THE PRIME SPACING THEOREMS

KIM E LUMBARD

This paper is dedicated to everyone who has gazed at the integers and seen the prime wonder.

ABSTRACT. The k-tuple Conjecture is true. The pure elimination sieve finds all prime instances of a k-tuple $\mathcal{H}$. Sieve function conjunction yields a $O(x^{1/2-\varepsilon})$ bound on the prime counting functions. Admissible k-tuples occur infinitely with asymptotic order $\sim \mathfrak{S}\, x/(\ln x)^k$. The maximum gap between primes is at most $O(\sqrt{x}/\ln x)$. Many conjectures are closed. Computation is confounded. We speculate on a potential approach to the Riemann Hypothesis.

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1. THE PURE ELIMINATION SIEVE FINDS ALL k-TUPLE PRIME INSTANCES

Given an admissible k-tuple $\mathcal{H}$, $k \in \mathbb{Z}^+$ [1]. Without loss of generality assume $0 \in \mathcal{H}$ with positive even integers. Call $L \equiv L(\mathcal{H}) = \max h \in \mathcal{H}$ the *length* of the tuple (one less than the other common definition). Let $\mathbb{H} \equiv \mathbb{H}(\mathcal{H}) \subset \mathbb{P}$ be the set of smallest prime instances p of $\mathcal{H}$, and $\pi_{\mathcal{H}}(x)$ count those

$$\pi_{\mathcal{H}}(x) = \sum_{p \le x}^{p\,\in\,\mathbb{H}} 1 \qquad \mathbb{H} = \{p \ni p + \mathcal{H} \subset \mathbb{P}\} \tag{1.1}$$

We ask: is $\pi_{\mathcal{H}}(x)$ unbounded? If so, what is its asymptotic order?

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Thanks to my college mentor Robert J McEliece (who showed me the joys of discrete counting problems) and Andrew M Odlyzko (for his patience and help with Mertens products).

Let p_a be the a^{th} prime and P_a be the a^{th} primorial, $P_a = \prod_{i=1}^{a} p_i$.

We mimic the Sieve of Eratosthenes [2] and use a *pure elimination sieve* (PES). At each step $a \in \mathbb{Z}^+$, we remove every integer that can't be in $\mathbb{H}$, crossing off up to k residues for each prime. $n+h$ is composite $\implies n \notin \mathbb{H}$. The PES eliminates all composite and some prime n, when $k \geq 2$. At step a we resolve all integers

$$n + h \equiv 0 \mod p_a \iff n \equiv -h \mod p_a \qquad \forall h \in \mathcal{H} \tag{1.2}$$

One of these will be p_a, the prime for that step. If that wasn't already crossed off, then $p_a \in \mathbb{H}^{(\mathbf{s})}$. All other residues can't be in $\mathbb{H}$ due to composite interference since p_a also divides $n + h + jP_a, j \in \mathbb{Z}^+$. Define our sieve counting function as

$$\pi_{\mathcal{H}}^{(\mathbf{s})}(x) = \sum_{p \leq x}^{p \in \mathbb{H}^{(\mathbf{s})}} 1 \qquad \mathbb{H}^{(\mathbf{s})} = \{p_a \not\equiv -h \mod p \quad \forall h \in \mathcal{H}, \forall p < p_a\} \tag{1.3}$$

Clearly $\mathbb{H} \subset \mathbb{H}^{(\mathbf{s})}$, as the sieve eliminates all integers not in $\mathbb{H}$, and only those. However, the sieve can declare false positives. How many errors might it make?

Theorem 1.1 (Exact Match Region)**.** *Let p_E be the prime satisfying*

$$p_E^2 - p_E \leq L \qquad p_{E+1}^2 - p_{E+1} > L \qquad p_0 = 0 \tag{1.4}$$

where L is the length of $\mathcal{H}$. Then the PES has at most E false positive errors

$$\pi_{\mathcal{H}}^{(\mathbf{s})}(x) - E \leq \pi_{\mathcal{H}}(x) \leq \pi_{\mathcal{H}}^{(\mathbf{s})}(x) \qquad x > 0 \qquad E \geq 0 \tag{1.5}$$

and these occur in the first E steps ($p \in \mathbb{H}^{(\mathbf{s})},\ p \notin \mathbb{H} \implies p \leq p_E$).

Proof. Recall we only need to check the prime factors of an integer n up to $\sqrt{n}$. p_a is assured when $p_a < p_a^2 - L$, as no integers are resolved between p_a and $p_a^2 - L$ at the a^{th} step. So after the E^{th} step all sieved $p \in \mathbb{H}^{(\mathbf{s})}$ are genuinely in $\mathbb{H}$, as are all sieved candidates in the region p_a to $p_a^2 - L$. □

The PES has compelling merits. It is simpler than more sophisticated methods [3] [4] [5] [6], counts all instances, and is nearly exact with trivial overcounting.

