On the Nicolas inequality involving primorial numbers

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March 4, 2017

ABSTRACT. In 1983, J.L. Nicholas demonstrated that the Riemann Hypothesis (RH) is equivalent to the statement that

$$\theta_k = \frac{N_k}{\varphi(N_k)\log\log N_k} > e^\gamma$$

for every positive integer k, where N_k denotes the primorial number of order k, $\varphi(n)$ the Euler totient function and γ the Euler-Mascheroni constant. In this note, we prove that the least upper bound for the entire sequence θ_k is greater than e^γ, so that for any large k, we cannot have $\theta_k \leq e^\gamma$, which Nicolas also showed to be sufficient for the RH to hold.

Introduction. The Riemann zeta function is the function of the complex variable s, defined in the half-plane $\Re(s) > 1$ by the absolutely convergent series

$$\zeta(s) := \sum_{n=1}^{\infty} n^{-s}$$

and in the whole complex plane by analytic continuation. As shown by Riemann, $\zeta(s)$ extends to $\mathbb{C}$ as a meromorphic function with only a simple pole at $s = 1$, with residue 1, and satisfies the functional equation $\xi(s) = \xi(1-s)$, where

$$\xi(s) = s(s-1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s)$$

In his epoch-making mermoir of 1859, Riemann obtained an analytic formula for the number of primes up to a preassigned limit. This formula is expressed in terms of the zeros of the zeta function, namely, the complex solutions ρ of the equation $\zeta(\rho) = 0$.

In this paper, Riemann introduces the function ξ (as defined above) of the complex variable t in terms of s with $s = \frac{1}{2} + it$, and shows that $\xi(t)$ is an even entire function of t whose zeros have imaginary part between $-\frac{i}{2}$ and $\frac{i}{2}$. He further states, without sketching a proof, that in the range between 0 and T the function has about

$$\frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi}$$

zeros, and then continues

Man findet nun in der that etwa so viel reelle wurzeln innerhalb diser grenzen, aund es ist sehrvwarscheinlich, dass alle wurzeln reell sind.

Which can be translated as:

Indeed, one finds between those limits many real zeros, and it is very likely that all zeros are real.

The statement that all zeros of $\xi(t)$ are real is the RH. The function $\zeta(s)$ has zeros at the negative even integers $-2n$, $n \geq 1$ and one refers to them as the trivial zeros. The other zeros are the complex numbers $\frac{1}{2} + i\alpha$, where α is a zero of $\xi(t)$. Thus, in terms of the function $\zeta(s)$, the problem can be stated as:

RIEMANN HYPOTHESIS (RH): *All non-trivial zeros of the analytic continuation of $\zeta(s)$ occur with real part $1/2$.*

There now exist a fairly large number of equivalent reformulations of the problem in the literature [1]. Our concern is on that of Nicolas [3], that states that the inequality

$$\frac{N_k}{\varphi(N_k)} > e^{\gamma}\log\log N_k$$

holds for all integers $k \geq 1$ if the RH is true.

Throughout this paper, N_k denotes the $k-th$ primorial number (the product of the first k primes), $\varphi(n)$ the Euler totient function and γ is the Euler-Mascheroni constant.

Conversely, if the RH is false, the inequality holds for infinitely many k, and is violated for infintely many k. Thus, to confirm the RH, it suffices to prove Nicolas' inequality for sufficiently large k.

Proof of the Nicolas Inequality

For every positive integer k, denote by N_k the product of first k primes, $\sigma(N_k)$ the sum of positive divisors of N_k and $\varphi(N_k)$ the Euler totient function at N_k.

Define $a_k = \frac{(N_k)^2}{\sigma(N_k)\varphi(N_k)}$, $b_k = \frac{\sigma(N_k)}{N_k \log\log N_k}$ and $c_k = a_k b_k$.

It is not very hard to verify that both a_k and b_k are bounded above [4], so that c_k is also bounded above. We want to investigate the magnitude of the least upper bound for c_k. That is, the minimal constant c such that $c_k \le c$ for every large k.
We therefore have to investigate the respective magnitudes of the minimal constants a, b such that $a_k \le a$ and $b_k \le b$ for every large k. Note that $c = ab$.
Observe that $a = \frac{\pi^2}{6}$ and $b > \frac{6e^\gamma}{\pi^2}$, where γ is the Euler-Mascheroni constant.
The first observation is quite straightforward, whereas the second can be explained by the following argument:
If we had

$$b_k = \frac{\sigma(N_k)}{N_k \log\log N_k} \le \frac{6e^\gamma}{\pi^2}$$

for every large k, then it would follow upon multiplying both sides by $a_k \le \frac{\pi^2}{6}$ that $c_k \le e^\gamma$ for every large k, contradicting an oscillation theorem of Nicolas [3].
Hence we indeed have $b > \frac{6e^\gamma}{\pi^2}$, so that we arrive at

$$c = ab = \frac{b\pi^2}{6} > e^\gamma$$

and since c was defined to be the *minimal* constant such that $c_k \le c$ for *every* large k, we conclude that the inequality

$$c_k = \frac{N_k}{\varphi(N_k) \log\log N_k} < e^\gamma$$

cannot hold for any large k, so that by Nicolas' criterion [3], we complete a proof of the RH.

Remark: We anticipate that the reader might still ask: *How can we be completely sure that* $a = \frac{\pi^2}{6}$ *?*

The following would be our response:

Firstly, notice that $a_k \leq \frac{\pi^2}{6}$, hence $a \neq \frac{\pi^2}{6}$ can be more precisely written as $a < \frac{\pi^2}{6}$. By our definition of a, it would follow that

$$a_\infty = \prod_{k=1}^{\infty} \frac{{p_k}^2}{{p_k}^2 - 1} \leq a < \frac{\pi^2}{6}$$

where p_k is the $k-th$ prime, which is false.

About the author: Tatenda Isaac Kubalalika is a financially disadvantaged but very ambitious prospective undergraduate student from Zimbabwe, who is also deeply passionate about study and research in the fields of analytic and algebraic number theory. Amongst other things, he hopes to be involved in the fascinating subject of Random Matrix Theory, and use it to analyse the suitably normalized spacings of the zeros of other automorphic $L-$ functions, in the spirit of Montgomery, Odlyzko, Rudnick, Sarnak *et al.*

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