Scott Hampton Contact: technocrat-ip@live.com

Forward

As of the date of this paper, there appears to be only conjecture as to the existence of more than three solutions to the Brocard Problem. This paper will conclusively demonstrate that no more than the three known solutions exist.

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Background

In the late 1800's, Henri Brocard posed a mathematical problem of the form,

$$n! + 1 = m^2 \tag{1}$$

Where the question was to find all integer values for n and m that satisfies the equation. There are three known solutions to this equation. The solutions, in the form of number pairs representing (n,m) are (4,5), (5,11), and (7,71).

Some mathematical solutions have generally demonstrated that only a finite number of solutions can exist; some have conjectured that there are no solutions other than the three known solutions; and numerical solutions to high values of n have not produces any new solutions.

Solution

This problem does not have a closed form solution. Therefore, patterns must be developed so that specific observations can be made without the need of performing an infinite number of calculations. This paper will demonstrate that as the factorial term (n!) becomes larger and larger that it diverges from the mathematical form necessary to be a solution of the Brocard problem. Moreover, this paper will show that significant divergence starts at n = 8, and therefore the three currently known solutions are the only solutions that can exist.

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Start with Equation 1 and rearrange it to the following form,

$$n! = m^2 - 1 \tag{2}$$

Then rearrange the right-hand side of Equation 2 to the product of two terms,

$$n! = (m+1)(m-1) \tag{3}$$

Since n! is the product of many numbers, assume those numbers can be broken into the product of two groups where X_1 and X_2 are integers,

$$n! = X_1 * X_2 \tag{4}$$

Equate the right-hand side of Equations 3 and 4,

$$(m+1)(m-1) = X_1 * X_2 \tag{5}$$

Solve for m as follows,

$$(m+1) = X_1 \longrightarrow m = X_1 - 1 \tag{6}$$

$$(m-1) = X_2 \longrightarrow m = X_2 + 1 \tag{7}$$

Equate the right-hand side of Equations 6 and 7,

$$X_1 - 1 = X_2 + 1 \tag{8}$$

Solve X_1 in terms of X_2,

$$X_1 = X_2 + 2 \tag{9}$$

Substitute X_1 in Equation 9 into Equation 4,

$$n! = (X_2 + 2) * X_2 \quad or \quad n! = X * (X+2) \tag{10}$$

Note that Equation 10 indicates that in order for Equation 1 to be true that n! must be able to be represented by two integer numbers with a numerical difference of two between the numbers. To verify that Equation 10 is a solution, substitute n! in Equation 10 into Equation 1.

$$X * (X+2) + 1 = X^2 + 2X + 1 = (X+1)^2 = m^2 \tag{11}$$

Now take a look at Table 1 (on the next page) at the three known solutions to Brocard's problem relative to Equation 10. In each case, the n! product can be represented as the product of two integers that differ in magnitude by two. Keep in mind that X must be an integer since this is a requisite part of the problem.

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n	n!	X(X+2)
4	4*3*2*1	(4*1)(3*2) = (4)(6)
5	5*4*3*2*1	(5*2*1)(4*3) = (10)(12)
7	7*6*5*4*3*2*1	(7*5*2*1)(6*4*3) = (70)(72)

Table 1 – Brocard's Solutions and Equation 10

But what of all the other values of n? Table 2 (on the next page) shows n! for the first 28 values of n, and shows the relationship between the two closest numbers that form the product of n!. The things that should be noted with respect to Table 2 and the relationship between the product of the two numbers are as follows:

1) For the lowest value of n, the difference between the two numbers is zero (0).
2) For only the three (3) known solutions to Brocard's problem does the difference between the two numbers equate to two (2).
3) In general, as n increases, the difference between the two numbers increases geometrically.

