Riemann Hypothesis and Primorial Number

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Abstract; In this paper we consider the Riemann hypothesis by the primorial numbers.
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1. Introduction and main result of paper

Let N be the set of the natural numbers. The function $\varphi(n)=n\cdot\prod_{p|n}\left(1-p^{-1}\right)$ is called the Euler's function of $n\in N$ ([3]). Here $p\,|\,n$ note p is the prime divisor of n. Robin showed in his paper [5] (also see [4])

[Robin Theorem] If the Riemann hypothesis (RH) is false, then there exist constants $0<\beta<1/2$ and $c>0$ such that $\sigma(n)\ge e^{\gamma}\cdot n\cdot\log\log n+c\cdot n\cdot\log\log n/(\log n)^{\beta}$ holds for infinitely many $n\in N$, where $\sigma(n)=\sum_{d|n}d$ is the divisor function of $n\in N$ ([5]) and $\gamma=0.577\cdots$ is Euler's constant ([3]).

From this we have

[Theorem 1] If there exists a constant $c_0\ge 1$ such that

$$n/\varphi(n)\le e^{\gamma}\cdot\log\log\left(c_0\cdot n\cdot\exp\left(\sqrt{\log n}\cdot(\log\log n)^2\right)\right) \qquad (*)$$

holds for any $n\ge 2$, then the RH is true.

For $n\in N$ $(n\ne 1)$ we define $\Phi_0(n)=\exp\left(\exp\left(e^{-\gamma}\cdot n/\varphi(n)\right)\right)/\left(n\cdot\exp\left(\sqrt{\log n}\cdot(\log\log n)^2\right)\right)$.

Then we give

[Theorem 2] For any $n\ge 2$ we have $\Phi_0(n)\le 24$.

[Corollary] For any $n\ge 5$ we have

$$\frac{n}{\varphi(n)}\le e^{\gamma}\cdot\log\log n+21.483\cdot\frac{(\log\log n)^2}{\sqrt{\log n}}.$$

2. Proof of Theorem 1

It is clear that $\sigma(n)\cdot\varphi(n)\le n^2$ for any $n\ge 2$. If (*) holds, but the RH is false, then

$$e^{\gamma}\cdot\log\log n+\frac{c\cdot\log\log n}{(\log n)^{\beta}}\le\frac{\sigma(n)}{n}\le\frac{n}{\varphi(n)}\le e^{\gamma}\cdot\log\log\left(c_0\cdot n\cdot\exp\left(\sqrt{\log n}\cdot(\log\log n)^2\right)\right)$$

holds for infinitely many $n\in N$. On the other hand, since $\log(1+t)\le t$ $(t>0)$, we have

$$\log\log\left(c_0\cdot n\cdot\exp\left(\sqrt{\log n}\cdot(\log\log n)^2\right)\right)=\log\left(\log n+\log c_0+\sqrt{\log n}\cdot(\log\log n)^2\right)=$$

$$=\log\log n+\log\left(1+\frac{\log c_0}{\log n}+\frac{(\log\log n)^2}{\sqrt{\log n}}\right)\le\log\log n+\frac{\log c_0}{\log n}+\frac{(\log\log n)^2}{\sqrt{\log n}}.$$

Therefore, for infinitely many $n\in N$ we have $e^{-\gamma}\cdot\frac{c\cdot\log\log n}{(\log n)^{\beta}}\le\frac{\log c_0}{\log n}+\frac{(\log\log n)^2}{\sqrt{\log n}}$. From this we have $0<e^{-\gamma}\cdot c\le\frac{1}{\log\log n}\cdot\frac{\log c_0}{(\log n)^{1-\beta}}+\frac{\log\log n}{(\log n)^{1/2-\beta}}\to 0\ (n\to\infty)$, but it is a contradiction. □

3. Reduction to the primorial number

Let $p_1=2,\ p_2=3,\ p_3=5,\cdots$ be first consecutive primes. Then p_m is $m-$th prime number. The number $\left(p_1\cdots p_m\right)$ is called the primorial number ([1,8]). Assume $n=q_1^{\lambda_1}\cdots q_m^{\lambda_m}$ is the prime factorization of $n\in N$. Here $q_1,\cdots,q_m$ are distinct primes and $\lambda_1,\cdots,\lambda_m$ are nonnegative integers ≥ 1. Put $\Im_m=p_1\cdots p_m$, then it is clear that $n\ge\Im_m$,

$$\frac{n}{\varphi(n)}=\prod_{i=1}^{m}\left(1-q_i^{-1}\right)^{-1}\le\prod_{i=1}^{m}\left(1-p_i^{-1}\right)^{-1}=\frac{\Im_m}{\varphi(\Im_m)}$$

and so $\Phi_0(n)\le\Phi_0(\Im_m)$. This shows that the boundedness of the function $\Phi_0(n)$ for $n\in N$ is reduced to one for the primorial numbers.

4. Some symbols

It is known $\sum_{p\le t}p^{-1}=\log\log t+b+E(t)$ by [6], where $E(t)=\mathrm{O}\left(\exp\left(-a_1\cdot\sqrt{\log t}\right)\right)$ $(a_1>0)$ and $b=\gamma+\sum_p\left[\log(1-1/p)+1/p\right]=0.26\cdots$ and t is a real number ≥ 2. Put $F_m=\Im_m/\varphi(\Im_m)$, then we have

$$\log(F_m)=-\sum_{i=1}^{m}\log(1-1/p_i)=-\sum_{i=1}^{m}\left[\log(1-1/p_i)+1/p_i\right]+\sum_{i=1}^{m}1/p_i=$$
$$=-\sum_{i=1}^{m}\left[\log(1-1/p_i)+1/p_i\right]+\log\log p_m+b+E(p_m)=$$
$$=-\sum_{i=1}^{m}\left[\log(1-1/p_i)+1/p_i\right]+\log\log p_m+\gamma+\sum_p\left[\log(1-1/p)+1/p\right]+E(p_m)=$$
$$=\log\log p_m+\gamma+E(p_m)+\varepsilon_0(p_m),$$

