# An basic evidence of both Catalan-Mihailescu and Fermat-Wiles theorems and generalization to Fermat-Catalan and Beal conjectures

**Jamel Ghanouchi**

jamel.ghanouchi@live.com

**Abstract**

( MSC=11D04) We begin with an equation, for example : $Y^p = X^q \pm Z^c$ and solve it.

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**Introduction**

The goal of this document is clearly to solve the Fermat-Catalan equation $Y^b = X^q \pm Z^c$. We have some solutions, they are :

$$1^m + 2^3 = 3^2$$

$$2^5 + 7^2 = 3^4$$

$$13^2 + 7^3 = 2^9$$

$$2^7 + 17^3 = 71^2$$

$$3^5 + 11^4 = 122^2$$

$$33^8 + 1549034^2 = 15613^3$$

$$1414^3 + 2213459^2 = 65^7$$

$$9262^3 + 15312283^2 = 113^7$$

$$17^7 + 76271^3 = 21063928^2$$

$$43^8 + 96222^3 = 30042907^2$$

If we study minutiously those solutions, it appears a common point, there is an exponent 2 in the formulas. It is not only the case of the Fermat-Catalan equation.
Effectively, this exponent 2 appears at least in two other diophantine equations : the Fermat equation, of course, but not only, it appears also in the Catalan equation and in some Pillai equations of the form $Y^p = X^q + a$.
Our goal, here, is to show and to prove formally, with the tools of the logic and algebra, how this exponent 2 appears in the equations !

**Resolution of Fermat-Catalan equation**

Let Fermat-Catalan equation :

$$Y^p = X^q + aZ^c$$

$a = \pm 1$
Now, let

$$w = \frac{\log(-aZ^c + \sqrt{Z^{2c} + 4Y^pX^q}) - \log(2Y^{p-2})}{\log(X)}$$

$w$ exists and

$$w\log(X) = \log(X^w) = \log(-aZ^c + \sqrt{Z^{2c} + 4Y^pX^q}) - \log(2Y^{p-2})$$

$$= \log(\frac{(-aZ^c + \sqrt{Z^{2c} + 4Y^pX^q})}{2Y^{p-2}})$$

Thus

$$X^w = \frac{-aZ^c + \sqrt{Z^{2c} + 4Y^pX^q}}{2Y^{p-2}}$$

We deduce

$$2X^wY^{p-2} + aZ^c = \sqrt{Z^{2c} + 4Y^pX^q}$$

Or

$$(2X^wY^{p-2} + aZ^c)^2 = Z^{2c} + 4Y^pX^q$$
$$= 4X^{2w}Y^{2p-4} + Z^{2c} + 4aZ^cX^wY^{p-2}$$

And

$$X^{2w}Y^{2p-4} + aZ^cX^wY^{p-2} = Y^pX^q$$

Or

$$X^wY^{p-2} + aZ^c = Y^2X^{q-w}$$

Hence

$$Y^2X^{q-w} - X^wY^{p-2} = aZ^c = Y^p - X^q = Y^2Y^{p-2} - X^wX^{q-w}$$

Thus

$$Y^2(X^{q-w} - Y^{p-2}) + X^w(X^{q-w} - Y^{p-2}) = 0$$
$$= (Y^2 + X^w)(X^{q-w} - Y^{p-2}) = 0$$

And

$$Y^{p-2} = X^{q-w}$$

Now, let

$$w' = \frac{\log(Y^q - X^p + \sqrt{(Y^q - X^p)^2 + 4Y^qX^p}) - \log(2X^{p-2})}{\log(Y)}$$

$w'$ exists and

$$w'\log(Y) = \log(Y^{w'}) = \log(Y^q - X^p + \sqrt{(Y^q - X^p)^2 + 4Y^qX^p}) - \log(2X^{p-2})$$
$$= \log(\frac{(Y^q - X^p + \sqrt{(Y^q - X^p)^2 + 4Y^qX^p})}{2X^{p-2}})$$

Thus

$$Y^{w'} = \frac{Y^q - X^p + \sqrt{(Y^q - X^p)^2 + 4Y^qX^p}}{2X^{p-2}}$$

We deduce

$$2Y^{w'}Y^{p-2} - (Y^q - X^p) = \sqrt{(Y^q - X^p)^2 + 4Y^qX^p}$$

Or

$$(2Y^{w'}X^{p-2} - (Y^q - X^p))^2 = (Y^q - X^p)^2 + 4Y^qX^p$$
$$= 4Y^{2w'}X^{2p-4} + (Y^q - X^p)^2 - 4(Y^q - X^p)Y^{w'}X^{p-2}$$