Table 2 only considers the value of n up to 28 since the magnitude of n! increase at a rapid pace, and since there is no indication or expectation (based on what is shown) that the magnitude difference between the two closest values that form the product of n! will ever decrease (to any appreciable degree) as n continues to increase. For an intuitive understanding of why the magnitude difference between the two closest values that form the product of n! continually increases as n increases, consider the following:

The n! value can be broken down into the product of prime numbers. Two's that come from even numbers, and odd numbers that come from even and odd numbers. These prime numbers have to be split into two groups to create two integer numbers of similar magnitude that when multiplied together equal the value of n!. For larger values of n!, some of the prime numbers can be split evenly between the two groups, but eventually the odd numbers will cause the need for a redistribution of the primes in order to keep the magnitudes of the two groups as equitable as possible. Although the fractional difference between the magnitudes of the two groups will become smaller as n! becomes larger, this small fraction multiplied by a very larger number will produce a substantial numerical difference between the two numbers. Therefore, the larger values of n! will necessarily create groups with greater difference in magnitude.

Based on the understanding given above, it can be concluded that there are no other solutions to the Brocard problem other than the three known solutions.

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Contact: technocrat-ip@live.com

n	n!	n! + 1	m	X_2	X_1	$X_1 - X_2$
1	1	2	1.41421356237309504880168872421	1	1	0
2	2	3	1.73205080756887729352744634151	1	2	1
3	6	7	2.64575131106459059050161575364	2	3	1
4	24	25	5	4	6	2
5	120	121	11	10	12	2
6	720	721	26.8514431641951043942027192538	24	30	6
7	5040	5041	71	70	72	2
8	40320	40321	200.800896412341745302705932848	192	210	18
9	362880	362881	602.396049123830160599763571263	576	630	54
10	3628800	3628801	1904 94120644181562100263554198	1890	1920	30
11	39916800	39916801	6317 97443806161659592300200717	6300	6336	36
12	479001600	479001601	21886.1052039873005815603370597	21600	22176	576
13	6227020800	6227020801	78911.474457140895575098505908	78848	78975	127
14	87178291200	87178291201	295259.701281769910280789923411	294840	295680	840
15	1307674368000	1307674368001	1143535.90586435019906650798751	1143072	1144000	928
16	20922789888000	20922789888001	4574143.62345576114508377798904	4572288	4576000	3712
17	355687428096000	355687428096001	18859677.3062531745959645292412	18849600	18869760	20160
18	6402373705728000	6402373705728001	80014834.2854498512020807444824	79968000	80061696	93696
19	121645100408832000	121645100408832001	348776576.634429395564539491958	348566400	348986880	420480
20	2432902008176640000	2432902008176640001	1559776268.62849788678955282268	1559376000	1560176640	800640
21	51090942171709440000	51090942171709440001	7147792818.1858656892148782722	7147140000	7148445696	1305696
22	1124000727777607680000	1124000727777607680001	33526120082.3717100758438700807	33522128640	33530112000	7983360
23	25852016738884976640000	25852016738884976640001	160785623545.40587668878028253	160758097500	160813154304	55056804
24	620448401733239439360000	620448401733239439360001	787685471322.938290082864725582	787652812800	787718131200	65318400
25	15511210043330985984000000	15511210043330985984000001	3938427356614.69145041432058101	3938264064000	3938590656000	326592000
26	403291461126605635584000000	403291461126605635584000001	20082117944245.9613193062398048	20080974513600	20083261440000	2286926400
27	10888869450418352160768000000	10888869450418352160768000001	104349745809073.977642050293	104348440350000	104351051284480	2610934480
28	304888344611713860501504000000	304888344611713860501504000001	552166953567228.487485881686753	552160113120000	552173794099200	13680979200

Table 2 - Brocard's Problem and n! Product Relationship (See Equation 4 and 10)

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Conclusions

The Brocard problem is evaluated by redefining n! as the product of two integer numbers. A set difference between the two numbers is required in order for (n! + 1) to be an exact square. It was shown that the product of the two integer numbers that form n! are of the correct relation to one another for only the three known solutions to the Brocard problem. For any other values of n, the relationship between the two integers that form n! will always be such that a solution does not exist.