1.1. A Twin Prime Example that readily Generalizes.

Let $\mathcal{H} = (0, 2)$, the twin prime 2-tuple [7]. Then on the first step

primes = $\mathbf{p_i}$	n	①	*2*	= *integer equivalence classes* of $\mathbb{Z}_{P_1}$
2	r_1	1	**0**	= residues mod p_a
			↑	= just resolved, ◯ = candidates

We declare $p_1 = 2 \in \mathbb{H}^{(\mathbf{s})}$, our one error $E = 1$. All integers $n > 2$ with residue $r_1 \equiv 0 = -2 \mod 2$ are resolved: none are in $\mathbb{H}$. Odd n are still candidates.

We form $\mathbb{Z}_{P_2}$ by duplicating $\mathbb{Z}_{P_1}$ $p_2 = 3$ times. After resolving we see the well known modulo 6 result

$\mathbf{p_i}$	n	*1*	*2*	*3*	*4*	⑤	*6*	$\mathbb{Z}_{P_2}$
2	r_1	1	**0**	1	**0**	1	**0**	
3	r_2	**1**	2	**0**	1	2	0	
		↑		↑	×		×	× = already done

We've resolved two entries, but only p_a can be part of an instance.

$$r = 3 = p_2 \implies \text{twin prime } \{3, 5\} \implies 3 \in \mathbb{H} \quad 3 + jP_2 \notin \mathbb{H}, j \geq 1$$
$$r = 1 \neq p_2 \implies 1 + jP_2 \notin \mathbb{H}, j \geq 0$$

We form $\mathbb{Z}_{P_3}$ by duplicating $\mathbb{Z}_{P_2}$ $p_3 = 5$ times

$\mathbf{p_i}$	n	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	*(11)*	*12*	*13*	*14*	*15*	*16*	*(17)*	*18*	*19*	*20*	*21*	*22*	*23*	*24*	*25*	*26*	*27*	*28*	*(29)*	*30*
2	r_1	1	**0**	1	**0**	1	**0**	1	**0**	1	**0**	1	**0**	1	**0**	1	**0**	1	**0**	1	**0**	1	**0**	1	**0**	1	**0**	1	**0**	1	**0**
3	r_2	**1**	2	**0**	1	2	0	**1**	2	**0**	1	2	0	**1**	2	**0**	1	2	0	**1**	2	**0**	1	2	0	**1**	2	**0**	1	2	0
5	r_3	1	2	3	4	**0**	1	2	3	4	0	1	2	3	4	0	1	2	3	4	0	1	2	**3**	4	0	1	2	3	4	0
				×		↑			×		×			×		×			×		×			↑		×			×		×

$$r = 5 = p_3 \quad \implies \quad \text{twin prime } 5 + \mathcal{H} \implies 5 \in \mathbb{H} \quad 5 + jP_3 \notin \mathbb{H}, j \geq 1$$
$$r = 23 \neq p_3 \quad \implies \quad 23 + jP_3 \notin \mathbb{H}, j \geq 0$$

Further iteration produces all remaining twin primes, though not at every step. To illustrate, 7, 13, 19, and 23 are primes but are eliminated as candidates along with all integers with those residues mod 30, because 9, 15, 21, and 25 mod 30 are composite. Only primes that are 11, 17, or 29 mod 30 are still viable candidates. All prime steps are taken, twin or not. At each step we disqualify a fraction of the primes, but a non-zero fraction always remains to consider, in an asymptotic sense a la Dirichlet's Theorem [8].

2. The Bound on Conjunctions of Sieve Functions is $O(x^{1/2-\varepsilon})$

Definition 2.1. Since residues can overlap, we may cross off less than the full k. Let $\nu_p \equiv \nu_p(\mathcal{H})$ count the *distinct residues* $-h$ mod p among $h \in \mathcal{H}$. $1 \leq \nu_p \leq k$ always and eventually $\nu_p = k$ once $p > L$. *Admissible* $\mathcal{H} \iff \nu_p < p\ \forall p$.

Definition 2.2. Define the *phi-$\mathcal{H}$ function* akin to the Legendre phi function $\phi(x, a)$ for the primes, namely that it counts all candidates still remaining at the a^{th} step of the sieve. The value at the primorial is

$$\phi_{\mathcal{H}}(P_a, a) = \prod_{p \leq p_a} (p - \nu_p) \underset{p_a > L}{=} \prod_{p \leq L} (p - \nu_p) \prod_{L < p \leq p_a} (p - k) \tag{2.1}$$

and the Exact Match Region in Theorem 1.1 means that for $a = \pi(\sqrt{x+L}) > E$

$$\pi_{\mathcal{H}}(x) - \pi_{\mathcal{H}}(\sqrt{x+L}) = \phi_{\mathcal{H}}(x, a) - \phi_{\mathcal{H}}(\sqrt{x+L}, a) \qquad p_a < x < p_a^2 - L \tag{2.2}$$

Our goal is to bound $\pi_{\mathcal{H}}(x)$ at x, which we achieve by developing the best bounded linear approximation Kmx to $\phi_{\mathcal{H}}(x, a)$ on the interval $[\sqrt{x+L}, x]$. Let us extract some constructive properties from phi-$\mathcal{H}$ that we'll need.

