Riemann Hypothesis and Euler Function

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April 15, 2017

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Abstract: The aim of this paper is to show the proof of the Riemann hypothesis (RH) by the primorial number. We find a new sufficient condition (SC) for the RH from well-known Robin theorem and prove that the SC holds unconditionally.
Keywords: *Euler function, Primorial number, Riemann hypothesis.*

I. Introduction and Main result of paper

Let N be the set of the natural numbers. The function

$$\varphi(n) = n \cdot \prod_{p|n} (1 - p^{-1})$$

is called the Euler function of $n \in N$ ([1]). Here $\varphi(1) = 1$ and $p|n$ denotes p is the prime divisor of n. The function $\sigma(n) = \sum_{d|n} d$ is the divisor function of $n \in N$. Here $d|n$ denotes d is the divisor of n. Robin showed in his paper [4] (also see [2]).

[**Robin Theorem**] If the Riemann hypothesis (RH) is false, then there exist constants $c > 0$ and $0 < \beta < 1/2$ such that

$$\frac{\sigma(n)}{n} \geq e^{\gamma} \cdot \log\log n + \frac{c \cdot \log\log n}{(\log n)^{\beta}} \tag{1.1}$$

holds for infinitely many $n \in N$, where $\gamma = 0.577\cdots$ is the Euler constant ([1]).

From this we have

[**Theorem 1**] If for any $n \geq 2$

$$\frac{n}{\varphi(n)} \leq e^{\gamma} \cdot \log\log(24 \cdot n \cdot \rho(n)) \tag{1.2}$$

holds, then the RH is true, where

$$\rho(n) = \exp(\sqrt{\log n} \cdot (\log\log n)^2).$$

For $n \in N (n \neq 1)$ we define the function

$$\Phi_0(n) = \frac{\exp(\exp(e^{-\gamma} \cdot n/\varphi(n)))}{n \cdot \rho(n)}. \tag{1.3}$$

Then we give

[**Theorem 2**] For any $n \geq 2$ we have $\Phi_0(n) \leq 24$.

[**Corollary**] For any $n \geq 5$ we have

$$\frac{n}{\varphi(n)} \leq e^{\gamma} \cdot \log\log n + 21.483 \cdot \frac{(\log\log n)^2}{\sqrt{\log n}}. \tag{1.4}$$

II. Proofs of Theorem 1 and Theorem 2

2.1. Proof of Theorem 1

It is clear that $\sigma(n)\cdot\varphi(n)\le n^2$ for any $n\ge 2$. If (1.2) holds, but the RH is false, then by the Robin Theorem,

$$e^{\gamma}\cdot\log\log n+\frac{c\cdot\log\log n}{(\log n)^{\beta}}\le\frac{\sigma(n)}{n}\le\frac{n}{\varphi(n)}\le e^{\gamma}\cdot\log\log(24\cdot n\cdot\rho(n))$$

holds for infinitely many $n\in N$. On the other hand, since $\log(1+t)\le t\ \ (t>0)$, we have

$$\log\log(24\cdot n\cdot\rho(n))=\log(\log 24+\log n+\sqrt{\log n}\cdot(\log\log n)^2)=$$

$$=\log\log n+\log\left(1+\frac{\log 24}{\log n}+\frac{(\log\log n)^2}{\sqrt{\log n}}\right)\le$$

$$\le\log\log n+\frac{\log 24}{\log n}+\frac{(\log\log n)^2}{\sqrt{\log n}}$$

and

$$1\le\frac{\mathrm{e}^{\gamma}\cdot c^{-1}\cdot\log 24}{(\log n)^{1-\beta}\cdot\log\log n}+\frac{\mathrm{e}^{\gamma}\cdot c^{-1}\cdot\log\log n}{(\log n)^{1/2-\beta}}\to 0\ (n\to\infty),\tag{2.1}$$

but it is a contradiction.

2.2. Reduction to the primorial number

Let $p_1=2,\ p_2=3,\ p_3=5,\cdots$ be the first consecutive primes. Then $p_m\ \ (m\in N)$ is m-th prime number. The number $(p_1\cdots p_m)$ is called the primorial number ([3, 7]). Assume that $n=q_1^{\lambda_1}\cdots q_m^{\lambda_m}$ is the prime factorization of $n\in N$. Here $q_1,\cdots,q_m$ are distinct primes, $\lambda_1,\cdots,\lambda_m$ are nonnegative integers ≥ 1 and $\omega(n):=m$ is the number of distinct prime factors of $n\in N([6])$. Put $\Im_m:=p_1\cdots p_m$, then it is clear that $n\ge\Im_m$,

$$\frac{n}{\varphi(n)}=\prod_{i=1}^{m}(1-q_i^{-1})^{-1}\le\prod_{i=1}^{m}(1-p_i^{-1})^{-1}=\frac{\Im_m}{\varphi(\Im_m)}\tag{2.2}$$

and so $\Phi_0(n)\le\Phi_0(\Im_m)$. This shows that the boundedness of the function $\Phi_0(n)$ for $n\in N\ (n\ne 1)$ is reduced to one for the primorial numbers. Now we put

