1)(t^2+t+1)\\
z &= 2(t^2+t+1)(t^2+4t+1)(t^2-1)^2\\
\sqrt{x^2+y^2} &= t(t+2)(2t+1)(t^2+1)(5t^2+8t+1)\\
\sqrt{y^2+z^2} &= 2(t^2-1)(t^2+t+1)(t^4+4t^3+8t^2+4t+1)\\
\sqrt{x^2+y^2+z^2} &= 2t^8+10t^7+33t^6+70t^5+94t^4+70t^3+33t^2+10t+2
\end{aligned}$$

7. Mamuka Meskhishvili (2015)

$$\begin{aligned}
x &= 16t^2(t^4-9)\\
y &= (t^4-10t^2+9)(t^4+2t^2+9)\\
z &= 4t(t^2+3)(t^4-10t^2+9)\\
\sqrt{x^2+z^2} &= 4t(t^2+3)(t^4-2t^2+9)\\
\sqrt{y^2+z^2} &= (t^4-1)(t^4-81)\\
\sqrt{x^2+y^2+z^2} &= t^8+46t^4+81
\end{aligned}$$

$$\begin{aligned}
x &= 16t^2(t^4-9)(t^4-2t^2+9)\\
y &= (t^4-10t^2+9)(t^8+46t^4+81)\\
z &= 4t(t^2-3)(t^4-10t^2+9)(t^4+2t^2+9)\\
\sqrt{x^2+z^2} &= 4t(t^2-3)(t^8+46t^4+81)\\
\sqrt{y^2+z^2} &= (t^4-2t^2+9)(t^8-82t^4+81)\\
\sqrt{x^2+y^2+z^2} &= (t^4-2t^2+9)(t^8+46t^4+81)
\end{aligned}$$

$$\begin{aligned}
x &= (t^4-1)(t^4-81)\\
y &= 4t(t^2-3)(t^4+2t^2+9)\\
z &= 16t^2(t^4-9)\\
\sqrt{x^2+z^2} &= t^8+46t^4+81\\
\sqrt{y^2+z^2} &= 4t(t^2-3)(t^4+10t^2+9)\\
\sqrt{x^2+y^2+z^2} &= (t^4-2t^2+9)(t^4+10t^2+9)
\end{aligned}$$

Searching for Perfect Cuboids

All the parametric systems listed above were checked for possible misprints using the symbolic math software DERIVE. Thus they are valid for any number t, real or complex, and in particular for any rational real number $t = h/k$. It has long been known that the Saunderson-Euler formulas will never generate a perfect cuboid, so it is pointless to search using them. In February 2015 Mamuka Meskhishvili asked me to conduct a computer search for a perfect cuboid using three systems he had recently published. Consider his first system

$$\begin{aligned} x &= 16t^2(t^4-9) \\ y &= (t^4-10t^2+9)(t^4+2t^2+9) \\ z &= 4t(t^2+3)(t^4-10t^2+9) \end{aligned}$$

in which we know that $\sqrt{x^2+z^2}$, $\sqrt{y^2+z^2}$, and $\sqrt{x^2+y^2+z^2}$ can be represented by polynomials in t. Since the degree of all six polynomials is at most eight, the system has degree eight. The object is to find a rational $t = h/k$ such that $\sqrt{x^2+y^2}$ is a rational number. Elementary algebra gives

$$\begin{aligned} x \cdot k^8 &= X = 16h^2k^2(h^4-9k^4) \\ y \cdot k^8 &= Y = (h^4-10h^2k^2+9k^4)(h^4+2h^2k^2+9k^4) \\ z \cdot k^8 &= Z = 4hk(h^2+3k^2)(h^4-10h^2k^2+9k^4) \end{aligned}$$

so that $(x^2+y^2)\cdot k^{16} = X^2+Y^2$ and hence $\sqrt{x^2+y^2}$ will be rational if the integer X^2+Y^2 is a perfect square. The formulas for x and y contain only even powers of t so we need consider only positive $t = h/k$. For $m = 2, 3, 4, \cdots$ let

$$R_m = \{h/k : 0 < h, 0 < k, h+k = m, \gcd(h,k) = 1\}$$

so that every reduced fraction $t = h/k$ belongs to R_{h+k}. For each m the computer checks every $t = h/k \in R_m$ and determines whether X^2+Y^2 is a perfect square. If this miracle were to happen, the computer then goes to a subroutine which computes Z, $g = \gcd(X,Y,Z)$, and $(A,B,C) = (X/g, Y/g, Z/g)$ and checks that A^2+B^2, A^2+C^2, B^2+C^2, and $A^2+B^2+C^2$ are indeed perfect squares. If so, we have found history's first perfect cuboid! (By the way the ABC subroutine, while actually called at most one time, is very useful in debugging the program and making sure that the primitive cuboids (A,B,C) are in fact genuine face cuboids.)

A single desktop computer completed $m \leq 600000$ in about three weeks, testing over 10^{11} values of X^2+Y^2 using Ubasic. Since run time is $O(m^2)$ reaching $m = 1000000$ should take about four more weeks. It should be noted that different m can be tested on different computers, so that multiple computers would greatly accelerate the search process. On the other hand, it should also be noted that X and Y are $O(m^d)$ where d is the degree of the parametric system. At $m = 600000$ the values of X^2+Y^2 average 94 decimal digits, so the odds that X^2+Y^2 is a perfect square are roughly one in 10^{47}.

My computers kept track of the number N of values $X^2 + Y^2$ tested in each of the three Meskhishvili systems, but there is a simple way to estimate N. The number of fractions h/k with $h + k = m$ is $m - 1$ so the total for $2 \leq m \leq M$ is

$$\sum_{m=2}^{M}(m-1) \;=\; \sum_{m=1}^{M-1} m \;=\; M(M-1)/2$$

and the probability that two random integers h and k satisfy $\gcd(h, k) = 1$ is $6/\pi^2$. Thus

$$N \;=\; \frac{M(M-1)}{2} \cdot \frac{6}{\pi^2} \;=\; 3M(M-1)/\pi^2 \;\approx M^2/3$$

and this is quite close to the current counts on each of my three computers.

Any reader aware of other similar parametric formulas is cordially invited to send them to me. I will test them with DERIVE and if they pass I will publish them in an update and I promise to use your name!

Bibliography

[1] Andrew Bremner: The rational cuboid and a quartic surface, **Rocky Mountain Journal of Mathematics, Vol 18. No.1, Winter 1988, 105-121.**

[2] Maurice Kraïtchik: *Théorie des Nombres*, Vol. III (1947)

[3] Allan J. MacLeod: Parametric expressions for a “nearly-perfect” cuboid (1991), PCTex document at http: //maths.paisley.ac.uk

[4] Mamuka Meskhishvili: Parametric solutions for a nearly-perfect cuboid, **Unsolved Problems in Number Theory, Logic, and Cryptography.** (The website can be found at *unsolvedproblems.org* .)

[5] Norihiko Narumiya and Hironori Shiga: On certain rational cuboid problems, **Nihonkai Math. J. Vol 12(2001), 75-88**.