And

$$Y^{2w'}X^{2p-4} - (Y^q - X^p)Y^{w'}X^{p-2} = Y^qX^p$$

Or

$$Y^{w'}X^{p-2} - (Y^q - X^p) = X^2Y^{q-w'}$$

Hence

$$X^2Y^{q-w'} - Y^{w'}X^{p-2} = -Y^q + X^p = -Y^{w'}Y^{q-w'} + X^2X^{p-2}$$

Thus

$$Y^{w'}(X^{p-2} - Y^{q-w'}) + X^2(X^{p-2} - Y^{q-w'}) = 0$$
$$= (Y^{w'} + X^2)(X^{p-2} - Y^{q-w'}) = 0$$

And

$$X^{p-2} = Y^{q-w'}$$

But

$$X^{(p-2)^2} = Y^{(p-2)(q-w')} = X^{(q-w)(q-w')}$$

Thus

$$(p-2)^2 = (q-w)(q-w')$$

And

$$X^{q-w} = Y^{p-2} = Y^{\sqrt{(q-w)(q-w')}}$$

Thus

$$X^{\sqrt{q-w}} = Y^{\sqrt{q-w'}}$$

But let

$$w_1 = q - \frac{(q-w)^2}{p-2}; \quad w_2 = q - \frac{(q-w')^2}{p-2}$$

Or

$$(q-w)^2 = (p-2)(q-w_1); \quad (q-w')^2 = (p-2)(q-w_2)$$

We have

$$(q-w)^2(q-w')^2 = (p-2)^4 = (p-2)^2(q-w_1)(q-w_2)$$

Thus

$$(q-w)(q-w') = (q-w_1)(q-w_2)$$

And if we suppose $(p-2)(q-w)(q-w') \neq 0$

$$(q-w)^2 = (p-2)(q-w_1) = \sqrt{(q-w)(q-w')}(q-w_1)$$

or

$$q - w_1 = (q-w)\sqrt{\frac{q-w}{q-w'}}$$

and

$$(q-w')^2 = (p-2)(q-w_2) = \sqrt{(q-w)(q-w')}(q-w_2)$$

$$q - w_2 = (q-w')\sqrt{\frac{q-w'}{q-w}}$$

But

$$Y^{q-w_2} = Y^{(q-w')\sqrt{\frac{q-w'}{q-w}}} = X^{(p-2)\sqrt{\frac{q-w'}{q-w}}}$$

$$= X^{\sqrt{(q-w)(q-w')\frac{q-w'}{q-w}}} = X^{q-w'} = X^{\sqrt{(p-2)(q-w_2)}}$$

Hence

$$Y^{q-w_2} = X^{q-w'}$$

And

$$X^{q-w_1} = X^{(q-w)\sqrt{\frac{q-w}{q-w'}}} = Y^{(p-2)\sqrt{\frac{q-w}{q-w'}}}$$

$$= Y^{\sqrt{(q-w)(q-w')\frac{q-w}{q-w'}}} = Y^{q-w} = Y^{\sqrt{(p-2)(q-w_1)}}$$

Hence

$$X^{q-w_1} = Y^{q-w}; \quad Y^{q-w_2} = X^{q-w'}$$

Then

$$X^{\sqrt[4]{q-w_1}} = Y^{\sqrt[4]{q-w_2}}$$

And

$$X^{\sqrt{p-2}} = Y^{\sqrt{q-w_2}}; \quad Y^{\sqrt{p-2}} = X^{\sqrt{q-w_1}}$$

But

$$\frac{w_2 - w'}{w - w_1} = \frac{q - w' - (q - w_2)}{q - w_1 - (q-w)} = \frac{q - w' - (q-w')\sqrt{\frac{q-w'}{q-w}}}{(q-w)\sqrt{\frac{q-w}{q-w'}} - (q-w)}$$

$$= \frac{(q-w')\sqrt{q-w'}}{(q-w)\sqrt{q-w}}$$

And

$$\frac{\sqrt{q-w}(w_2-w)}{\sqrt{q-w'}(w'-w_1)} = \frac{\sqrt{q-w}(q-w) - \sqrt{q-w}(q-w_2)}{\sqrt{q-w'}(q-w_1) - \sqrt{q-w'}(q-w')}$$

$$= \frac{\sqrt{q-w}(q-w) - \sqrt{q-w'}(q-w')}{(q-w)\sqrt{q-w} - \sqrt{q-w'}(q-w')} = 1$$

