Perfect Cuboid Inequalities

Abstract

A new approach is given for resolving the Perfect Cuboid problem: proving many special cases impossible using inequality relationships. These results are directly applicable to reducing the work for a computer search. There is also the prospect that all other cases can be proved impossible by extending this approach.

1. Perfect Cuboid Formulation

A Perfect Cuboid is

[1.1] $X^2 + Y^2 = T^2$
[1.2] $X^2 + Z^2 = U^2$
[1.3] $Y^2 + Z^2 = V^2$
[1.4] $X^2 + V^2 = W^2$

where X, Y, Z are the positive integer edge lengths of the cuboid,
T, U, V are the positive integer face diagonal lengths,
and W is the positive integer length of the space diagonal.

Equation **[1.4]** is equivalent to $X^2+Y^2+Z^2 = W^2$.

If X, Y, Z all have a factor in common, then from **[1.1],[1.2],[1.3]**,
T, U, V all have that factor, and then from **[1.4]**, W has that factor,
so that factor can be cancelled in all terms producing a smaller solution.

If X, Y, Z do not have a factor in common,
then at least one of X, Y, Z is odd.

By symmetry, assume X is odd.

2. Pythagorean Triangle Solution

A Perfect Cuboid contains within it several Pythagorean triangles.
Here is a complete solution for a Pythagorean triangle using a different formula than is usually given.

This solution includes both primitive and non-primitive cases.

Rerrange **[1.4]** to become

[2.1] $X^2 = (W - V)(W + V)$

Note that $(W - V)$ and $(W + V)$ are divisors of X^2
and since X is odd, those divisors are odd.
Also, since $W > V$,

$(W - V) < X$

and

$(W + V) > X$.

Assign a new odd variable A.

[2.2] $A = W - V$

A is a divisor of X^2, $A < X$, and from **[2.1]** and **[2.2]**,

[2.3] $X^2/A = W + V$

Add and subtract **[2.2]** and **[2.3]** to solve for V and W.

[2.4] $V = (X^2/A - A)/2$

[2.5] $W = (X^2/A + A)/2$

Repeat the above with **[1.1]** and **[1.2]**
and assign B and C, two more odd divisors of X^2
that are each less than X, to solve for Y, T, Z, U.

[2.6] $Y = (X^2/B - B)/2$

[2.7] $T = (X^2/B + B)/2$

[2.8] $Z = (X^2/C - C)/2$

[2.9] $U = (X^2/C + C)/2$

3. Divisor Formulation

Equation **[1.3]** still needs to be solved.
Substitute the solutions for Y, Z, V into **[1.3]** and cancel some 2's to get

$$[3.1] \quad (X^2/A - A)^2 = (X^2/B - B)^2 + (X^2/C - C)^2$$

where A, B, C are odd divisors of X^2 that are each less than odd X.

Equation **[3.1]** is the main subject of this paper.
Choosing special cases of the prime factorization of X
determines the possible A, B, C divisors.
In this paper, many of these cases will be proved impossible
using inequality arguments.

4. Divisor Inequalities

Since **[3.1]** is a Pythagorean triangle,
its hypotenuse must be larger than each of its legs.
The relative magnitude can be determined based on the
relative magnitudes of its A, B, C divisors.

Theorem
If $A < B$, then $(X^2/A - A) > (X^2/B - B)$.

Proof
If $A < B$,
then $1/A > 1/B$.
Multiply both sides by X^2.

$$X^2/A > X^2/B$$

Add B to the left and A to the right,
which doesn't reverse the inequality since $B > A$.

$$(X^2/A + B) > (X^2/B + A)$$

Subtract $A + B$ from both sides.

$$(X^2/A - A) > (X^2/B - B)$$

So Equation **[3.1]** requires $A < B$ and $A < C$.
B and C are swappable, so without loss of generality,
[3.1] requires that

$$[4.1] \quad 0 < A < B < C < X$$

5. Stronger Divisor Inequality for C

Rearrange **[3.1]** into a quadratic equation in X^2.

$$(A^2C^2+A^2B^2-B^2C^2)X^4 - 2(ABC)^2X^2 + (ABC)^2(C^2+B^2-A^2) = 0$$

Use the quadratic formula to solve for X^2.

$$X^2 = \frac{(ABC)^2 \pm (ABC)^2 sqrt[(C^2-A^2)(B^2-A^2)(B^2+C^2)]}{(A^2C^2+A^2B^2-B^2C^2)}$$

Since X^2 is positive, the numerator and denominator of the fraction must be either both positive or both negative.
Since $A < B < C$, the argument of the $sqrt$ is always positive and thus the $sqrt[]$ is always real.