$$C_m:=\Phi_0(\Im_m)\ \ (m\ge 1).$$

2.3. Proof of Theorem 2

Let $n=q_1^{\lambda_1},\cdots,q_m^{\lambda_m}$ be the prime factorization of any $n\ge 2$. Then $\Phi_0(n)\le C_m$.
If $1\le m\le 4$, we see from the computation by MATLAB (see table 1)

$$C_1=\frac{\exp(\exp(e^{-\gamma}\cdot 2))}{2\cdot\exp(\sqrt{\log 2}\cdot(\log\log 2)^2)}=9.6680\cdots,$$

$$C_2=\frac{\exp(\exp(e^{-\gamma}\cdot 2\cdot 3/2))}{(2\cdot 3)\exp(\sqrt{\log(2\cdot 3)}\cdot(\log\log(2\cdot 3))^2)}=23.1516\cdots,$$

$$C_3=\frac{\exp(\exp(e^{-\gamma}\cdot 2\cdot(3/2)\cdot(5/4)))}{(2\cdot 3\cdot 5)\exp(\sqrt{\log(2\cdot 3\cdot 5)}\cdot(\log\log(2\cdot 3\cdot 5))^2)}=7.7386\cdots,$$

$$C_4 = \frac{\exp(\exp(e^{-\gamma}\cdot 2\cdot(3/2)\cdot(5/4)\cdot(7/6)))}{(2\cdot 3\cdot 5\cdot 7)\exp(\sqrt{\log(2\cdot 3\cdot 5\cdot 7)}\cdot(\log\log(2\cdot 3\cdot 5\cdot 7))^2)} = 0.8317\cdots.$$

If $m \geq 5$, then we have $C_m < 1$ by the Lemma 1 (also see below). Therefore we have

$$\Phi_0(n) \leq \Phi_0(\Im_m) = C_m \leq \max_{m\geq 1}\{C_m\} \leq 24. \tag{2.3}$$

2.4. Proof of Corollary

From the theorem 1, then for any $n \geq 5$ we have

$$\frac{n}{\varphi(n)} \leq e^{\gamma}\cdot\log\log n + e^{\gamma}\cdot\left(\frac{\log 24}{\log n} + \frac{(\log\log n)^2}{\sqrt{\log n}}\right) =$$

$$= e^{\gamma}\cdot\log\log n + e^{\gamma}\cdot\left(1 + \frac{\log 24}{\sqrt{\log n}\cdot(\log\log n)^2}\right)\cdot\frac{(\log\log n)^2}{\sqrt{\log n}} \leq$$

$$\leq e^{\gamma}\cdot\log\log n + 21.483\cdot\frac{(\log\log n)^2}{\sqrt{\log n}}.$$

III. Lemma 1 for Boundedness of Sequence $\{C_m\}$

In the proof of the theorem 2, it was used that the sequence $\{C_m\}$ is bounded. So we here would prove the following Lemma 1.

[**Lemma 1**] For any $m \geq 5$ we have $C_m < 1$.

For the proof of the Lemma 1, we need some estimates.

3.1. Some symbols

It is known that by [5],

$$\sum_{p\leq t} p^{-1} = \log\log t + b + E(t), \tag{3.1}$$

where $t \geq 1$ is a real number and p is the prime number,

$$E(t) = O(\exp(-a_1\cdot\sqrt{\log t}))\ \ (a_1 > 0)\ \ [5]$$

and

$$b = \gamma + \sum_{p}(\log(1-1/p) + 1/p) = 0.261497212\cdots$$

is the Mertens' constant [8]. Put $F_m := \Im_m/\varphi(\Im_m)$, then

$$\log(F_m) = -\sum_{i=1}^{m}(\log(1-1/p_i) + 1/p_i) + \sum_{i=1}^{m} 1/p_i =$$

$$= \log\log p_m + \gamma + E(p_m) + \varepsilon(p_m),$$

where

$$\varepsilon(p_m) = \sum_{p>p_m}(\log(1-1/p) + 1/p) = O(1/p_m).$$

From this we have

$$(e^{-\gamma}\cdot F_m) = \log p_m\cdot e_0, \quad \exp(e^{-\gamma}\cdot F_m) = p_m\cdot e_0', \tag{3.2}$$

where

$$e_0 = \exp(E(p_m) + \varepsilon(p_m)), \qquad e_0' = \exp(\log p_m \cdot (e_0 - 1)).$$

Similarly, we easily have

$$(e^{-\gamma} \cdot F_{m-1}) = (\log p_{m-1}) \cdot e_1, \quad \exp(e^{-\gamma} \cdot F_{m-1}) = p_{m-1} \cdot e_1', \tag{3.3}$$

where

$$e_1 = \exp(E(p_{m-1}) + \varepsilon(p_{m-1})), \qquad e_1' = \exp(\log p_{m-1} \cdot (e_1 - 1)).$$

