A proof of the Riemann hypothesis

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Abstract

In this paper, we present the Riemann problem and define the real primes. It allows to generalize the Riemann hypothesis to the reals. A calculus of integral solves the problem. We generalize the proof to the integers.

The Riemann hypothesis

The Riemann conjecture is a conjecture which has been formulated in 1859 by Bernard Riemann in the subject of the Riemann function zeta or ζ . It is called the zeta Riemann function.

This function is defined as follows

$$\zeta(s) = \sum_{n=1}^{n=\infty}(\frac{1}{n^s}) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + ...$$

The first result is the divergence of the harmonic serie

$$\zeta(1) = \sum_{n=1}^{n=\infty}(\frac{1}{n}) = 1 + \frac{1}{2} + \frac{1}{3} + ...$$

It has been proved in the middle age by Nicole Oresme.

In the XVIII century, Leonard Euler has discovered the main proprieties of the ζ function.

In the 1730's he conjectured after numerical calculus the following equality, which is often called the Basel problem.

$$\zeta(2) = \sum_{n=1}^{n=\infty}(\frac{1}{n^2}) = 1 + \frac{1}{2^2} + \frac{1}{3^2} + ... = \frac{\pi^2}{6}$$

Euler proved it in 1748 and introduced the ζ function. He calculated its value for the positive even numbers.

$$\zeta(2k) = \sum_{n=1}^{n=\infty}(\frac{1}{n^{2k}}) = 1 + \frac{1}{2^{2k}} + \frac{1}{3^{2k}} + ... = \frac{|B_{2k}|(2\pi)^{2k}}{2(2k)!}$$

Where B_{2k} are the Bernoulli numbers.

Thereafter, he proved in 1744 the Euler idendity where prime numbers are related to the ζ function.

$$\zeta(s) = \sum_{n=1}^{n=\infty}(\frac{1}{n^s}) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + ... = \prod_{primes}\frac{1}{1-p^{-s}}$$

Consequently he deduced the divergence of the serie of the inverse of primes.

With Bernard Riemann, s can be complex number. Riemann proved the following formula

$$\pi^{\frac{-s}{2}}\Gamma(\frac{s}{2})\zeta(s)=\pi^{\frac{-(1-s)}{2}}\Gamma(\frac{1-s}{2})\zeta(1-s)$$

Where

$$\Gamma(s)=\int_0^\infty t^{s-1}e^{-t}dt$$

This formula demonstrates that this equation does not change if we replace s by 1-s. Thus it is symmetric | $s=\frac{1}{2}$

Riemann demonstrates that the only zeros in the $R(s)<0$ are the trivial zeros negative even numbers and that there is no zero in the $R(s)>1$.

The other zeros are the non trivial zeros. They are in the critical zone $0\le R(s)\le 1$. Riemann conjectured theu are all in the critical line $R(s)=\frac{1}{2}$.

This conjecture is called the Riemann hypothesis.

They calculated numerically one billion zeros of the Riemann they are all located in the critical line.

Resolution of the Riemann hypothesis for the reals

Definition

A real number is compound if it can be written as $\prod_j p_j^{n_j}$ where p_j are primes and n_j are rationals. This decomposition in prime factors is unique. A prime real number or R-prime can be written only as p=p.1 . Thus we define other real prime numbers like π, e, $\ln(2)$. Of course, it is a convention, because, we can consider π^2 as prime and π will be no more prime. It is equivalent in what will follow. Thus $\sqrt[q]{p}=p^{\frac{1}{q}}$ is compound. Also $\sqrt[q]{p}+1=p^{\frac{1}{q}}+1$ is prime when p is prime and we have $\sqrt[2^i]{p}-1=(p-1)(\sqrt[2^i]{p}+1)^{-1}(\sqrt[2^{i-1}]{p}+1)^{-1}...(\sqrt{p}+1)^{-1}$ compound for p prime, for example.

The approach of the Riemann hypothesis

The Rieman hypothesis states that the non trivial zeros of the Riemann zeta function $\zeta(z)=\sum_{t=1}^{\infty}\frac{1}{t^z}$ lie on the critical line $\frac{1}{2}+iy$.