There are two solutions of the quadratic equation, the "+" solution and the "-" solution.

For the "+" solution, the numerator is positive, thus the denominator must also be positive which requires that

$$A^2C^2+A^2B^2-B^2C^2 > 0$$

or

$$\boxed{[5.1] \quad C^2 < A^2B^2/(B^2-A^2)}$$

For the "-" solution, the numerator is negative, thus the denominator must be negative which requires that

$$C^2 > A^2B^2/(B^2-A^2)$$

But if this were true, then $C^2 > X^2$, making the value of Z negative in **[2.8]**.
(See the proof of this on the next page).

So only the "+" solution is valid for the quadratic equation to make all positive variables for the Perfect Cuboid, and thus **[5.1]** must be used.

5. Stronger Divisor Inequality for C (continued)

Theorem
In **[3.1]**, if $C^2 > A^2B^2/(B^2-A^2)$, then $C^2 > X^2$.

Proof
Rearrange **[3.1]**.

$$\frac{B^2(X^2-A^2)^2 - A^2(X^2-B^2)^2}{A^2B^2} = \frac{(X^2-C^2)^2}{C^2}$$

Multiply out, refactor and rearrange.

$$\text{[5.2]} \quad \frac{C^2}{A^2B^2/(B^2-A^2)} = \frac{(X^2-C^2)^2}{(X^4-A^2B^2)}$$

Since

$$C^2 > A^2B^2/(B^2-A^2),$$

it follows from **[5.2]** that

$$(X^2-C^2)^2 > (X^4-A^2B^2)$$

or

$$\text{[5.3]} \quad \frac{(C^4+A^2B^2)}{2C^2} > x^2$$

Since

$$A < B < C,$$

it follows that

$$C^4 > A^2B^2$$

thus

$$2C^4 > (C^4+A^2B^2)$$

and, dividing both sides by $2C^2$,

$$\text{[5.4]} \quad C^2 > \frac{(C^4+A^2B^2)}{2C^2}$$

therefore, from **[5.3]** and **[5.4]**,

$$C^2 > X^2$$

6. Stronger Divisor Inequality for B

Since $B < C$, the limit inequality for C^2 must also apply to B^2.
Thus

$$B^2 < A^2B^2/(B^2-A^2)$$

producing

$$\boxed{[6.1] \quad B^2 < 2A^2}$$

[6.1] will be used to easily eliminate a large number of choices for A and B.

7. $X = p^n$

Theorem
The odd edge of a Perfect Cuboid cannot be a power of a prime.

Proof
Suppose $X = p^n$, where p is an odd prime.
The divisors of X^2 are all powers of p.
If $A = p^a$ and $B = p^b$, where $a < b$, then B^2 is at least p^2 times that of A^2.
Since $p \geq 3$, this violates inequality **[6.1]**.

8. A Useful Inequality

If p is an odd prime, then

$$p^2 > 2$$

or

$$0 < p^2 - 2$$

Add p^4 to both sides.

$$p^4 < p^4 + p^2 - 2$$

Divide both sides by (p^2-1).

$$\boxed{[8.1] \quad p^4/(p^2-1) < p^2+2}$$

[8.1] will be used in section 9 and several times in section 11.

9. $X = pq$

Theorem
The odd edge of a Perfect Cuboid cannot be the product of two different primes.

Proof
Assume $p < q$.
The divisors of X^2 less than X are $1,\ p,\ q,\ p^2$.
There are two cases for their relative magnitude.
If $q < p^2$, then $1 < p < q < p^2$.
if $q > p^2$, then $1 < p < p^2 < q$.

It is required to assign A, B, C so that $A < B < C$.
If $A = 1$ and $B = p$ or greater, inequality **[6.1]** is violated
because p^2 is more than twice 1, since $p \geq 3$.

So for both cases, we must have $A = p$.
But $B = p^2$ violates **[6.1]** because p^4 is more than twice p^2, since $p \geq 3$.

The only assignment left: $A = p,\ B = q,\ C = p^2$.
But this violates inequality **[5.1]** as follows.
Since $p < q$,

$q^2 - p^2$ is at least $(p+2)^2 - p^2 = 2p+4$.

Since $p \geq 3$,

[9.1] $q^2 - p^2 \geq 10$.