Hence

$$\frac{w_2 - w'}{w - w_1} = (\frac{w_2 - w}{w' - w_1})^3$$

$$X^{q-w'} = Y^{q-w_2}$$

But

$$X^{\sqrt[3]{w_2-w'}}Y^{\sqrt[3]{w_1-w}} = X^{\sqrt{\frac{q-w}{q-w'}}\sqrt[3]{w_1-w}}Y^{-\sqrt[3]{w-w_1}}$$
$$= Y^{\sqrt[3]{w-w_1}}Y^{-\sqrt[3]{w-w_1}} = 1$$

But let

$$q-w = u(q-w'); \quad q-w_1 = v(q-w_2)$$

We have

$$\frac{q-w_1}{q-w_2} = v = (\frac{q-w}{q-w'})^2 = u^2$$

Thus $v = u^2$. And

$$(q-w_1)(q-w_2) = u^2(q-w_2)^2 = (p-2)^2 = (q-w)(q-w') = u(q-w')^2$$

Thus

$$(q-w')^2 = u(q-w_2)^2$$

And

$$(q-w)^2 = u^2(q-w')^2 = u^3(q-w_2)^2$$

And

$$\frac{(q-w)^2}{(q-w_1)^2} = \frac{u^2(q-w_2)^2}{u^4(q-w_2)^2} = \frac{1}{u}$$

But

$$(q-w)\sqrt{\frac{q-w}{q-w'}}(q-w')^2 = (q-w')\sqrt{\frac{q-w'}{q-w}}(q-w)^2$$
$$= (q-w_1)(q-w')^2 = (q-w_2)(q-w)^2$$

And

$$(p-2)^2(q-w')^2 = (p-2)^3(q-w_2) = (q-w_1)(q-w_2)(q-w')^2 = (q-w_2)^2(q-w)^2$$
$$= \frac{1}{u}(q-w_1)^2(q-w_2)^2 = \frac{1}{u}(p-2)^4$$

And

$$(p-2)^2(q-w)^2 = (p-2)^3(q-w_1) = (q-w_1)(q-w_2)(q-w)^2 = (q-w_1)^2(q-w')^2$$
$$= u(q-w_1)^2(q-w_2)^2 = u(p-2)^4$$

We deduce

$$p-2 = u(q-w_2) = \frac{1}{u}(q-w_1) = \sqrt{u}(q-w') = \sqrt{\frac{1}{u}}(q-w)$$

But

$$Y^{\sqrt{q-w'}} = X^{\sqrt{q-w}} = X^{\sqrt{u(q-w')}}$$
$$Y = X^{\sqrt{u}}$$

And

$$X^{\sqrt[3]{\frac{w_2-w'}{w-w_1}}} = Y = X^{\sqrt{u}}$$

Hence

$$w_2-w' = \sqrt{u^3}(w-w_1) = (q-w')-(q-w_2) = \frac{1}{u}(q-w)-\frac{1}{u^2}(q-w_1) = (\frac{1}{u}-\frac{\sqrt{u}}{u^2})(q-w)$$
$$= \sqrt{u^3}(q-w_1-(q-w)) = \sqrt{u^3}(\sqrt{u}-1)(q-w)$$

Hence

$$(u-\sqrt{u}) = \sqrt{u}(\sqrt{u}-1) = u^2\sqrt{u^3}(\sqrt{u}-1)$$

Or

$$(\sqrt{u}-1)\sqrt{u}(u^3-1) = 0$$

It means that $u = 1$. But as $GCD(X,Y) = 1$ and

$$Y^{\sqrt{q-w'}} = Y^{\sqrt{q-w}} = X^{\sqrt{q-w}}$$

Then $q-w=q-w'=0$ or $p=2$ and $q=w=w'=w_1=w_2$.
This calculus is available if we replace $p-2$ by $p-3$ and it leads as this last case does not exclude the case $p-2$ to $p=2$ or $p=3$. The two cases lead then to $p=2$ and ($p=2$ or $p=3$) which means $p=2$. The same calculation is available to $p=4$ and we have $p=2$ and ($p=2$ or $p=3$ or $p=4$) and it means $p=2$. Etc... Until infinity. The only solution is $p=2$.