Definition 2.3. Let $s(x)$ be an integer counting function with $s(0) = 0$. $s(x)$ is a *sieve function* if it is periodic with finite period $\rho = \prod p$ and $r = s(\rho)$, $0 < r < \rho$. ρ must be single primes with no multiplicity; two sieve functions $s_1(x)$ and $s_2(x)$ are *compatible* if $\gcd(\rho_1, \rho_2) = 1 \iff \rho = \rho_1 \rho_2$ is a valid period.

We interpret a sieve function staying flat at n to mean that integer is eliminated, while when it jumps n is still a possible candidate. Observe $0 < m = r/\rho < 1$.

$\phi_{\mathcal{H}}(x, a)$ has period $\rho = P_a$ and $\phi_{\mathcal{H}}(0, a) = 0$. From Equation (2.1) we know $\phi_{\mathcal{H}}(P_a, a) < P_a$ and is non-zero for admissible $\mathcal{H} \iff \phi_{\mathcal{H}}(x, a)$ is a sieve function.

Theorem 2.4 (Sieve Function Bound). *A sieve function $s(x)$ is bounded by*

$$-\lambda < s(x) - mx \leq +\upsilon \qquad \lambda = -s(l^-) + ml \geq 0 \qquad \upsilon = s(u) - mu \geq 0 \tag{2.3}$$

Either bound can only be tight when $s(n)$ jumps; l and u are those spots.

Proof. Since $s(0) = 0$ and is periodic, then $s(n\rho) = nr$. Thus, mx intersects it infinitely often at $n\rho$. Consider all lines $mx + b$. The bound lines have to hit a corner to be tight, i.e. $s(n)$ jumps there. If u is the upper corner, then $(u, s(u))$ is on the upper bound line. If l is the lower corner, then $(l, s(l - \varepsilon))$ is on the lower bound line. Solving for b in each case gives the results.

[The ε and non-symmetric inequalities are because counting functions favor the top of a step when they jump, which is a niggling yet negligible subtlety. The peculiar $-\lambda$ and $+\upsilon$ are to emphasize positivity and make the bounds clearer.] ☐

Definition 2.5. Let $s_1(x)$ and $s_2(x)$ be two compatible sieve functions. Their *conjunction sieve function* $c(x) \equiv (s_1 \wedge s_2)(x)$ jumps at n iff both $s_1(n)$ and $s_2(n)$ jump (still a candidate). $c(x)$ has period $\rho = \rho_1\rho_2$ and from the magic of primality $r = r_1 r_2$. Conjunction is commutative, associative, and may be done in any order.

["$\wedge$" is logical AND, not to be confused with the von Mangoldt function $\Lambda(n)$.]

Theorem 2.6 (Prime Conjunction)**.** *Let $s_1(x)$ and $s_2(x)$ be two compatible sieve functions. Then*

$$\lim_{x\to\infty} \frac{s_1(x)\, s_2(x)}{x(s_1 \wedge s_2)(x)} = 1 \qquad \Longrightarrow \qquad c(x) = (1 + \epsilon(x))\frac{s_1(x)\, s_2(x)}{x} \tag{2.4}$$

For convenience denote $(1 + \epsilon(x))$ as (1_ϵ). The conjunction is bounded by

$$\begin{aligned} -(1_\epsilon)(m_2\lambda_1 + m_1\lambda_2 - \tfrac{1}{x}\lambda_1\lambda_2) &< c(x) - (1_\epsilon)m_1m_2x \\ &\leq +(1_\epsilon)(m_2\upsilon_1 + m_1\upsilon_2 + \tfrac{1}{x}\upsilon_1\upsilon_2) \end{aligned} \tag{2.5}$$

and (1_ϵ) is explicitly

$$(1 + \epsilon(x)) = 1 + \frac{b - m_1b_2 - m_2b_1}{m_1m_2x} + O\left(\frac{1}{x^2}\right) \tag{2.6}$$

where $-\lambda_i < b_i \leq +\upsilon_i$ and b uses the bounds from Equation (2.5)*.*

Proof. From Theorem 2.4 $s_i(x) \sim m_ix$ and $c(x) \sim m_1m_2x$, so the limit is simply balancing lead order terms. The limit also gives us $\epsilon(x) \to 0$ as $x \to \infty$.

Expressing $s_i(x) = m_ix + b_i$, the product becomes

$$\frac{s_1(x)\, s_2(x)}{x} = m_1m_2x + (m_1b_2 + m_2b_1) + \frac{b_1b_2}{x} \tag{2.7}$$

For the conjunction upper bound, use $b_i = \upsilon_i$; for the lower, use $b_i = -\lambda_i$ and note $(-\lambda_1)(-\lambda_2) = +\lambda_1\lambda_2$. Using $c(x) = m_1m_2x + b$ and solving gives (1_ϵ)

$$(1_\epsilon) = \frac{m_1m_2x + b}{m_1m_2x + (m_1b_2 + m_2b_1) + \frac{1}{x}b_1b_2} = \frac{1 + \frac{b}{m_1m_2x}}{1 - \frac{-(m_1b_2+m_2b_1)}{m_1m_2x} + O\left(\frac{1}{x^2}\right)} \tag{2.8}$$

Use $1/(1 - y) = 1 + y + \ldots$, multiply, and only keep the two $1/x$ terms. ☐

Though it looks innocent and innocuous, the Prime Conjunction Theorem 2.6 is the crux of a new form of prime analysis. Let's leverage it for some amazing results.