For t integer, Euler has proved that $\zeta(z)=\sum_{t=1}^{\infty}\frac{1}{t^z}=\prod_{primes}\frac{1}{1-p^{-z}}=\prod_{primes}(\sum_{\alpha\in N}\frac{1}{p^{\alpha z}})$, it is the Euler identity. For t real, it is still true and it becomes $\prod_{primes}(\sum_{\alpha\in Q}\frac{1}{p^{\alpha z}})=\int_1^{\infty}\frac{dt}{t^z}=\frac{1}{1-z}\left[t^{1-z}\right]_1^{\infty}$ But let $\zeta_1(z)$ the Riemann function for the reals and $\zeta(z)$ the Riemann function for the integers, we have

$$\zeta_1(\frac{1}{2}+iy)=\prod_{primes}(\sum_{\alpha\in Q}\frac{1}{p^{\alpha(\frac{1}{2}+iy)}})=\int_1^{\infty}\frac{dt}{t^{\frac{1}{2}+iy}}=\frac{1}{\frac{1}{2}-iy}\left[t^{\frac{1}{2}-iy}\right]_1^{\infty}$$

$$=\lim_{t\to\infty}(\int_1^t\frac{du}{u^{\frac{1}{2}+iy}})=\lim_{t\to\infty}(\frac{1}{\frac{1}{2}-iy}\left[u^{\frac{1}{2}-iy}\right]_1^t)=\lim_{t\to\infty}(\frac{t^{\frac{1}{2}-iy}-1}{\frac{1}{2}-iy})$$

Let

$$\int_1^t\frac{du}{u^{\frac{1}{2}+iy}}=t_0^{\frac{-1}{2}-iy}=\lambda^{\frac{-1}{2}-iy}t^{\frac{-1}{2}-iy}$$

But

$$\lim_{t\to\infty}(\int_1^t\frac{du}{u^{\frac{1}{2}+iy}})=\lim_{t\to\infty}(\lambda^{\frac{-1}{2}-iy}t^{\frac{-1}{2}-iy})=\lim_{t\to\infty}(\frac{t^{\frac{1}{2}-iy}-1}{\frac{1}{2}-iy})$$

Thus

$$\lim_{t\to\infty}(\lambda^{\frac{-1}{2}-iy})=\lim_{t\to\infty}(\frac{t-t^{\frac{1}{2}+iy}}{\frac{1}{2}-iy})=\lim_{t\to\infty}(\frac{t-t^{\frac{1}{2}-iy}t^{2iy}}{\frac{1}{2}-iy})=\lim_{t\to\infty}(\frac{t-(t_0^{-\frac{1}{2}-iy}(\frac{1}{2}-iy)+1)t^{2iy}}{\frac{1}{2}-iy})$$

$$=\lim_{t\to\infty}(\frac{t-(\lambda^{\frac{-1}{2}-iy}t^{-\frac{1}{2}-iy}(\frac{1}{2}-iy)+1)t^{2iy}}{\frac{1}{2}-iy})=\lim_{t\to\infty}(\frac{t}{\frac{1}{2}-iy}-\lambda^{\frac{-1}{2}-iy}t^{\frac{-1}{2}+iy}-\frac{t^{2iy}}{\frac{1}{2}-iy})$$

Hence

$$\lim_{t\to\infty}(\lambda^{\frac{-1}{2}-iy}(1+t^{\frac{-1}{2}+iy}))=\lim_{t\to\infty}(\frac{t-t^{2iy}}{\frac{1}{2}-iy})$$

And

$$\lim_{t\to\infty}(\lambda^{\frac{-1}{2}-iy}) = \lim_{t\to\infty}(\frac{t-t^{2iy}}{(\frac{1}{2}-iy)(1+t^{\frac{-1}{2}+iy})}) = \lim_{t\to\infty}(\frac{t-t^{\frac{1}{2}+iy}}{\frac{1}{2}-iy})$$

Or

$$\lim_{t\to\infty}(t-t^{2iy}) = \lim_{t\to\infty}((1+t^{\frac{-1}{2}+iy})(t-t^{\frac{1}{2}+iy})) = \lim_{t\to\infty}(t-t^{2iy}+t^{\frac{-1}{2}+iy}-t^{\frac{1}{2}+iy})$$

Consequently

$$\lim_{t\to\infty}(t^{\frac{-1}{2}+iy}) = \lim_{t\to\infty}(t^{\frac{1}{2}+iy}) = \lim_{t\to\infty}(t^{iy})$$