**Resolution of Catalan equation**

Let Catalan equation :

$$Y^p = X^q + 1$$

Let

$$w = \frac{\log(-1+\sqrt{1+4Y^pX^q}) - \log(2) - (p-2)\log(Y)}{\log(X)}$$

$w$ exists as we see. But

$$w\log(X) = \log(X^w) = \log(-1+\sqrt{1+4Y^pX^q}) - \log(2) - \log(Y^{p-2})$$

$$= \log(\frac{-1+\sqrt{1+4Y^pX^q}}{2Y^{p-2}})$$

Thus

$$2X^wY^{p-2}+1 = \sqrt{1+4Y^pX^q}$$

Or

$$(2X^wY^{p-2}+1)^2 = 1+4Y^pX^q$$
$$= 1+4X^{2w}Y^{2p-4}+4X^wY^{p-2}$$

We deduce

$$Y^pX^q - X^{2w}Y^{2p-4} = X^wY^{p-2}$$

Hence

$$Y^2X^{q-w} - X^wY^{p-2} = 1 = Y^p - X^q = Y^{p-2}Y^2 - X^wX^{q-w}$$

Or

$$Y^2(X^{q-w}-Y^{p-2}) + X^w(X^{q-w}-Y^{p-2}) = 0$$
$$= (Y^2+X^w)(X^{q-w}-Y^{p-2}) = 0$$

And as $GCD(X,Y)=1$ it leads to $p-2=q-w=0$. And

$$w = q = \frac{\log(-1+\sqrt{1+4Y^2X^q}) - \log(2)}{\log(X)} \in \mathbb{N}$$

This equation leads to $(X,q)=(2,3)$. Ko Chao has already solved the case $p=2$.

**Resolution of Fermat equation**

Let Fermat equation :

$$Y^n = X^n + Z^n$$

Let here too

$$w = \frac{\log(-Z^n+\sqrt{Z^{2n}+4Y^nX^n}) - \log(2) - (n-2)\log(Y)}{\log(X)}$$

$w$ exists as we see. But

$$w\log(X) = \log(X^w) = \log(-Z^n+\sqrt{Z^{2n}+4Y^nX^n}) - \log(2) - \log(Y^{n-2})$$

$$= \log(\frac{-Z^n+\sqrt{Z^{2n}+4Y^nX^n}}{2Y^{n-2}})$$

Thus

$$2X^wY^{n-2}+Z^n = \sqrt{Z^{2n}+4Y^nX^n}$$

Or

$$(2X^wY^{n-2}+Z^n)^2 = Z^{2n}+4Y^nX^n$$
$$= Z^{2n}+4X^{2w}Y^{2n-4}+4X^wY^{n-2}Z^n$$

We deduce

$$Y^nX^n - X^{2w}Y^{2n-4} = X^wY^{n-2}Z^n$$

Hence

$$Y^2X^{n-w} - X^wY^{n-2} = Z^n = Y^n - X^n = Y^{n-2}Y^2 - X^wX^{n-w}$$

Or

$$Y^2(X^{n-w} - Y^{n-2}) + X^w(X^{n-w} - Y^{n-2}) = 0$$
$$= (Y^2 + X^w)(X^{n-w} - Y^{n-2}) = 0$$

And as $GCD(X, Y) = 1$ it leads to $n - 2 = n - w = 0$. And

$$2 = n = \frac{\log(-1 + \sqrt{1 + 4Y^2X^2}) - \log(2)}{\log(X)} \in \mathbb{N}$$

This equation leads to the solutions of Fermat equation for $n > 1$ as we will see.

**Resolution of Fermat- Catalan equation**

The only solution, in all cases, is $p = 2$.
And $Y^2 = X^q + aZ^c$. Thus, Fermat-Catalan equation is available for

$$q = \frac{\log(-aZ^c + \sqrt{Z^{2c} + 4Y^2X^q}) - \log(2)}{\log(X)} \in \mathbb{N}$$

$$= \frac{\log(Y^q - X^2 + \sqrt{(Y^q - X^2)^2 + 4Y^qX^2}) - \log(2)}{\log(Y)} \in \mathbb{N}$$

If we try successively $q = 3$ and $q = 4$, tc..., we will find the $X, Y$ which satisfy the equations.
Example

$$Z^c = 1^c$$
$$q = \frac{\log(-1 + \sqrt{1 + 4Y^2X^q}) - \log(2)}{\log(X)} \in \mathbb{N}$$

Implies $X = 2$, $Y = 3$ and $q = 3$.

$$1^c + 2^3 = 3^2$$

$$aZ^c = -2^5 = -32$$
$$q = \frac{\log(2^5 + \sqrt{1024 + 4Y^2X^q}) - 0.69}{\log(X)} \in \mathbb{N}$$