Theorem 2.7. *The phi-$\mathcal{H}$ function is bounded by*

$$-\lambda(x) < \phi_{\mathcal{H}}(x, a(x)) - (K_\epsilon)m(x)x < \upsilon(x) \tag{2.9}$$

where $a(x) = \pi(\sqrt{x + L})$, $\lambda(x)$ and $\upsilon(x)$ are both $O\left(m(x)\, a(x)\right)$, and

$$(K_\epsilon) \equiv (K_\epsilon)(x) = \prod^{a-1}(1 + \epsilon(x)) \qquad m(x) = \prod_{p\leq\sqrt{x+L}} \left(\frac{p - \nu_p}{p}\right) \tag{2.10}$$

Proof. $\phi_{\mathcal{H}}(x,a)$ is the conjunction of the sieve functions $s_p(x)$ for the first a primes. Using Theorem 2.4, a single prime has $\rho = p$, $r = p - \nu_p$, $m_p = (p-\nu_p)/p$, and once $p > L+1$ then $l = 1$, $u = p - L - 1$. Since $s_p(l^-) = 0$ and $s_p(u) = p - L - 1$

$$-\frac{p-\nu_p}{p} < s_p(x) - \frac{p-\nu_p}{p}x \le +\frac{p-L-1}{p}\nu_p \qquad p > L+1 \tag{2.11}$$

For $p \le L+1$ either bound may go as high as $\frac{p-1}{p}\nu_p$. Recall $1 \le \nu_p \le k$ from Definition 2.1 and $\sum_{p<x}\frac{1}{p} \to \ln\ln x + M$, the Meissel-Mertens Constant [9]. Thus,

$$\begin{array}{lcll} \lambda_\Sigma(x) & \equiv & \sum \lambda_p & = a(x) - k\,\ln\ln\sqrt{x+L} + O\left(1\right) \\ \upsilon_\Sigma(x) & \equiv & \sum \upsilon_p & = k\,a(x) - k\,(L+1)\ln\ln\sqrt{x+L} + O\left(1\right) \end{array} \tag{2.12}$$

which are both clearly $O\left(a(x)\right)$. [The $O\left(1\right)$ constants also depend on $\mathcal{H}$.]

Since p_i and P_{i-1} are compatible when $i \ge 2$, apply Theorem 2.6 $a-1$ times with $s_1(x) = s_p(x)$ a single prime and $s_2(x) = (\bigwedge_{q<p} s_q)(x)$ the conjunction of all primes before p. Use induction on Equation (2.4) to get (K_ϵ) and $m(x)$

$$\phi_{\mathcal{H}}(x,a) = (K_\epsilon)\frac{1}{x^{a-1}}\prod_{p\le p_a} s_p(x) = (K_\epsilon)m(x)x\prod_{p\le p_a}(1+\frac{b_p}{m_p x}) \tag{2.13}$$

We handle the bounds as before, setting $b_p = \upsilon_p$. Expanding the product we get

$$\prod = 1 + \frac{1}{x}\sum_i \frac{\upsilon_i}{m_i} + \frac{1}{x^2}\sum_{i<j}\frac{\upsilon_i}{m_i}\frac{\upsilon_j}{m_j} + \frac{1}{x^3}\sum_{i<j<l}\frac{\upsilon_i}{m_i}\frac{\upsilon_j}{m_j}\frac{\upsilon_l}{m_l} + \ldots \tag{2.14}$$

Let us relax the bound. First let $A = \max_p 1/m_p$ and pull it outside each sum, then include higher multiplicity cross terms for a simpler expression

$$\prod \le 1 + \frac{A}{x}\upsilon_\Sigma(x) + \frac{A^2}{x^2}\upsilon_\Sigma^2(x) + \frac{A^3}{x^3}\upsilon_\Sigma^3(x) + \ldots \le \frac{1}{1-\frac{A}{x}\upsilon_\Sigma(x)} \tag{2.15}$$

This yields an upper bound of the stated order

$$\upsilon(x) = A(K_\epsilon)\,m(x)\,\upsilon_\Sigma(x) + O\left(\frac{m(x)\,a^2(x)}{x}\right) \tag{2.16}$$

Using $b_p = -\lambda_p$ for the lower bound gives Equation (2.14) alternating signs. Forgive even degree positive terms and a similar result holds with $O\left(m(x)\,a^3(x)/x^2\right)$. □