where $\varepsilon_0(p_m)=\sum_{p>p_m}\left[\log(1-1/p)+1/p\right]$. From this we have

$$\left(e^{-\gamma}\cdot F_m\right)=\log p_m\cdot e_0,\qquad \exp\left(e^{-\gamma}\cdot F_m\right)=p_m\cdot e_0',$$

where $e_0=\exp\left(E(p_m)+\varepsilon_0(p_m)\right)$ and $e_0'=\exp\left(\log p_m\cdot(e_0-1)\right)$. Similarly, we have

$$\left(e^{-\gamma}\cdot F_{m-1}\right)=\log p_{m-1}\cdot e_1,\qquad \exp\left(e^{-\gamma}\cdot F_{m-1}\right)=p_{m-1}\cdot e_1',$$

where $e_1=\exp\left(E(p_{m-1})+\varepsilon_0(p_{m-1})\right)$ and $e_1'=\exp\left(\log p_{m-1}\cdot(e_1-1)\right)$.

We recall the Chebyshev's function $\vartheta(t)=\sum_{p\le t}\log p$ ([3]). Then by the prime number theorem ([3]), it is known that $\vartheta(t)=t\cdot(1+\theta(t))$ where $\theta(t)=\mathrm{O}\left(\exp\left(-a_2\cdot\sqrt{\log t}\right)\right)(a_2>0)$. Then we see $\log\Im_m=p_m\cdot\alpha_0$ and $\log\Im_{m-1}=p_{m-1}\cdot\alpha_1$, where $\alpha_0=1+\theta(p_m)$ and $\alpha_1=1+\theta(p_{m-1})$.

Now we put $N_i=\sqrt{\log\Im_{m-i}}\cdot\left(\log\log\Im_{m-i}\right)^2$ $(i=0,1)$ and $C_m=\Phi_0(\Im_m)(m\ge 1)$.

5. Some numerical estimates

5.1. An estimate of e_1 and e_1'

We put $p=p_{m-1},\ p_0=p_m$ below. For the theoretical calculation we assume $p\ge e^{14}$. The discussion for $p\le e^{14}$ is supported by MATLAB. Since $\left(e^{-\gamma}\cdot F_{m-1}\right)=\log p\cdot e_1<\log p+1/\log p\,(p\ge 2)$ by (3.30) of [6], we respectively have $e_1<1+1/\log^2 p<1.0052\left(p\ge e^{14}\right)$, $e_1'<\exp(1/\log p)<1.075\ \left(p\ge e^{14}\right)$ and $e_1\cdot e_1'<1.08\left(p\ge e^{14}\right)$.

5.2. An estimate of $\left(e_1\cdot e_1'\right)$

Since if $e_1\le 1$ then $e_1'\le 1$, we have $e_1\cdot e_1'\le 1$. On the other hand, it is known that by (3.17), (3.20) of [6], $\left(-1/\log^2 t\right)\le E(t)=\sum_{p\le t}p^{-1}-b-\log\log t\le\left(1/\log^2 t\right)$ $(t>1)$. Hence, since $\varepsilon_0(p)<0$, if $e_1>1$, then we have $0<a:=E(p)+\varepsilon_0(p)<1/\log^2 p\le 0.0052\left(p\ge e^{14}\right)$ and so

$$e_1 = 1 + a + \sum_{n=2}^{\infty} a^n / n! \le 1 + a + a^2 / (2\cdot(1-a)) \le 1 + a + 0.51\cdot a^2 .$$

We have $e_1 \cdot e_1' = \exp(a + \log p\cdot(e_1 - 1)) \le 1 + b + b^2/(2\cdot(1-b))$, where

$$b = (1+\log p)\cdot a + 0.51\cdot \log p \cdot a^2 \le 0.113 \ (p \ge e^{14}).$$

Therefore we have

$$e_1\cdot e_1' \le 1 + (1+\log p)\cdot(E(p)+\varepsilon_0(p)) + 0.61\cdot(1+\log p)^2\cdot(E(p)+\varepsilon_0(p))^2 \ (e_1>1,\ p\ge e^{14}).$$

5.3. An estimate of $K_0 := p_0\cdot(e_0' - \alpha_0) - p\cdot(e_1' - \alpha_1)$

It is clear that $p_0\cdot\alpha_0 - p\cdot\alpha_1 = \log\Im_m - \log\Im_{m-1} = \log p_0$ and

$$E(p_0) - E(p) = \left(\sum_{i=1}^{m} 1/p_i - \log\log p_m - b\right) - \left(\sum_{i=1}^{m-1} 1/p_i - \log\log p_{m-1} - b\right) =$$

$$= 1/p_m - \log\log p_m + \log\log p_{m-1} = \frac{1}{p_0} - \log\left(\frac{\log p_0}{\log p}\right)$$

and $\varepsilon_0(p_0) - \varepsilon_0(p) = -\log(1 - 1/p_0) - 1/p_0$. From this we have

$$\frac{e_0}{e_1} = \left(\frac{\log p}{\log p_0}\right)\cdot\left(1 + \frac{1}{p_0 - 1}\right) \text{ and } \frac{e_0'}{e_1'} = \frac{p}{p_0}\cdot\exp\left(\frac{\log p\cdot e_1}{p_0 - 1}\right).$$

Thus we have

$$K_0 = p\cdot e_1'\cdot\left(\frac{p_0\cdot e_0'}{p\cdot e_1'} - 1\right) - \log p_0 = p\cdot e_1'\cdot\left(\exp\left(\frac{\log p\cdot e_1}{p_0-1}\right) - 1\right) - \log p_0 = \log p_0\cdot(\mu\cdot e_1' - 1),$$

where $\mu = \dfrac{p}{\log p_0}\cdot\left(\exp\left(\dfrac{\log p\cdot e_1}{p_0 - 1}\right) - 1\right)$. Hence we get

$$\mu \le \frac{p}{\log p}\cdot\left(\exp\left(\frac{\log p\cdot e_1}{p}\right) - 1\right) \le \frac{p}{\log p}\cdot\left(\frac{\log p\cdot e_1}{p} + \frac{1}{2}\cdot\left(\frac{\log p\cdot e_1}{p}\right)^2 \Big/ \left(1 - \frac{\log p\cdot e_1}{p}\right)\right) \le$$

$$\le e_1 + \frac{1}{2}\cdot\frac{\log p\cdot e_1}{p} \Big/ \left(1 - \frac{\log p\cdot e_1}{p}\right) \le e_1 + 0.503\cdot\frac{\log p}{p} \ (e_1 > 1,\ p \ge e^{14})$$

and $\mu\cdot e_1' - 1 \le (e_1\cdot e_1' - 1) + 0.55\cdot\dfrac{\log p}{p} \ (e_1>1,\ p\ge e^{14})$.