Implies $X = 3$, $q = 4$ and $Y = 7$.

$$2^5 + 7^2 = 3^4$$

$$aZ^c = -7^3 = -343$$
$$q = \frac{\log(343 + \sqrt{117649 + 4Y^2X^q}) - 0.69}{\log(X)} \in \mathbb{N}$$

Implies $X = 2$, $q = 9$ and $Y = 13$.

$$13^2 + 7^3 = 2^9$$

$$aZ^c = 17^3 = 4913$$
$$q = \frac{\log(-4913 + \sqrt{24137569 + 4Y^2X^q}) - 0.69}{\log(X)} \in \mathbb{N}$$

Implies $X = 2$, $q = 7$ and $Y = 71$.

$$2^7 + 17^3 = 71^2$$

$$aZ^c = 11^4 = 14641$$
$$q = \frac{\log(-14641 + \sqrt{14614^2 + 4Y^2X^q}) - 0.69}{\log(X)} \in \mathbb{N}$$

Implies $X=3$, $q=5$ and $Y=122$.

$$3^5+11^4=122^2$$

$$aZ^c=-33^8$$

$$q=\frac{\log(33^8+\sqrt{33^{16}+4Y^2X^q})-0.69}{\log(X)}\in\mathbb{N}$$

Implies $X=15613$, $q=3$ and $Y=1549034$.

$$33^8+1549034^2=15613^3$$

$$aZ^c=-1414^3$$

$$q=\frac{\log(1414^3+\sqrt{1414^6+4Y^2X^q})-0.69}{\log(X)}\in\mathbb{N}$$

Implies $X=65$, $q=7$ and $Y=2216459$.

$$1414^3+2213459^2=65^7$$

$$aZ^c=-9262^3$$

$$q=\frac{\log(9262^3+\sqrt{9262^6+4Y^2X^q})-0.69}{\log(X)}\in\mathbb{N}$$

Implies $X=113$, $q=7$ and $Y=15312283$.

$$9262^3+15312283^2=113^7$$

$$aZ^c=17^7$$

$$q=\frac{\log(-17^7+\sqrt{17^{14}+4Y^2X^q})-0.69}{\log(X)}\in\mathbb{N}$$

Implies $X=76271$, $q=3$ and $Y=21063928$.

$$17^7+76271^3=21063928^2$$

$$aZ^c=43^8$$

$$q=\frac{\log(-43^8+\sqrt{43^{16}+4Y^2X^q})-0.69}{\log(X)}\in\mathbb{N}$$

Implies $X=96222$, $q=3$ and $Y=30042907$.

$$43^8+96222^3=30042907^2$$

For Fermat equation, we have $q=n=2=w$ and

$$2=\frac{\log(-aZ^2+\sqrt{Z^4+4Y^2X^2})-\log(2)}{\log(X)}\in\mathbb{N}$$

Example :

$$aZ^2=11^4$$

$$2=\frac{\log(-11^4+\sqrt{11^8+4Y^2X^2}-0.69}{\log X)}$$

$$\log(2X^2)=\log(-11^4+\sqrt{11^8+4Y^2X^2})$$

$$(2X^2+11^4)^2=11^8+4Y^2X^2=11^8+4X^4+4(11^4)X^2$$

$$Y^2=X^2+(11^4)$$

$$(Y-X)(Y+X)=11^4$$

$$Y+X=11^3=1331$$

$$Y-X=11$$

$$2Y=1342$$

$$2X=1320$$

$$Y=671$$

$$X = 660$$

Or

$$671^2 = 660^2 + 11^4$$
$$= (11(61))^2 = (11(60))^2 + 11^4$$
$$61^2 + 60^2 = 11^2$$

Or

$$aZ^2 = 13^{12}$$
$$Y^2 = X^2 + 13^{12}$$
$$(Y - X)(Y + X) = 13^{12}$$
$$Y + X = 13^9$$
$$Y - X = 13^3$$
$$2Y = 13^9 + 13^3 = 13^3(13^6 + 1)$$
$$2X = 13^9 - 13^3 = 13^3(13^6 - 1)$$
$$4Y^2 = (13^6)(13^6 + 1)^2 = (13^6)(13^6 - 1)^2 + 4(13^{12})$$
$$(13^6 + 1)^2 = (13^6 - 1)^2 + (2(13^3))^2$$

Etc...

**Conclusion**

Fermat-Catalan equation $Y^p = X^q \pm Z^c$ has solutions only for $p = 2$. We have shown a way to solve it.

# References


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