Theorem 2.8 ($\mathcal{H}$-Counting Function Bound)**.** *$\pi_{\mathcal{H}}(x)$ is bounded by*

$$-\lambda(x) < \pi_{\mathcal{H}}(x) - [(K_\epsilon)m(x)x + \pi_{\mathcal{H}}(\sqrt{x+L}) - \phi_1] < \upsilon(x) \qquad a(x) > E \tag{2.17}$$

with functions as from Theorem 2.7, E as from Theorem 1.1, and

$$\begin{array}{lcl} \lambda(x) & = & A_\lambda(K_\epsilon)\,m(x)[\,a(x) - k\,\ln\ln\sqrt{x+L} + O\left(1\right)] + o\left(1\right) \\ \upsilon(x) & = & A_\upsilon(K_\epsilon)\,m(x)[\,k\,a(x) - k\,(L+1)\ln\ln\sqrt{x+L} + O\left(1\right)] + o\left(1\right) \end{array} \tag{2.18}$$

where $A = \max(A_\lambda, A_\upsilon) \le \max_{p\le k+1} p/(p-\nu_p) \le k+1$ and $A \to 1$ as $x \to \infty$.

Proof. Rearrange Equation (2.2) and apply Theorem 2.7 to get the bounds

$$\pi_{\mathcal{H}}(x) = \phi_{\mathcal{H}}(x,a(x)) + \pi_{\mathcal{H}}(\sqrt{x+L}) - \phi_{\mathcal{H}}(\sqrt{x+L},a(x)) \tag{2.19}$$

Note $\phi_1 = \phi_{\mathcal{H}}(\sqrt{x+L},a) = \phi_{\mathcal{H}}(1,a)$ is just 0 or 1 if residue 1 is eliminated or not. Average the whole weighted sum in Equation (2.14) with $A_\upsilon\upsilon_\Sigma = \sum \upsilon_p/m_p$ instead of bounding individual terms. Since $1/m_p \to 1$ and the excess is $O\left(\ln\ln\sqrt{x+L}\right)$ small, eventually $A_\upsilon \to 1$, as does A_λ and A. □

3. The k-Tuple Conjecture is True

Let us convert everything asymptotically into the usual elementary functions.

Theorem 3.1. *Mertens products can be re-expressed*

$$\prod_{k<p\le x}\left(\frac{p-k}{p}\right)=\frac{e^{S(k)}e^{\delta_k(x)}}{e^{k\gamma}\left(\ln x\right)^k}\qquad S(k)=\sum_{p\le k}\frac{k}{p}\qquad S(1)=0 \tag{3.1}$$

where $\gamma\approx 0.57721$ *is the Euler-Mascheroni Constant* [10] *and* $\delta_k(x)\to 0$ *as* $x\to\infty$.

Proof. The excellent treatment [11] on Mertens Theorem [12] has explicit error:

$$\prod_{p\le x}\left(\frac{p-1}{p}\right)=\frac{e^{\delta(x)}}{e^{\gamma}\ln x}\qquad |\delta(x)|<\beta(x)=\frac{4}{\ln(x+1)}+\frac{2}{x\ln x}+\frac{1}{2x} \tag{3.2}$$

where $e^{\gamma}\approx 1.78107$ and $\delta(x)\to 0$ as $x\to\infty$. Log both sides and expand the logs

$$\sum_{p\le x}\ln\left(1+\frac{-1}{p}\right)=\sum_{p\le x}\frac{-1}{p}+O\left(\frac{1}{p^2}\right)=-\gamma-\ln\ln x+\delta(x) \tag{3.3}$$

Repeating for our k products

$$\sum_{k<p\le x}\ln\left(1+\frac{-k}{p}\right)=\sum_{k<p\le x}\frac{-k}{p}+O\left(\frac{1}{p^2}\right)=S(k)-k\gamma-k\ln\ln x+\delta_k(x) \tag{3.4}$$

with $S(k)$ and $\delta_k(x)$ as given. Exponentiate both sides to finish. □

The behavior of (K_ϵ) bears special mention. When a is fixed, the finite product $(K_\epsilon)\to 1$ as $x\to\infty$, giving the slope $m(p_a^2-L)$. When $a=\pi(\sqrt{x+L})$, then (K_ϵ) becomes an infinite product that converges to a different constant.

Theorem 3.2. *Let* (K_ϵ) *be as Theorems 2.7 and 2.8. Then*

$$(K_\epsilon)\to\frac{e^{k\gamma}}{2^k}\qquad x\to\infty \tag{3.5}$$

Proof. From the PNT [13] [14] $\pi(x)\sim\mathrm{Li}(x)\sim x/\ln x$, so $O\left(a(x)\right)=O\left(\sqrt{x}/\ln x\right)$. Since the lead order terms in Theorem 2.8 must balance, we deduce from $k=1$

$$(K_\epsilon)\to\frac{e^{\gamma}}{2}\approx 0.89<1\qquad x\to\infty\qquad\Longrightarrow\qquad (K_\epsilon)=\frac{e^{\gamma}}{2}e^{\delta(x)}\qquad \delta(x)\to 0 \tag{3.6}$$