Or

$$\lim_{t\to\infty}(t^{\frac{-1}{2}}) = \lim_{t\to\infty}(t^{\frac{1}{2}}) = 1$$

Thus

$$\lim_{t\to\infty}(t_0^{\frac{-1}{2}}) = \lim_{t\to\infty}(\frac{t^{\frac{1}{2}}-1}{\frac{1}{2}}) = 0$$

We deduce that

$$\lim_{t\to\infty}(t_0^{\frac{-1}{2}-iy}) = \lim_{t\to\infty}(\frac{t^{\frac{1}{2}-iy}-1}{\frac{1}{2}-iy}) = 0; \forall y \mid \lim_{t\to\infty}(t_0^{-iy}) \neq 0$$

And for every m

$$\lim_{t\to\infty}(\lambda^{\frac{-1}{m}-iy})=\lim_{t\to\infty}(\frac{t-t^{\frac{1}{m}+iy}}{\frac{m-1}{m}-iy})=\lim_{t\to\infty}(\frac{t-t^{\frac{m-1}{m}-iy}t^{2iy+\frac{2-m}{m}}}{\frac{m-1}{m}-iy})=\lim_{t\to\infty}(\frac{t-(t_0^{-\frac{1}{2}-iy}(\frac{m-1}{m}-iy)+1)t^{2iy+\frac{2-m}{m}}}{\frac{m-1}{m}-iy})$$

$$=\lim_{t\to\infty}(\frac{t-(\lambda^{\frac{-1}{m}-iy}t^{-\frac{1}{m}-iy}(\frac{m-1}{m}-iy)+1)t^{2iy+\frac{2-m}{m}}}{\frac{m-1}{m}-iy})=\lim_{t\to\infty}(\frac{t}{\frac{m-1}{m}-iy}-\lambda^{\frac{-1}{2}-iy}t^{\frac{1-m}{m}+iy}-\frac{t^{2iy+\frac{2-m}{m}}}{\frac{m-1}{m}-iy})$$

$$\lim_{t\to\infty}(\lambda^{\frac{-1}{2}-iy}(1+t^{\frac{1-m}{m}+iy}))=\lim_{t\to\infty}(\frac{t-t^{2iy+\frac{2-m}{m}}}{\frac{m-1}{m}-iy})$$

$$\lim_{t\to\infty}(\lambda^{\frac{-1}{2}-iy})=\lim_{t\to\infty}(\frac{t-t^{2iy+\frac{2-m}{m}}}{(\frac{m-1}{m}-iy)(1+t^{\frac{1-m}{m}+iy})})=\lim_{t\to\infty}(\frac{t-t^{\frac{1}{m}+iy}}{\frac{m-1}{m}-iy})$$

$$\lim_{t\to\infty}(t-t^{2iy+\frac{2-m}{m}})=\lim_{t\to\infty}((1+t^{\frac{1-m}{m}+iy})(t-t^{\frac{1}{m}+iy}))=\lim_{t\to\infty}(t-t^{2iy+\frac{2-m}{m}}+t^{\frac{1-m}{m}+iy}-t^{\frac{1}{m}+iy})$$

$$\lim_{t\to\infty}(t^{\frac{1-m}{m}+iy})=\lim_{t\to\infty}(t^{\frac{1}{m}+iy})=\lim_{t\to\infty}(t^{iy+\frac{2-m}{2m}})$$

It is true for every m, particularly in the infinity, thus

$$\lim_{m\to\infty}(\lim_{t\to\infty}(t^{\frac{1-m}{m}+iy}))=\lim_{m\to\infty}(\lim_{t\to\infty}(t^{\frac{1}{m}+iy}))=\lim_{m\to\infty}(\lim_{t\to\infty}(t^{iy+\frac{2-m}{2m}}))$$

$$=\lim_{t\to\infty}(t^{-1+iy})=\lim_{t\to\infty}(t^{iy})=\lim_{t\to\infty}(t^{iy-\frac{1}{2}})=1$$

And it is impossible as it means that the Riemann function is equal to 0 in all the segment between 0 and 1. Thus m=2 defines the zeros of the Riemann function !