5.4. An estimate of $(1+\log p)\cdot E(p)$

Put $f(t) = t\cdot(\log t\cdot E(t) - \theta(t))$, $g(t) = \sqrt{t}\cdot\log^2(t\cdot\alpha_1)$ and $d(t) = \dfrac{f(t)}{g(t)}$ $(p \le t \le p+1)$, where t is a real number and $\alpha_1 = 1 + \theta(p)$ is a positive constant such that $(1 - 1/14) \le \alpha_1 \le (1 + 1/14)$. Then both $f(t)$ and $g(t)$ are continuously differentiable functions on the interval $(p,\ p+1)$. In fact, since the functions $\sum_{p\le t}(1/p) - b$ and $\vartheta(t) = \sum_{p\le t}\log p$ are constants on $(p,\ p+1)$, we have

$$E'(t) = \left(\sum_{p\le t} p^{-1} - b - \log\log t\right)' = \frac{-1}{t\cdot\log t},\quad \theta'(t) = \left(\frac{\vartheta(t)}{t} - 1\right)' = -\frac{\vartheta(t)}{t^2} = -\frac{1}{t} - \frac{\theta(t)}{t}$$

and hence $f'(t) = (1+\log t)\cdot E(t)$, where $f'(t)$ is the derivative of the function $f(t)$ and so on. Thus the function $d(t)$ is also continuously differentiable on the interval $(p,\ p+1)$, since $g(t) > 0$. Now we will arbitrary take $x_1,\ x_2$ such that $p < x_1 < x_2 < p+1$, and fix it. Then we have

$$d(x_2) - d(x_1) = \frac{1}{g_2}\cdot\left[(f(x_2) - f(x_1)) - d_1\cdot(g(x_2) - g(x_1))\right]$$

and hence $\int_{x_1}^{x_2} F(t)\cdot dt = 0$, where $g_2 = g(x_2)$, $d_1 = d(x_1)$, $F(t) = d'(t) - \frac{1}{g_2}\cdot\left(f'(t) - d_1\cdot g'(t)\right)$ and

$$g'(t) = \frac{\log^2(t\cdot\alpha_1)}{2\cdot\sqrt{t}}\cdot\left(1+\frac{4}{\log(t\cdot\alpha_1)}\right) \text{ for } t\in(x_1,\, x_2).$$

By the mean value theorem for integrals of [7], there exists a point ξ_0 such that $x_1 < \xi_0 < x_2$ and $\int_{x_1}^{x_2} F(t)\cdot dt = F(\xi_0)\cdot(x_2 - x_1) = 0$. On the other hand, since $d'(t) = \frac{1}{g(t)}\cdot\left(f'(t) - d(t)\cdot g'(t)\right)$ and $f'(t) = d(t)\cdot g'(t) + d'(t)\cdot g(t)$ for any $t\in(x_1,\, x_2)$, we have

$$F(t) = \left(\frac{1}{g(t)} - \frac{1}{g_2}\right)\cdot f'(t) - \left(\frac{d(t)}{g(t)} - \frac{d_1}{g_2}\right)\cdot g'(t) = \frac{1}{g_2}\cdot\left[\left(g_2 - g(t)\right)\cdot d'(t) - \left(d(t) - d_1\right)\cdot g'(t)\right]$$

and hence $F(\xi_0) = \frac{1}{g_2}\cdot\left[\left(g_2 - g(\xi_0)\right)\cdot d'(\xi_0) - \left(d(\xi_0) - d_1\right)\cdot g'(\xi_0)\right] = 0$.

5.4.1. Proof of $d''(t) < 0$

As above mentioned, since $\left(-1/\log^2 t\right) \le E(t) \le \left(1/\log^2 t\right)$ $(t > 1)$ and $t \ge p \ge e^{14}$, we easily see

$$f''(t) = \frac{E(t)}{t} - \frac{1}{t}\cdot\left(1+\frac{1}{\log t}\right) < 0,\quad g''(t) = -\frac{\log^2(t\cdot\alpha_1)}{4\cdot t\cdot\sqrt{t}}\cdot\left(1-\frac{8}{\log^2(t\cdot\alpha_1)}\right) < 0$$

and hence $f''(t) < g''(t)$ for any $t\in(x_1,\, x_2)$, where $f''(t)$ is the second-order derivative function of $f(t)$. From this we have $d''(t) < 0$ for any $t\in(x_1,\, x_2)$, where

$$d''(t) = \frac{1}{g(t)}\cdot\left[f''(t) - d(t)\cdot g''(t) - 2\cdot g'(t)\cdot d'(t)\right].$$

In fact, it is clear that $d''(t) < 0$ is equivalent to $A < 1 + 1/\log t$, where $\beta = 1 + \frac{4}{\log(t\cdot\alpha_1)}$ and

$$A = E(t) + \frac{f(t)}{4\cdot t}\cdot\left(1-\frac{8}{\log^2(t\cdot\alpha_1)}\right) - f'(t)\cdot\beta + \frac{f(t)}{2\cdot t}\cdot\beta^2.$$

On the other hand, it is known that $\left(-1/\log t\right) \le \theta(t) \le \left(1/\log t\right)(t \ge 41)$ by (3.15), (3.16) of [6]. And since $\alpha_1 \ge (1 - 1/14)$ and $t \ge p \ge e^{14}$, we have $\log(t\cdot\alpha_1) = \log t + \log\alpha_1 \ge 13.925 > 0$ and hence

$$|A| \le \frac{1}{\log^2 t} + \frac{1}{2\cdot\log t} + \left(\frac{2}{\log t} + \frac{1}{\log^2 t}\right)\cdot\left(1+\frac{4}{\log(t\cdot\alpha_1)}\right)^2 \le 0.3 < 1 \quad \left(t \ge p \ge e^{14}\right).$$

This shows that $d''(t) < 0$ for any $t\in(x_1,\, x_2)$.