Return to Equation (2.6) where $s_1(x)=s_p(x)$, $s_2(x)=$ cumulative conjunction, $m_1=1-\nu_p/p\to 1$, $m_2=O\left((\ln x)^{-k}\right)$, and $|b_1|\le k<<|b,b_2|\le O\left(m(x)a(x)\right)$. Thus, m_2b_1 becomes negligible and $b=(1+\varepsilon')b_2$ are comparable. The worst case is $b\sim b_2$ near the bound $m(x)a(x)$, cancelling $m(x)=m_1m_2$ in the denominator

$$(1+\epsilon(x))=1+(1-m_1)O\left(\frac{a(x)}{x}\right)=1+\frac{\nu_p}{p}O\left(\frac{1}{\sqrt{x}\ln x}\right) \tag{3.7}$$

More precisely, each (1_ϵ) term is $1/x^{C_p}$, where $1/2+\varepsilon<C_p(x)\le 1$. We massage (K_ϵ) and use a similar argument as Theorem 3.1 to get

$$(K_\epsilon)=\prod\left[1+\frac{k}{p}O\left(\frac{1}{x^{C_p}}\right)\right]=\frac{e^{k\gamma}}{2^k}\frac{e^{\delta_k(x)}}{e^{S(k,x)}}\qquad S(k,x)=\sum_{p\le k}\frac{k}{p}O\left(\frac{1}{x^{C_p}}\right) \tag{3.8}$$

The $1/x^{C_p}$ forces $S(k,x)\to 0$ as $x\to\infty$, along with finite ν_p to k corrections. □

Taking stock, we see the bound on our counting functions is a very fine $x^{1/2-\varepsilon}$.

Theorem 3.3 (k-Tuple Proof)**.** *Given an admissible k-tuple $\mathcal{H}$, then*

$$\pi_{\mathcal{H}}(x) \sim \mathfrak{S}\frac{x}{(\ln x)^k} \quad = (K_\epsilon)m(x)x \qquad x \to \infty \tag{3.9}$$

where $\mathfrak{S}$ is the singular series, $S(k) = \sum_{p \leq k} \frac{k}{p}$, and $\mathfrak{M}$ is a convenience

$$\mathfrak{S} \equiv \mathfrak{S}(\mathcal{H}) = e^{S(k)}\mathfrak{M} \qquad \mathfrak{M} \equiv \mathfrak{M}(\mathcal{H}) = \prod_{p \leq k} \frac{p - \nu_p}{p} \prod_{k < p \leq L} \frac{p - \nu_p}{p - k} \tag{3.10}$$

Observe inadmissible $\mathcal{H} \iff \mathfrak{M} = 0 = \mathfrak{S} \equiv$ admissible $\mathcal{H} \iff \mathfrak{M}, \mathfrak{S} \in \mathbb{R}^+$.

Proof. Recall $\nu_p = k$ when $p > L$ from Definition 2.1. Then from Theorem 3.1

$$m(x) = \prod_{p \leq \sqrt{x+L}} \left(\frac{p - \nu_p}{p}\right) = \mathfrak{M} \prod_{k < p \leq \sqrt{x+L}} \left(\frac{p-k}{p}\right) \to \frac{2^k}{e^{k\gamma}} \mathfrak{S}\frac{1}{(\ln x)^k} \tag{3.11}$$

as $x \to \infty$. Apply Theorem 2.8 and the sub-orders vanish. The first constant cancels (K_ϵ) from Theorem 3.2, leaving just the singular series. □

3.1. The Maximum Gap between Instances is at most $O(\sqrt{x}/\ln x)$.

Theorem 3.4 (Square Root Prime Gap)**.** *Let $G_{\mathcal{H}}(x)$ be the maximum gap around x between prime instances of $\mathcal{H}$. Then $G_{\mathcal{H}}(x)$ is bounded by*

$$G_{\mathcal{H}}(x) \leq A(\lambda_\Sigma(x) + \upsilon_\Sigma(x)) + o\,(1) \quad \sim 2(k+1)\frac{\sqrt{x}}{\ln x} \qquad x \to \infty \tag{3.12}$$

Proof. Since run=rise/slope the maximum run is limited by the bounded rise

$$G_{\mathcal{H}}(x) \leq \frac{\lambda(x) + \upsilon(x)}{(K_\epsilon)m(x)} = A(\lambda_\Sigma(x) + \upsilon_\Sigma(x)) + O\left(\frac{a^2(x)}{x}\right) \tag{3.13}$$

The denominator cancels from Equation (2.16) and $A \to 1$ from Theorem 2.8. Equation (2.12) gives a slightly tighter bound, if needed. □

4. Numerous Conjectures are Closed and Affected

The k-tuple Conjecture is a central result from which many others follow. For example, we've also disproven the second Hardy-Littlewood Conjecture $\pi(x+y) \leq \pi(x) + \pi(y)$, as the two are incompatible [15]. [Note k-tuple is known by other names, aka the first Hardy-Littlewood Conjecture, Prime Constellation, etc.]

The Polignac Conjecture [16] is k-tuple for $k = 2$. Since $\mathcal{H} = (0, 2n)$ is always admissible, all even separations occur infinitely between pairs of primes, including well known special cases like Twin Primes, Cousin Primes, Sexy Primes, etc.

Goldbach-like conjectures [17] on the differences of primes follow. To illustrate, Polignac implies every even integer is the difference of two primes (infinitely, both consecutively and non-consecutively to boot).

