The 3×3 Magic Square with All Entries as Perfect Squares Problem

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I would like to thank our Lord Jesus Christ for the beauty of math as well as the solution to this problem. All glory is given to God for the solution to this problem.

Abstract

This article will prove that a 3×3 magic square with all entries as unique integer perfect squares does not exist.

1 Introduction

A 3×3 magic square is a square whose entries are all positive integers and the sums of the rows, columns, and main diagonals have the same value. For example,

$$\begin{matrix} 4 & 9 & 2 \\ 3 & 5 & 7 \\ 8 & 1 & 6 \end{matrix}$$

is a magic square since the sum of each row, column, and main diagonal is 15.

Rows:	$4+9+2=15$	$3+5+7=15$	$8+1+6=15$
Columns:	$4+3+8=15$	$9+5+1=15$	$2+7+6=15$
Main Diagonals:	$4+5+6=15$	$2+5+8=15$	

The goal of this paper is to show that a 3×3 magic square with all perfect squares as entries does not exist.

2 Theorem and Proof

Theorem 1. *A 3×3 magic square with all entries as unique integer perfect squares does not exist.*

Proof. Assume a magic square exists that has the form given below.

$$\begin{matrix} a^2 & b^2 & c^2 \\ d^2 & e^2 & f^2 \\ g^2 & h^2 & i^2 \end{matrix} \tag{1}$$

In order for this to be a magic square, the sums of the rows, columns, and main diagonals must add up to the same value.

The sums of the rows in (1) are:

$$a^2+b^2+c^2 \qquad d^2+e^2+f^2 \qquad g^2+h^2+i^2.$$

The sums of the columns in (1) are:

$$a^2+d^2+g^2 \qquad b^2+e^2+h^2 \qquad c^2+f^2+i^2.$$

The sums of the main diagonals in (1) are:

$$a^2 + e^2 + i^2 \qquad\qquad c^2 + e^2 + g^2.$$

By the assumption that (1) is a magic square, all the above eight equations must have the same value and, therefore, must equal each other. This fact gives us several conditions that *must be* met in order to have a magic square. Two of these conditions are given by setting the sum of row 1 equal to the sum of column 1 and the sum of column 2 equal to the sum of main diagonal 2.

$$\begin{aligned} a^2 + b^2 + c^2 = a^2 + d^2 + g^2 \qquad &\Longrightarrow \qquad b^2 = d^2 + g^2 - c^2 \\ b^2 + e^2 + h^2 = c^2 + e^2 + g^2 \qquad &\Longrightarrow \qquad b^2 = c^2 + g^2 - h^2 \end{aligned}$$

This then gives the following condition.

$$d^2 + g^2 - c^2 = c^2 + g^2 - h^2 \qquad \Longrightarrow \qquad d^2 = 2c^2 - h^2$$

This gives us a relation among d, c, and h, namely,

$$\begin{aligned} d &= 7 \\ c &= 5 \\ h &= 1. \end{aligned}$$

From this, we have the following two ratios that relate c to d and h to d that must hold in order for (1) to be a magic square.

$$\frac{c}{d} = \frac{5}{7} \qquad \Longrightarrow \qquad c^2 = \frac{25}{49}d^2 \tag{2}$$

$$\frac{h}{d} = \frac{1}{7} \qquad \Longrightarrow \qquad h^2 = \frac{1}{49}d^2 \tag{3}$$

By substituting these ratios into (1), we get a new magic square.

$$\begin{matrix} a^2 & b^2 & \frac{25}{49}d^2 \\ d^2 & e^2 & f^2 \\ g^2 & \frac{1}{49}d^2 & i^2 \end{matrix} \tag{4}$$

Setting the sum of row 1 equal to the sum of column 1 from (4), we see that

$$a^2 + b^2 + \frac{25}{49}d^2 = a^2 + d^2 + g^2 \qquad \Longrightarrow \qquad b^2 = \frac{24}{49}d^2 + g^2.$$

We have already found a value for d, namely, 7. However, if we use $d = 7$, there are no integers that satisfy the above relation that would be unique. However, choosing $d = 14$ will give us unique integers. This gives the relation

$$b^2 = 96 + g^2.$$

Since $d = 14$, this would make $c = 10$ and $h = 2$ from (2) and (3). Since all entries must be unique, we can choose $g = 5$ and $b = 11$. This would then give the following ratios.

$$\begin{aligned}
\frac{b}{d} &= \frac{11}{14} &&\implies && b^2 = \frac{121}{196}d^2\\
\frac{c}{d} &= \frac{10}{14} &&\implies && c^2 = \frac{100}{196}d^2\\
\frac{g}{d} &= \frac{5}{14} &&\implies && g^2 = \frac{25}{196}d^2\\
\frac{h}{d} &= \frac{2}{14} &&\implies && h^2 = \frac{4}{196}d^2.
\end{aligned}$$

Substituting these ratios into (1) we now get a new magic square given by the following.

$$\begin{matrix} a^2 & \frac{121}{196}d^2 & \frac{100}{196}d^2\\ d^2 & e^2 & f^2\\ \frac{25}{196}d^2 & \frac{4}{196}d^2 & i^2 \end{matrix} \tag{5}$$

Setting the sum of column 1 equal to the sum of row 3 in (5), we get

$$a^2 + d^2 + \frac{25}{196}d^2 = \frac{25}{196}d^2 + \frac{4}{196}d^2 + i^2 \qquad \implies \qquad a^2 + \frac{192}{196}d^2 = i^2.$$

With $d = 14$, this has a solution of $a = 8$ and $i = 16$, which are unique to the square. This gives the new ratios

$$\begin{aligned}
\frac{a}{d} &= \frac{8}{14} &&\implies && a^2 = \frac{64}{196}d^2\\
\frac{i}{d} &= \frac{16}{14} &&\implies && i^2 = \frac{256}{196}d^2.
\end{aligned}$$

Substituting these ratios into (5), we obtain the following new magic square.

$$\begin{matrix} \frac{64}{196}d^2 & \frac{121}{196}d^2 & \frac{100}{196}d^2\\ d^2 & e^2 & f^2\\ \frac{25}{196}d^2 & \frac{4}{196}d^2 & \frac{256}{196}d^2 \end{matrix} \tag{6}$$

This is no longer a magic square. If we set the sum of row 1 equal to the sum of column 3 in (6), we obtain what is given below.

$$\frac{64}{196}d^2 + \frac{121}{196}d^2 + \frac{100}{196}d^2 = \frac{100}{196}d^2 + f^2 + \frac{256}{196}d^2 \qquad \implies \qquad \frac{285}{196}d^2 = \frac{356}{196}d^2 + f^2 \qquad \implies \qquad f^2 = \frac{-71}{196}d^2 < 0.$$

By definition of a magic square, all entries must be positive integers, and since d^2 is positive, this implies the entry $f^2 < 0$, which is a contradiction. Therefore, a 3×3 magic square with all perfect entries does not exist.

□