Inequality **[5.1]** requires

$$p^4 < p^2q^2/(q^2-p^2)$$

or

[9.2] $q^2 < p^4/(p^2-1)$

Combining **[9.2]** with the useful inequality **[8.1]**,

$$q^2 < p^2 + 2$$

or

$$q^2 - p^2 < 2$$

but this violates **[9.1]**.

10. Scale Factors

Another way to eliminate many combinations of A, B, C divisors
is by comparing the scale factors of the terms in **[3.1]**.
An example of a scale factor is the value in common with
X^2/A and A in the term $(X^2/A - A)$.

Suppose $X = pq^2$ and $A = q^2$,
then $X^2/A = p^2q^2$ and the common factor is q^2.

Since $V = (X^2/A - A)$, it follows that q is a prime factor of V.
Also, q is a factor of X since A is a factor of X^2.

In this case, the value of $(X^2/A - A)$ could be represented by
$q^2(x^2/a - a)$ where $x = p$ and $a = 1$. This consists of smaller values
of the odd edge and the divisor multiplied by a scale factor.

Equation **[3.1]** is a representation of **[1.3]** where $V^2 = Y^2 + Z^2$.
If two of V, Y, Z have a factor in common, so must the third.
But if this results in V, X, Y, Z all having a common factor,
and the A, B, C assignment actually makes a Perfect Cuboid,
it will not be a primitive cuboid; it will be a scaled cuboid.
The same solution could have been discovered with a smaller value of X
containing at least one less of the common prime factor.

Therefore, it not necessary to check A, B, C assignments where two or more
of the scale factors of the terms in **[3.1]** have a prime in common.

11. $X = pq^2$

The divisors of X^2 that are less than X depend on the relative magnitude of p and q. For each case, the divisors are listed in increasing order, followed by their scale factors, then followed by all A, B, C assignments that don't violate **[6.1]** and don't have a common factor in their scalings.

Case: $q^4 < p$
Divisor: $1,\ q,\ q^2,\ q^3,\ q^4,\ p,\ pq$
Scaling: $1,\ q,\ q^2,\ q\ ,\ 1\ ,\ p,\ pq$
[no assignments]

Case: $q^3 < p < q^4$
Divisor: $1,\ q,\ q^2,\ q^3,\ p,\ q^4,\ pq$
Scaling: $1,\ q,\ q^2,\ q\ ,\ p,\ 1\ ,\ pq$
$A = q^3,\ B = p,\ C = q^4$

Case: $q^2 < p < q^3$
Divisor: $1,\ q,\ q^2,\ p,\ q^3,\ pq,\ q^4$
Scaling: $1,\ q,\ q^2,\ p,\ q\ ,\ pq,\ 1$
$A = q^2,\ B = p,\ C = q^4$
$A = p,\ B = q^3,\ C = q^4$

Case: $q < p < q^2$
Divisor: $1,\ q,\ p,\ q^2,\ pq,\ p^2,\ q^3$ or $q^3,\ p^2$
Scaling: $1,\ q,\ p,\ q^2,\ pq,\ 1\ ,\ q$ or $q\ ,\ 1$
$A = q,\ B = p,\ C = p^2$
$A = p,\ B = q^2,\ C = p^2$

Case: $p^2 < q$
Divisor: $1,\ p,\ p^2,\ q,\ pq,\ p^2q,\ q^2$
Scaling: $1,\ p,\ 1\ ,\ q,\ pq,\ q\ ,\ q^2$
[no assignments]

Case: $p < q < p^2$
Divisor: $1,\ p,\ q,\ p^2,\ pq,\ q^2,\ p^2q$
Scaling: $1,\ p,\ q,\ 1\ ,\ pq,\ q^2,\ q$
$A = p,\ B = q,\ C = p^2$

There were 7 taken 3 at a time = 35 possible A, B, C assignments for each case, but there are now only 6 assignments that survived the initial filtering.
The next filter uses inequality **[5.1]** with some help from **[8.1]**.

11. $X = pq^2$ (continued)

Case: $q^3 < p < q^4$; $A = q^3$, $B = p$, $C = q^4$

Inequality **[5.1]** requires

$$q^8 < q^6p^2/(p^2-q^6)$$

or

$$p^2 < q^8/(q^2-1)$$

which is a little more than q^6.

The filtered result narrows the range for p^2

$$q^6 < p^2 < q^8/(q^2-1)$$

Case: $q^2 < p < q^3$; $A = q^2$, $B = p$, $C = q^4$

Inequality **[5.1]** requires

$$q^8 < q^4p^2/(p^2-q^4)$$

or

$$p^2 < q^8/(q^4-1)$$

From an adaptation of **[8.1]**,

$$q^8/(q^4-1) < q^4+2$$

thus

$$p^2 < q^4+2$$

which violates $q^2 < p$, since $q \geq 3$.