5.4.2. Proof of $F'(t) < 0$ under $d'(t)\cdot g(t)\cdot t \ge 1$

We here assume $d'(t)\cdot g(t)\cdot t \ge 1$ for any $t\in(x_1,\, x_2)$ and we will call it the condition (d) below. Then we have $F'(t) < 0$ for any $t\in(x_1,\, x_2)$ under the condition (d), where

$$F'(t) = \frac{1}{g_2}\cdot\left[\left(g_2 - g(t)\right)\cdot d''(t) - \left(d(t) - d_1\right)\cdot g''(t) - 2\cdot d'(t)\cdot g'(t)\right].$$

In fact, since $g(t) > 0$ for any $t\in(x_1,\, x_2)$, it is clear that $F'(t) < 0$ is equivalent to

$$\left(g_2 - g(t)\right)\cdot d''(t) + \left(d(t) - d_1\right)\cdot\left(-g''(t)\right) < 2\cdot d'(t)\cdot g'(t).$$

Since $g(t)$ is the increasing function on the interval $(x_1,\, x_2)$ and so $(g_2 - g(t))\cdot d''(t) < 0$, it is sufficient to show $(d(t)-d_1)\cdot(-g''(t)) < 2\cdot d'(t)\cdot g'(t)$. By the mean value theorem of [7], there exists a point t_1 such that $x_1 < t_1 < t$ and $d(t) - d(x_1) = d'(t_1)\cdot(t - x_1)$. From the condition (d), we have $d'(t) > 0$ and hence $d'(t_1)\cdot(t-x_1) \le d'(t_1)\cdot(x_2 - x_1) \le d'(t_1)$, because $x_2 - x_1 \le p + 1 - p = 1$. Also by the mean value theorem of [7], there exists a point t_2 such that $t_1 < t_2 < t$ and $d'(t_1) - d'(t) = d''(t_2)\cdot(t_1 - t)$. On the other hand, for any $t \in (x_1,\, x_2)$ we have

$$\left(-d''(t)\cdot g(t)\cdot t\right) = 1 + \frac{1}{\log t} - A \le 1 + 0.072 + 0.3 \le 1.5 \ \left(t \ge p \ge e^{14}\right).$$

From this, since $(-d''(t)) > 0$, we have

$$\left(-d''(t_2)\right)\cdot(t - t_1) \le -d''(t_2) \le \frac{1.5}{g(t_2)\cdot t_2} = \frac{1.5}{g(t)\cdot t}\cdot\left(1 + \frac{t - t_2}{t_2}\right)\cdot\left(1 + \frac{g(t) - g(t_2)}{g(t_2)}\right) \le$$

$$\le \frac{1.5}{g(t)\cdot t}\cdot\left(1 + \frac{1}{t_2}\right)\cdot\left(1 + \frac{g'(t_2)}{g(t_2)}\right) \le \frac{1.5}{g(t)\cdot t}\cdot\left(1 + \frac{1}{t_2}\right)^2 \le \frac{1.5}{g(t)\cdot t}\cdot\left(1 + \frac{1}{14}\right)^2 \le \frac{2}{g(t)\cdot t}.$$

By the condition (d), we have

$$\left(d(t) - d_1\right)\cdot\left(-g''(t)\right) \le d'(t_1)\cdot\left(-g''(t)\right) \le \left[d'(t) + \left(-d''(t_2)\right)\right]\cdot\left(-g''(t)\right) \le$$

$$\le \left[d'(t) + 2/\left(g(t)\cdot t\right)\right]\cdot\left(-g''(t)\right) \le 3\cdot d'(t)\cdot\left(-g''(t)\right) < 2\cdot d'(t)\cdot g'(t)$$

and hence $F'(t) < 0$.

5.4.3. An estimate for the point ξ_0 of $F(\xi_0) = 0$ under the condition (d)

Here we will obtain $\xi_0 = (x_1 + x_2)/2$ for the point ξ_0 of $F(\xi_0) = 0$ under the condition (d). By the mean value theorem for integrals of [7] and by

$$\int_{x_1}^{x_2} F(t)\cdot dt = \int_{x_1}^{\xi_0} F(t)\cdot dt + \int_{\xi_0}^{x_2} F(t)\cdot dt = 0,$$

there exist points $\lambda_1,\, \lambda_2$ such that $x_1 < \lambda_1 < \xi_0 < \lambda_2 < x_2$ and

$$F(\lambda_1)\cdot(\xi_0 - x_1) + F(\lambda_2)\cdot(x_2 - \xi_0) = 0. \qquad \text{(F1)}$$

Since $F'(t) < 0$ for $t \in (x_1,\, x_2)$ under the condition (d), we here have $F(\lambda_1) > F(\xi_0) = 0 > F(\lambda_2)$. On the other hand, we denote by $y(x)$ the line passing through the points $(x_1,\, F(\lambda_1))$ and $(x_2,\, F(\lambda_2))$, then we have

$$y(x) = \frac{F(\lambda_2) - F(\lambda_1)}{x_2 - x_1}\cdot(x - x_1) + F(\lambda_1).$$

If the line $y(x)$ intersects the line $y = 0$ at the point x_0, then $y(x_0) = 0$ and hence we have

$$F(\lambda_1)\cdot(x_2 - x_0) + F(\lambda_2)\cdot(x_0 - x_1) = 0. \qquad \text{(F2)}$$