The Square Root Prime Gap unconditionally improves the θ bound to $\frac{1}{2} - \varepsilon$ from 0.525 [18] [19]. Any conjecture of "Does a prime exist in {this linear interval}?" is true. Several results of this kind are already known [20] [21] [22] [23].

Moreover, the finer square root order conjectures have been proven:

- Oppermann - there are primes between $n(n-1)$, n^2, and $n(n+1)$ [24]
- Andrica - the next prime gap is bounded by $2\sqrt{p} + 1$ [25] [26]
- Legendre - there is a prime between perfect squares [27]
- Brocard - there are at least four primes between prime squares [28]

Keep in mind that equivalents of these conjectures are true for admissible k-tuples. In one fell swoop we have proven they hold not only for the primes but for all k. [When generalizing we must qualify "eventually after {this condition}" because of possible initial finite failure a la E in Theorem 1.1.] In fact, we expect that many prime results translate to k-tuples, especially if there is a clean "sieve-based" interpretation. For example, we could prove a k-Dirichlet's Theorem using an $\mathcal{H}$-totient function, thereby establishing valid arithmetic progressions, etc.

5. Practical Computation of $\pi_{\mathcal{H}}(x)$ has a Serious Confound

Thus far, our method has been the goose that laid the golden results. But it suffers from a major flaw: it's useless to actually compute $\pi_{\mathcal{H}}(x)$. In Equation (2.13) (K_ϵ) is defined using the *true value* of $\phi_{\mathcal{H}}(x,a)$... the goal of computation! Our slick analytic trick is at the same time an epic practical drawback.

Let's work through what an implementation would look like. Define

$$(5.1)\quad \sigma(x) = \frac{1}{x^{a-1}} \prod_{p<\sqrt{x+L}} s_p(x) \quad\implies\quad \pi_{\mathcal{H}}(x) = (K_\epsilon)\sigma(x) + \pi_{\mathcal{H}}(\sqrt{x+L}) - \phi_1$$

[Note the Mertens product $m(x)$ was just a tool used to establish asymptotic order, the limit of (K_ϵ), and bounded behavior. It isn't needed for calculation.]

More specifically for the primes $k=1$, $L=0$, $a=\pi(\sqrt{x})$

$$(5.2)\qquad \pi(x) = (K_\epsilon)\frac{1}{x^{a-1}} \prod_{p<\sqrt{x}} \left[(p-1)\left\lfloor \frac{x}{p} \right\rfloor + (\lfloor x \rfloor \mod p)\right] + a - 1$$

The $\sigma(x)$ product terms involve simple divisor / remainder operations on $n = \lfloor x \rfloor$.

The convergence of $(K_\epsilon) = \phi/\sigma$ to $e^\gamma/2 \approx .890536$ is log order slow

x	$\phi(x,a)$	$\sigma(x)$	(K_ϵ)	$\frac{e^\gamma}{2}\sum\frac{(n-1)!}{(\ln x)^n}$
10^1	3	3.50000	0.857143	1.613222
10^2	22	23.0480	0.954530	1.270127
10^3	158	153.819	1.027181	1.095073
10^4	1,205	1204.70	1.000245	1.022861
10^5	9,528	9655.23	0.986823	0.987375
10^6	78,331	80971.2	0.967393	0.967376
10^7	664,134	696,029	0.954175	0.954408
10^8	5,760,227	6,088,507	0.946082	0.945249
10^9	50,844,133	54,166,914	0.938657	0.938413
10^{10}	455,042,919	487,529,410	0.933365	0.933108
10^{11}	4,118,027,520	4,433,046,839	0.928938	0.928867
10^{12}	37,607,833,520	40,638,211,759	0.925430	0.925397
10^{13}	346,065,309,192	375,126,848,134	0.922529	0.922506
10^{14}	3,204,941,086,223	3,483,377,464,535	0.920067	0.920059
10^{15}	29,844,568,470,712	32,511,482,032,519	0.917970	0.917960
10^{16}	279,238,335,272,470	304,797,216,193,497	0.916145	0.916141

Apart from (K_ϵ), calculation error would be very manageable. $\phi_1, \lambda_p, \upsilon_p$, and m_p are exact and trivial. Choosing $x = n$ ensures $\sigma(x)$ is rational and incurs no more than numerical error. $\pi_{\mathcal{H}}(\sqrt{x+L})$ might introduce some real error, but it would still be an acceptable sub-square root. (K_ϵ) is the stumbling block.

5.1. **We Conclude with Observations and Further Work.** In 1901, von Koch proved that the Reimann Hypothesis gave the best possible bound for $\pi(x)$ [29]. $\mathrm{Li}(x)$ was $O\left(\sqrt{x}\ln x\right)$ away, we couldn't do any better, and consequently complex analysis became a primary focus of attention. So how can we have found a superior function $(K_\epsilon)m(x)x$, bounded a mere $O\left(\sqrt{x}/(\ln x)^2\right)$ away, using only a simple sieve? Von Koch appears to cast serious doubt on our result.