We have proved that the non trivial zeros of the Riemann function for the reals lie in the critical line ! So the hypothesis is proved for the real numbers. The Riemann hypothesis is important because it gives information about the zeros of the Riemann function and the distribution of those zeros are related to real primes !

The generalization to the integers

We have

$$\zeta_1(z)=\zeta(z)+A=B\zeta(z)$$
$$\zeta(z)=\zeta_1(z)+A'=B'\zeta_1(z)$$

The interpretation is what follows

$$\int_1^\infty \frac{dt}{t^z} = \frac{1}{1-z}\left[t^{1-z}\right]_1^\infty = \sum_{t=1}^\infty \frac{1}{t^z} + \sum_{\alpha\in Q^-}^\infty \frac{1}{t'^{\alpha z}} + \sum_{\alpha\in Q^+\setminus N}^\infty \frac{1}{t'^{\alpha z}}$$
$$= \sum_{t=1}^\infty \frac{1}{t^z}.B = \prod_{primes}(\sum_{\alpha\in N}\frac{1}{p^{\alpha z}}).B = \sum_{t=1}^\infty \frac{1}{t^z} + A$$

But

$$\frac{\zeta_1(z)}{\zeta(z)} = \frac{AB(B'-1)}{A'B'(B-1)} = \frac{-B(B'-1)}{B'(B-1)} = \frac{A'(B-1)}{A(B'-1)} = \frac{-(B-1)}{B'-1}$$

Thus

$$(B'-1)\zeta(z) + A(B'-1) = -(B-1)\zeta_1(z) - A'(B-1)$$
$$= -B(B-1)\zeta(z) + A(B-1) = B'(B'-1)\zeta_1(z) - A'(B'-1)$$

Hence

$$\zeta(z) = \frac{A(B-B')}{B'-1+B(B-1)}$$
$$\zeta_1(z) = \frac{-A'(B-B')}{B-1+B'(B'-1)}$$

And

$$\frac{\zeta_1(z)}{\zeta(z)} = \frac{-A'}{A}\frac{B'-1+B(B-1)}{B-1+B'(B'-1)} = \frac{B'-1+B(B-1)}{B-1+B'(B'-1)} = \frac{-(B-1)}{B'-1}$$

We deduce

$$(B'-1)^2 + (B-1)^2 = -(B+B')(B-1)(B'-1)$$

It means

$$(B'-1)^2 + (B-1)^2 - 2(B-1)(B'-1) = (-B-B'-2)(B-1)(B'-1)$$
$$= (B-B')^2 = (B+1-(B'+1))^2 = (B+1)^2 + (B'+1)^2 - 2(B+1)(B'+1)$$

Or

$$(B+1)(B+1+(B-1)(B'-1)-(B'+1)) + (B'+1)(B'+1+(B-1)(B'-1)-(B+1)) = 0$$

If

$$B = 0 \Rightarrow (-B'-B'+1) + (B'+1)(B'-B'+1) = 0 = (1-2B') + B'+1 = 2-B' \Rightarrow B' = 2$$

And if

$$B' = 0 \Rightarrow (B+1)(B-B+1) + (-B-B+1) = 0 = 2-B \Rightarrow B = 2$$

It means that if

$$\zeta(z) = 0 = B'\zeta_1(z) \Rightarrow B' = 0 \Rightarrow B = 2 \Rightarrow \zeta_1(z) = B\zeta(z) = 2\zeta(z) = 0 \Rightarrow \zeta_1(z) = 0$$

And if

$$\zeta_1(z)=0=B\zeta(z)\Rightarrow B=0\Rightarrow B'=2\Rightarrow\zeta(z)=B'\zeta_1(z)=2\zeta_1(z)=0\Rightarrow\zeta(z)=0$$

We have the equivalence

$$\zeta(z)=0\Leftrightarrow\zeta_1(z)=0$$

But

$$\zeta_1(\frac{1}{2}+iy)=0\Leftrightarrow\zeta(\frac{1}{2}+iy)=0$$

Thus the Riemann hypothesis is proved !

Thus the non trivial zeros of the Riemann function zeta lie in the critical line like for the reals ! It is the proof of the Riemann hypothesis !

Conclusion

We have generalized the concept of prime to the reals. It allowed to prove the conjecture for the reals and for the integers. Thus, we have proved the Riemann hypothesis.

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