From (F1) and (F2), we have $\dfrac{F(\lambda_1)}{-F(\lambda_2)} = \dfrac{(x_2 - \xi_0)}{(\xi_0 - x_1)} = \dfrac{(x_0 - x_1)}{(x_2 - x_0)}$ and so $x_1 + x_2 = x_0 + \xi_0$. From (F1) and (F2), we also obtain

$$\left[F(\lambda_1) + F(\lambda_2)\right]\cdot(x_2 - x_1) = \left[F(\lambda_1) - F(\lambda_2)\right]\cdot(x_0 - \xi_0),$$

$$F(\lambda_1) + F(\lambda_2) = -F(\lambda_2)\cdot\frac{x_0 - \xi_0}{\xi_0 - x_1} = F(\lambda_1)\cdot\frac{x_0 - \xi_0}{x_0 - x_1}.$$

Since $F(\lambda_1)-F(\lambda_2)>0$, we have two quadratic equations with respect to ξ_0 and x_0;

$$\begin{cases} \xi_0^2-(x_0+x_1+\delta_0)\cdot\xi_0+(x_1+\delta_0)\cdot x_0=0, \\ x_0^2-(\xi_0+x_1+\delta_1)\cdot x_0+(x_1+\delta_1)\cdot\xi_0=0, \end{cases}$$

where $\delta_0=\dfrac{-F(\lambda_2)}{F(\lambda_1)-F(\lambda_2)}\cdot(x_2-x_1)$ and $\delta_1=\dfrac{F(\lambda_1)}{F(\lambda_1)-F(\lambda_2)}\cdot(x_2-x_1)$. Here since $\delta_1-\delta_0=x_0-\xi_0$, we first have $x_0+x_1+\delta_0=\xi_0+x_1+\delta_1$. Next from (F1), (F2) and $x_1+x_2=x_0+\xi_0$ we also have $(x_1+\delta_0)\cdot x_0=(x_1+\delta_1)\cdot\xi_0$. This shows that above two equations have common roots. Thus we have $x_0=\xi_0=(x_1+x_2)/2$.

5.4.4. An estimate of $(1+\log p)\cdot E(p)$

By the mean value theorem of [7], there exist points η_1 and η_2 such that $x_1<\eta_1<\xi_0<\eta_2<x_2$ and $d(\xi_0)-d(x_1)=d'(\eta_1)\cdot(\xi_0-x_1)$, $g(x_2)-g(\xi_0)=g'(\eta_2)\cdot(x_2-\xi_0)$. Then we have $g'(\xi_0)\ge g'(\eta_2)$, since $g'(t)$ is the decreasing function on $(x_1,\, x_2)$. If the condition (d) holds, then $d'(t)>0$ for any $t\in(x_1,\, x_2)$, hence by $F(\xi_0)=0$ we have

$$d'(\xi_0)=\frac{d(\xi_0)-d(x_1)}{g(x_2)-g(\xi_0)}\cdot g'(\xi_0)=d'(\eta_1)\cdot\frac{\xi_0-x_1}{x_2-\xi_0}\cdot\frac{g'(\xi_0)}{g'(\eta_2)}\ge d'(\eta_1),$$

but this is a contradiction to that $d'(t)$ is the decreasing function, because $d''(t)<0$ for any $t\in(x_1,\, x_2)$. Thus the condition (d) is impossible. From this we have that there exists a point t_0 such that $x_1<t_0<x_2$ and $d'(t_0)\cdot g(t_0)\le t_0^{-1}$, and so $f'(t_0)\le d(t_0)\cdot g'(t_0)+t_0^{-1}$. Since x_1 and x_2 were arbitrary taken, it is clear that $t_0\to p$ as $x_2\to p$ and we have $f'(p)\le d(p)\cdot g'(p)+1/p$ as $t_0\to p$. Therefore we have

$$(1+\log p)\cdot E(p)\le d(p)\cdot\frac{\log^2(p\cdot\alpha_1)}{2\cdot\sqrt{p}}\cdot\left(1+\frac{4}{\log(p\cdot\alpha_1)}\right)+\frac{1}{p}.$$

5.5. An estimate of $G_0:=(\log p_0\cdot R(\Im_{m-1})-(N_0-N_1))/N_0$

Here $R(\Im_{m-1}):=\dfrac{(\log\log\Im_{m-1})^2}{2\cdot\sqrt{\log\Im_{m-1}}}\cdot\left(1+\dfrac{4}{\log\log\Im_{m-1}}\right)$. It is known that $p_{k+1}^2\le p_1\cdot p_2\cdots p_k$ for $p_k\ge 7$ by 246p. of [2] and hence $\dfrac{\log p_0}{\log\Im_{m-1}}<\dfrac{1}{2}$ $(p\ge e^{14})$. Since $\log(1+t)\ge t\cdot(1-t/2)$ $(0<t<1/2)$, first we have

$$N_0-N_1=\left(\sqrt{\log\Im_m}-\sqrt{\log\Im_{m-1}}\right)\cdot(\log\log\Im_m)^2+\sqrt{\log\Im_{m-1}}\cdot\left((\log\log\Im_m)^2-(\log\log\Im_{m-1})^2\right)\ge$$

$$\ge\frac{\log p_0}{2\cdot\sqrt{\log\Im_m}}\cdot(\log\log\Im_{m-1})^2+\sqrt{\log\Im_{m-1}}\cdot 2\cdot\log\log\Im_{m-1}\cdot(\log\log\Im_m-\log\log\Im_{m-1})=$$

$$=\frac{\log p_0}{2\cdot\sqrt{\log\Im_m}}\cdot(\log\log\Im_{m-1})^2+\sqrt{\log\Im_{m-1}}\cdot 2\cdot\log\log\Im_{m-1}\cdot\log\left(1+\frac{\log p_0}{\log\Im_{m-1}}\right)\ge$$