Case: $q^2 < p < q^3$; $A = p$, $B = q^3$, $C = q^4$

Inequality **[5.1]** requires

$$q^8 < p^2q^6/(q^6-p^2)$$

or

$$q^8/(q^2+1) < p^2$$

which is a little less than q^6.

The filtered result narrows the range for p^2

$$q^8/(q^2+1) < p^2 < q^6$$

Case: $q < p < q^2$; $A = q$, $B = p$, $C = p^2$

Inequality **[5.1]** requires

$$p^4 < q^2p^2/(p^2-q^2)$$

or

$$p^4/(p^2+1) < q^2$$

Take

$$p^4-1 < p^4$$

and divide both sides by p^2+1

$$p^2-1 < p^4/(p^2+1)$$

thus

$$p^2-1 < q^2$$

but this violates $q < p$ since $q \geq 3$.

11. $X = pq^2$ (continued)

Case: $q < p < q^2$ $\quad A = p,\ B = q^2,\ C = p^2$

Inequality **[5.1]** requires

$$p^4 < p^2q^4/(q^4-p^2)$$

or

$$q^4 < p^4/(p^2-1)$$

and then using **[8.1]**

$$q^4 < p^2+2$$

violating $p < q^2$ since $q \geq 3$.

Case: $p < q < p^2;$ $\quad A = p,\ B = q,\ C = p^2$

Inequality **[5.1]** requires

$$p^4 < p^2q^2/(q^2-p^2)$$

or

$$q^2 < p^4/(p^2-1)$$

and then using **[8.1]**

$$q^2 < p^2+2$$

violating $p < q$ since $p \geq 3$.

There are now two cases that survived the three filters.
The next step is to plug them into **[3.1]**.

Case: $A = q^3,\ B = p,\ C = q^4;$ $\quad q^6 < p^2 < q^8/(q^2-1)$

Equation **[3.1]** requires

$$(p^2q - q^3)^2 = (pq^4 - p)^2 + (p^2 - q^4)^2$$

or more simply

$$(p^2q^2 - 1)(p^2 - q^6) = (p^2 - q^4)^2$$

Case: $A = p,\ B = q^3,\ C = q^4;$ $\quad q^8/(q^2+1) < p^2 < q^6$

Equation **[3.1]** requires

$$(pq^4 - p)^2 = (p^2q - q^3)^2 + (p^2 - q^4)^2$$

or more simply

$$(p^2q^2 - 1)(q^6 - p^2) = (p^2 - q^4)^2$$

There are two approaches to make more progress. Extend the set of inequalities so that these last two cases can be proved impossible or do a very efficient computer search, described on the next page.

12. Targeted Computer Searches

For a case that hasn't been proved impossible, a computer search that hones in on just its pattern can be very efficient. For example, the first unproved case above requires $q^6 < p^2 < q^8/(q^2-1)$. This means that p is a little more than q^3, but not too much more, so there is a narrow range of p values for each q.

```
for each odd q
  L = floor(q^8/(q^2-1))
  p = q^3 + 2
  while p^2 < L
    plug p,q into [3.1] and test
    p = p + 2
```

Since $X = pq^2$ and $p > q^3$, $X > q^5$.
When $q \geq 101$, $X > 10$ billion.
When $q \geq 479$, $X > 25$ trillion.

13. Divisor Count

There were 4 divisors for $X = pq$ and 7 divisors for $X = pq^2$, which were reasonably easy to determine. But in more complicated cases, it would be nice to have a check on the expected number of divisors to make sure that none will be missed. So here is a formula.

Given the prime factorization of odd X,

$$X = (p^a)(q^b)(r^c)(...$$

where $p, q, r, ...$ are different odd primes,
the number of divisors of X^2 less than X

$$N = (M - 1) / 2$$

where

$$M = (2a + 1)(2b + 1)(2c + 1)(...$$

Proof
The factor p^a in X becomes p^{2a} in X^2.
Powers of the prime p can be in a divisor in $2a$ ways or not at all, thus there are $(2a + 1)$ ways of using it.
The same is true for all the other primes of X so that the total number of divisors is the product M.

However, one of those divisors will equal X, so 1 has to be subtracted from M.
Then since X^2/D is less than X if D is greater than X, exactly half of the $(M - 1)$ divisors are less than X.