One of my math professors at Caltech had a favorite saying: "If you want to approximate elephants, use elephant functions." His point: results are improved by a closer fit between model and problem. $\mathrm{Li}(x)$ is still the best approximation among monotonic smooth continuous functions. The previous section illustrates we haven't found a better one. We've just re-represented $\pi(x)$ itself in a way that is jagged enough for a snug fit, yet tractable enough to analyze. Von Koch doesn't apply, as $(K_\epsilon)m(x)x$ isn't smooth, continuous, or even an independent function.

We'd like to understand (K_ϵ) better: rate of convergence, if it approaches from above or below, how it sawtooths and oscillates, etc. We know (K_ϵ) and $\sigma(x)$ have log expansions, as the two multiplied must match the $\pi(x) \sim \mathrm{Li}(x)\ (n-1)!/(\ln x)^n$ expansion. Our interest in (K_ϵ) goes beyond a computational approximation; it may offer a possible route to attack the Riemann Hypothesis [30]. Corollary 3 on p.341 of Schoenfeld's Sharper Bounds II [31] has a Mertens product form

$$\left| e^\gamma \ln x \prod_{p\le x}\left(\frac{p-1}{p}\right) - 1\right| < \frac{3\ln x + 5}{8\pi\sqrt{x}} \qquad x > 8 \quad \Longleftrightarrow \quad \text{RH true} \tag{5.3}$$

and we could readily demonstrate via Theorem 2.8 and Equation (2.15)

$$\left| (K_\epsilon)\frac{x}{\pi(x)} \prod_{p\le\sqrt{x}}\left(\frac{p-1}{p}\right) - 1\right| < \frac{\pi(\sqrt{x})}{\pi(x)}(1 + o\,(1)) \qquad x \ge 2 \tag{5.4}$$

Thus we see the error is in the expansion of $(K_\epsilon)/\pi(x)$; this approach is viable only if it's better than log. In the (fingers crossed) event we can prove the Riemann Hypothesis via something like $(K_\epsilon) \sim \mathrm{Li}(x)/m(x)\,x$, we'll probably also be able to show the k-Riemann analogues $\mathfrak{S}\,\mathrm{Li}_{\mathcal{H}}(x)$ are $O\left(x^{1/2+\varepsilon}\right) + E$ close to $\pi_{\mathcal{H}}(x)$

$$\mathrm{Li}_{\mathcal{H}}(x) = \int_L^x \prod_{h\in\mathcal{H}} \frac{1}{\ln(t-h)}\,dt \quad \sim \quad \mathrm{Li}_k(x) = \int_0^x \frac{1}{(\ln t)^k}\,dt \tag{5.5}$$

Sieve variations amplified cause $\pi_{\mathcal{H}}(x)$ to "swing" unevenly. We can develop an improved analog of Littlewood's Theorem [32]. Theorem 2.4 has l and u where the Sieve Function Bound is tight. With some algebra we can calculate l_a and u_a at the a^{th} sieve step and model the swing of $\pi_{\mathcal{H}}(x)$. This would enable us to more accurately place the counting function (e.g. near u we are close to the upper bound) and give a subtle order correction to the slope.

Since our bounds are already asymptotically tight, the game is now predicting where within them $\pi_{\mathcal{H}}(x)$ falls. Note that our "bounds" are more of a "spread" from the conjunction operator, crudely treating each b_p as an independent event. A further refinement would be to express each $b_p \equiv b_p(x)$ and see whether more precision can be squeezed from their interdependence. One also wonders if the bound pinching around $p^2 - L$ give any substantial gains or insights.

As an intriguing curio, notice that the bounds in Theorem 2.8 become *tighter* as k increases. That means that instances become sparser but more regular, and the initial appearance(s) of $\mathcal{H}$ with small L, large k should be highly predictable.

We are completely convinced that the max gap is really $G_{\mathcal{H}}(x) \leq O\left((\ln x)^{k+1}\right)$, i.e. Cramer's [33] and Kourbatov's [34] Conjectures are true. Here is the thought experiment why. Given an admissible k-tuple $\mathcal{H}$, create two families of admissible $(k+1)$-tuples: one with $\mathcal{H}$ at the front and a lone prime stuck to the end, and the other with $\mathcal{H}$ shifted to the end and the lone prime is the new 0. Both families may be made with unboundedly large $L_{k+1} > L$. All the $(k+1)$-tuples occur on average every $O\left((\ln x)^{k+1}\right)$ so {insert careful density argument that distinguishes family members} the original k-tuple must occur at least that often. The constant is probably something like $\mathfrak{S}/\min \mathfrak{S}_{k+1}$.

We speculate that maximal gaps occur on u to l legs where slope is minimized, probably right before striking l. Connecting log order max gaps to precise movement of $\pi_{\mathcal{H}}(x)$ would be another major milestone in prime spacing [35].

The pure elimination sieve is an old technique infused with some stimulating new analysis. It's time to expand the method, dust off Hardy [36] and Shanks [37], and start proving prime conjectures.

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1715 Mcminn St, Aliquippa, PA 15001
E-mail address: `kel@alumni.caltech.edu`