$$\ge\frac{\log p_0}{2\cdot\sqrt{\log\Im_m}}\cdot(\log\log\Im_{m-1})^2+2\cdot\log\log\Im_{m-1}\cdot\frac{\log p_0}{\sqrt{\log\Im_{m-1}}}\cdot\left(1-\frac{\log p_0}{2\cdot\log\Im_{m-1}}\right)$$

and

$$\log p_0 \cdot R(\Im_{m-1}) - (N_0 - N_1) \le \log p_0 \cdot \frac{(\log\log \Im_{m-1})^2}{2\cdot\sqrt{\log \Im_{m-1}}} - \frac{\log p_0}{2\cdot\sqrt{\log \Im_m}} \cdot (\log\log \Im_{m-1})^2 +$$

$$+\log p_0 \cdot \frac{2\cdot \log\log \Im_{m-1}}{\sqrt{\log \Im_{m-1}}} - \log p_0 \cdot \frac{2\cdot\log\log \Im_{m-1}}{\sqrt{\log \Im_{m-1}}} \cdot \left(1 - \frac{\log p_0}{2\cdot \log \Im_{m-1}}\right) \le$$

$$\le \frac{\log p_0}{2} \cdot \left(\frac{1}{\sqrt{\log \Im_{m-1}}} - \frac{1}{\sqrt{\log \Im_m}}\right) \cdot (\log\log \Im_{m-1})^2 + \frac{\log^2 p_0}{(\log \Im_{m-1})^{3/2}} \cdot \log\log \Im_{m-1} \le$$

$$\le \frac{\log^2 p_0}{(\log \Im_{m-1})^{3/2}} \cdot (\log\log \Im_{m-1})^2 \cdot \left(\frac{1}{4} + \frac{1}{\log\log \Im_{m-1}}\right).$$

On the other hand, it is known $p_{k+1}^2 \le 2\cdot p_k^2$ for $p_k \ge 7$ by 247p. of[2] and $t - t/\log t < \vartheta(t)\,(t \ge 41)$ by (3.16) of [6]. So $\log p_0 \le \log p \cdot \left(1 + \log\sqrt{2}/\log p\right)$ and if $p \ge e^{14}$ then we have $\alpha_1 \ge (1 - 1/14)$ and

$$G_0 \le \frac{\log^2 p_0}{(\log \Im_{m-1})^{3/2}} \cdot (\log\log \Im_{m-1})^2 \cdot \left(\frac{1}{4} + \frac{1}{\log\log \Im_{m-1}}\right) \cdot \frac{1}{N_0} \le$$

$$\le \frac{\log^2 p_0}{(\log \Im_{m-1})^2} \cdot \left(\frac{1}{4} + \frac{1}{\log\log \Im_{m-1}}\right) \le$$

$$\le \frac{\log^3 p}{p\cdot\alpha_1^2} \cdot \left(1 + \frac{\log\sqrt{2}}{\log p}\right)^2 \cdot \left(\frac{1}{4} + \frac{1}{\log p + \log\alpha_1}\right) \cdot \frac{1}{p\cdot\log p} \le \frac{0.01}{p\cdot\log p}\left(p \ge e^{14}\right).$$

5.6. An estimate of $S(p') := \sum_{p' \le p \le +\infty} 1/(p\cdot\log p)$

Put $s(t) = \sum_{p\le t} 1/p = \log\log t + b + E(t)$. Then we have

$$S(p') = \int_{p'}^{+\infty} \frac{1}{\log t} \cdot ds(t) = \int_{p'}^{+\infty} \frac{1}{\log t} \cdot \left(\frac{dt}{t\cdot\log t} + dE(t)\right) =$$

$$= \int_{p'}^{+\infty} \frac{dt}{t\cdot\log^2 t} + \frac{E(t)}{\log t}\Big|_{p'}^{+\infty} + \int_{p'}^{+\infty} \frac{E(t)}{t\cdot\log^2 t} \cdot dt \le$$

$$\le -\frac{1}{\log t}\Big|_{p'}^{+\infty} + \frac{E(t)}{\log t}\Big|_{p'}^{+\infty} + \int_{p'}^{+\infty} \frac{1}{t\cdot\log^4 t} \cdot dt \le$$

$$\le \frac{1}{\log p'} - \frac{E(p')}{\log p'} + \int_{p'}^{+\infty} \frac{1}{t\cdot\log^4 t} \cdot dt \le$$

$$\le \frac{1}{\log p'} + \frac{1}{\log^3 p'} - \frac{1}{3\cdot\log^3 t}\Big|_{p'}^{+\infty} = \frac{1}{\log p'} + \frac{4}{3\cdot\log^3 p'}$$

and $S(p') \ge -\frac{1}{\log t}\Big|_{p'}^{+\infty} + \frac{E(t)}{\log t}\Big|_{p'}^{+\infty} - \int_{p'}^{+\infty} \frac{1}{t\cdot\log^4 t} \cdot dt \ge \frac{1}{\log p'} - \frac{4}{3\cdot\log^3 p'}$. If p' is a first prime $\ge e^{14}$, then $p' = 1202609$ and it is 93118-th prime. And we have $0.06 \le S(p') \le 0.08$.

Now we are ready for the proof of the following lemma.

Lemma. For any $m \ge 4$ we have $C_m < 1$.

proof. Let $D_m = p_m \cdot (e_0' - \alpha_0)/\left(\sqrt{p_m\cdot\alpha_0} \cdot \log^2(p_m\cdot\alpha_0)\right)(m \ge 4)$. Then $C_m < 1$ is equivalent to $D_m < 1$.

And we here have $D_m < 1$ for $7 \le p_m \le e^{14}$ and $D_m \le a_m := 1 - 11\cdot S(p_m)$ for any $p_m \ge e^{14}$. In fact, it is easy to see that for $7 \le p_m \le e^{14}$ by MATLAB (see the table 1 and the table 2)

$$\Re_m := \log\left(e^{-\gamma}\cdot F_m\right) - \log\log\left(\log\Im_m + \sqrt{\log\Im_m}\cdot\left(\log\log\Im_m\right)^2\right) < 0 .$$

Next, $p' = 1202609$ then we have $D_{93118} = 0.01038\cdots \le 0.1 \le 1 - 11\cdot S\left(p'\right) \le 0.4 < 1$.

Now assume $p \ge e^{14}$ and $D_{m-1} \le a_{m-1}$. Let us see $D_m \le a_m$. We have

$$D_m = \frac{p_0\cdot\left(e_0' - \alpha_0\right)}{N_0} = \frac{1}{N_0}\cdot\left(p\cdot\left(e_1' - \alpha_1\right) + K_0\right) = D_{m-1}\cdot\frac{N_1}{N_0} + \frac{K_0}{N_0} \le$$

$$\le a_{m-1}\cdot\frac{N_1}{N_0} + \frac{1}{N_0}\cdot\log p_0\cdot\left(\mu\cdot e_1' - 1\right) \le a_{m-1} + b_{m-1},$$

where $b_{m-1} = \left(\log p_0\cdot\left(\mu\cdot e_1' - 1\right) - a_{m-1}\cdot\left(N_0 - N_1\right)\right)/N_0$. We have to obtain $b_{m-1} \le 11/\left(p\cdot\log p\right)$. By the assumption $D_{m-1} \le a_{m-1}$, we have

$$e_1' \le \alpha_1 + a_{m-1}\cdot\frac{\sqrt{p\cdot\alpha_1}\cdot\log^2\left(p\cdot\alpha_1\right)}{p} = \alpha_1\cdot\left(1 + a_{m-1}\cdot\frac{\log^2\left(p\cdot\alpha_1\right)}{\sqrt{p\cdot\alpha_1}}\right)$$

and by taking logarithm of both sides we have $\log e_1' = \log p\cdot\left(e_1 - 1\right) \le \theta\left(p\right) + a_{m-1}\cdot\dfrac{\log^2\left(p\cdot\alpha_1\right)}{\sqrt{p\cdot\alpha_1}}$.

We also have

$$e_1 \le 1 + \frac{1}{\log p}\cdot\left(\theta\left(p\right) + a_{m-1}\cdot\frac{\log^2\left(p\cdot\alpha_1\right)}{\sqrt{p\cdot\alpha_1}}\right),\ E\left(p\right) + \varepsilon_0\left(p\right) \le \frac{1}{\log p}\cdot\left(\theta\left(p\right) + a_{m-1}\cdot\frac{\log^2\left(p\cdot\alpha_1\right)}{\sqrt{p\cdot\alpha_1}}\right).$$

Thus we see $p\cdot\log p\cdot E\left(p\right) - p\cdot\theta\left(p\right) \le \dfrac{a_{m-1}}{\sqrt{\alpha_1}}\cdot\sqrt{p}\cdot\log^2\left(p\cdot\alpha_1\right) - p\cdot\log p\cdot\varepsilon_0\left(p\right)$ and

$$d\left(p\right) = \frac{f\left(p\right)}{g\left(p\right)} = \frac{p\cdot\log p\cdot E\left(p\right) - p\cdot\theta\left(p\right)}{\sqrt{p}\cdot\log^2\left(p\cdot\alpha_1\right)} \le \frac{a_{m-1}}{\sqrt{\alpha_1}} - \frac{p\cdot\log p\cdot\varepsilon_0\left(p\right)}{\sqrt{p}\cdot\log^2\left(p\cdot\alpha_1\right)}.$$

By above 5.4, we have

$$\left(1 + \log p\right)\cdot E\left(p\right) \le d\left(p\right)\cdot\frac{\log^2\left(p\cdot\alpha_1\right)}{2\cdot\sqrt{p}}\cdot\left(1 + \frac{4}{\log\left(p\cdot\alpha_1\right)}\right) + \frac{1}{p} \le$$

$$= a_{m-1}\cdot\frac{\log^2\left(p\cdot\alpha_1\right)}{2\cdot\sqrt{p\cdot\alpha_1}}\cdot\left(1 + \frac{4}{\log\left(p\cdot\alpha_1\right)}\right) - \frac{\log p\cdot\varepsilon_0\left(p\right)}{2}\cdot\left(1 + \frac{4}{\log\left(p\cdot\alpha_1\right)}\right) + \frac{1}{p} \le$$

$$\le a_{m-1}\cdot\frac{\log^2\left(p\cdot\alpha_1\right)}{2\cdot\sqrt{p\cdot\alpha_1}}\cdot\left(1 + \frac{4}{\log\left(p\cdot\alpha_1\right)}\right) - \left(1 + \log p\right)\cdot\varepsilon_0\left(p\right) + \frac{1}{p},$$

since $\varepsilon_0\left(p\right) < 0$ and $\dfrac{\log p}{2}\cdot\left(1 + \dfrac{4}{\log\left(p\cdot\alpha_1\right)}\right) \le \left(1 + \log p\right)$ $\left(p \ge e^{14},\ \alpha_1 \ge 1 - 1/14\right)$. Thus we see

$$\left(1 + \log p\right)\cdot\left(E\left(p\right) + \varepsilon_0\left(p\right)\right) \le a_{m-1}\cdot\frac{\log^2\left(p\cdot\alpha_1\right)}{2\cdot\sqrt{p\cdot\alpha_1}}\cdot\left(1 + \frac{4}{\log\left(p\cdot\alpha_1\right)}\right) + \frac{1}{p}.$$

If $e_1 > 1$, then, since $0 < a_{m-1} \le 1$, we also have

$$\left(1 + \log p\right)^2\cdot\left(E\left(p\right) + \varepsilon_0\left(p\right)\right)^2 \le \left(\frac{\log^2\left(p\cdot\alpha_1\right)}{2\cdot\sqrt{p\cdot\alpha_1}}\cdot\left(1 + \frac{4}{\log\left(p\cdot\alpha_1\right)}\right) + \frac{1}{p}\right)^2 \le$$

$$\le \frac{\log^4\left(p\cdot\alpha_1\right)}{p\cdot\alpha_1}\cdot\left(\frac{1}{2} + \frac{2}{\log\left(p\cdot\alpha_1\right)} + \frac{\sqrt{\alpha_1}}{\sqrt{p}\cdot\log^2\left(p\cdot\alpha_1\right)}\right)^2 \le 0.4143\cdot\frac{\log^4\left(p\cdot\alpha_1\right)}{p\cdot\alpha_1}.$$

and

$$\log p_0\cdot(\mu\cdot e_1'-1)-a_{m-1}\cdot(N_0-N_1)\le$$
$$\le\log p_0\cdot(1+\log p)\cdot(E(p)+\varepsilon_0(p))-a_{m-1}\cdot(N_0-N_1)+0.55\cdot\frac{\log^2 p_0}{p}+$$
$$+0.61\cdot\log p_0\cdot(1+\log p)^2\cdot(E(p)+\varepsilon_0(p))^2\le$$
$$\le G_0\cdot N_0+0.253\cdot\log p_0\cdot\frac{\log^4(p\cdot\alpha_1)}{p\cdot\alpha_1}+\frac{\log^2 p_0}{p}\cdot\left(0.55\cdot+\frac{1}{\log p_0}\right).$$

Finally, we have

$$b_{m-1}\le G_0+0.253\cdot\log p_0\cdot\frac{\log^4(p\cdot\alpha_1)}{p\cdot\alpha_1\cdot N_1}+0.622\cdot\frac{\log^2 p_0}{p\cdot N_1}\le$$
$$\le G_0+0.253\cdot\frac{\left(\log p+\log\sqrt{2}\right)}{\sqrt{p\cdot\alpha_1^{3/2}}}\cdot\frac{\log p\cdot(\log p+\log\alpha_1)^2}{p\cdot\log p}+$$
$$+0.622\cdot\frac{\log p}{\sqrt{p\cdot\alpha_1}}\cdot\left(1+\frac{\log\sqrt{2}-\log\alpha_1}{\log p+\log\alpha_1}\right)^2\cdot\frac{1}{p\cdot\log p}\le$$
$$\le\frac{0.01}{p\cdot\log p}+\frac{10.251}{p\cdot\log p}+\frac{0.01}{p\cdot\log p}\le\frac{11}{p\cdot\log p}\ \left(p\ge e^{14}\right).$$

Next, if $e_1\le 1$ then we have $b_{m-1}\le 0.55\cdot\frac{\log^2 p_0}{p\cdot N_1}\le\frac{0.01}{p\cdot\log p}\ \left(p\ge e^{14}\right)$. □

6. Proof of Theorem 2

Let $n=q_1^{\lambda_1}\cdots q_m^{\lambda_m}$ be the prime factorization of any natural number $n\ge 2$. Then it is clear $p_m\le q_m$. If $7\le p_m\le e^{14}$, then we have $C_m<1$, since $\Re_m<0$ (see the table 1 and the table 2), and if $p_m\ge e^{14}$ then we have $C_m<1$ by the Lemma. Therefore we have $\Phi_0(n)\le\Phi_0(\Im_m)=C_m\le\max_{m\ge1}\{C_m\}\le 24$. □

7. Proof of Corollary

From the theorem 1 and theorem 2, for $n\ge 5$ we have

$$\frac{n}{\varphi(n)}\le e^{\gamma}\cdot\log\log n+e^{\gamma}\cdot\left(1+\frac{\log 24\cdot(\log\log n)^{-2}}{\sqrt{\log n}}\right)\cdot\frac{(\log\log n)^2}{\sqrt{\log n}}\le$$
$$\le e^{\gamma}\cdot\log\log n+21.483\cdot\frac{(\log\log n)^2}{\sqrt{\log n}}.$$

8. Note and Algorithm

The table 1 shows the values $C_m=\Phi_0(\Im_m)$ and $\Re_m$ to $\omega(n)=m$ of $n\in N$. There are only values of C_m and $\Re_m$ for $1\le m\le 10$ here. But it is not difficult to verify them for $31\le p_m\le e^{14}$. Note, if more informations, then it should be taken $\Re_m<0$, not $C_m<1$, for $263\le p_m\le e^{14}$, by reason of the limited values of MATLAB 6.5. The table 2 shows the valu1es $\Re_m$ for $93109\le m\le 93118$.
Of course, all the values in the table 1 and the table 2 are approximate.

The algorithm for $\Re_m$ to $\omega(n)=m$ by MATLAB is as follows:

```
Function Phi-Index, clc, gamma=0.57721566490153286060; format long
P= [2, 3, 5, 7,…,1202609];  M=length(P);
for m=1:M; p=P(1:m); q=1-1./p; F=-gamma+log(prod(1./q)); N1=sum(log(p.^1)); N2=(N1)^(1/2);
N3=(log(N1))^2; N4=N2*N3; N5=N1+N4;  m, Pm, Rm=F-log(log(N5)),  end
```

Table 1

m	p_m	C_m	$\Re_m$
1	2	9.66806133818849	-
2	3	23.15168798263150	0.73259862957209
3	5	7.73864609733096	0.14633620860732
4	7	0.83171792006862	-0.00636141995881
5	11	0.01114282713904	-0.09308687002330
6	13	1.102119966548700e-004	-0.12730939385590
7	17	3.834259945131073e-007	-0.15077316854133
8	19	1.397561045763582e-009	-0.15960912308179
9	23	2.821898264763264e-012	-0.16612788105591
10	29	2.081541289212468e-015	-0.17415284347098

Table 2

m	p_m	$\Re_m$
93109	1202477	-0.01154791933871
93110	1202483	-0.01154786567870
93111	1202497	-0.01154781201949
93112	1202501	-0.01154775835370
93113	1202507	-0.01154770468282
93114	1202549	-0.01154765103339
93115	1202561	-0.01154759738330
93116	1202569	-0.01154754372957
93117	1202603	-0.01154749009141
93118	1202609	-0.01154743644815

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