We recall the Chebyshev's function $\vartheta(t) = \sum_{p \le t} \log p$ ([1]). By the prime number theorem ([1]),

$$\vartheta(p_m) = p_m \cdot (1 + \theta(p_m)), \tag{3.4}$$

where

$$\theta(p_m) = O(\exp(-a_2 \cdot \sqrt{\log p_m}))\ (a_2 > 0)\ \ [5].$$

Then we see

$$\log \Im_m = p_m \cdot \alpha_0, \qquad \log \Im_{m-1} = p_{m-1} \cdot \alpha,$$

where

$$\alpha_0 = 1 + \theta(p_m), \qquad \alpha = 1 + \theta(p_{m-1}).$$

Now put

$$N_i := \sqrt{\log \Im_{m-i}} \cdot (\log \log \Im_{m-i})^2 \quad (i = 0,\ 1).$$

Then we have

$$N_0 = \sqrt{p_m \cdot \alpha_0} \cdot \log^2(p_m \cdot \alpha_0), \qquad N_1 = \sqrt{p_{m-1} \cdot \alpha} \cdot \log^2(p_{m-1} \cdot \alpha).$$

3.2. An estimate of e_1 and e_1'

We put $p = p_{m-1},\ \ p_0 = p_m$ below. For the theoretical calculation we assume $p \ge e^{14}$. The discussion for $p \le e^{14}$ is supported by MATLAB. Since

$$(e^{-\gamma} \cdot F_{m-1}) = (\log p) \cdot e_1 < \log p + \frac{1}{\log p} \quad (p \ge 2) \tag{3.5}$$

by (3.30) of [5], we respectively have

$$e_1 < 1.0052 \quad (p \ge e^{14}), \qquad e_1' < 1.075 \quad (p \ge e^{14}),$$

$$(e_1 \cdot e_1') < 1.08 \quad (p \ge e^{14}). \tag{3.6}$$

3.3. An estimate of $(e_1 \cdot e_1')$

Since if $e_1 \le 1$ then $e_1' \le 1$, we have $(e_1 \cdot e_1') \le 1$. On the other hand, it is known that

$$(-1/\log^2 t) \le E(t) \le (1/\log^2 t) \quad (t > 1) \tag{3.7}$$

by (3.17) and (3.20) of [5]. Hence, since $\varepsilon(p) < 0$, if $e_1 > 1$, then

$$0 < r := E(p) + \varepsilon(p) < \frac{1}{\log^2 p} \le 0.0052 \quad (p \ge e^{14})$$

and

$$e_1 = 1 + r + \sum_{n=2}^{\infty} \frac{r^n}{n!} \le 1 + r + \frac{r^2}{2 \cdot (1 - r)} \le 1 + r + 0.503 \cdot r^2,$$

$$e_1 \cdot e_1' = \exp(r + (\log p) \cdot (e_1 - 1)) \leq 1 + h + \frac{h^2}{2 \cdot (1-h)},$$

where

$$h = (1 + \log p) \cdot r + 0.503 \cdot \log p \cdot r^2 \leq 0.1125 \quad (p \geq e^{14}).$$

Therefore we have

$$(e_1 \cdot e_1' - 1) \leq (1 + \log p) \cdot (E(p) + \varepsilon(p)) +$$
$$+ 0.6 \cdot (1 + \log p)^2 \cdot (E(p) + \varepsilon(p))^2 \ (e_1 > 1,\ p \geq e^{14}). \tag{3.8}$$

3.4. An estimate of $V_0 := p_0 \cdot (e_0' - \alpha_0) - p \cdot (e_1' - \alpha)$

It is clear that $p_0 \cdot \alpha_0 - p \cdot \alpha = \log p_0$ and, since

$$E(t) = \sum_{p \leq t} p^{-1} - \log\log t - b,$$

we have

$$E(p_0) - E(p) = \frac{1}{p_0} - \log\left(\frac{\log p_0}{\log p}\right),$$

$$\varepsilon(p_0) - \varepsilon(p) = -\log\left(1 - \frac{1}{p_0}\right) - \frac{1}{p_0}.$$

From this

$$\frac{e_0}{e_1} = \left(\frac{\log p}{\log p_0}\right) \cdot \left(1 + \frac{1}{p_0 - 1}\right), \qquad \frac{e_0'}{e_1'} = \frac{p}{p_0} \cdot \exp\left(\frac{\log p \cdot e_1}{p_0 - 1}\right).$$

Thus we have

$$V_0 = p \cdot e_1' \cdot \left(\frac{p_0 \cdot e_0'}{p \cdot e_1'} - 1\right) - \log p_0 = \log p_0 \cdot (\mu \cdot e_1' - 1), \tag{3.9}$$

where

$$\mu = \frac{p}{\log p_0} \cdot \left(\exp\left(\frac{\log p \cdot e_1}{p_0 - 1}\right) - 1\right).$$

Since

$$\mu \leq e_1 + \frac{1}{2} \cdot \frac{\log p \cdot e_1}{p} \cdot \left(1 - \frac{\log p \cdot e_1}{p}\right)^{-1} \leq$$
$$\leq e_1 + 0.503 \cdot \frac{\log p}{p}, \quad (e_1 > 1,\ p \geq e^{14})$$

we have

$$\mu \cdot e_1' - 1 \leq (e_1 \cdot e_1' - 1) + 0.55 \cdot \frac{\log p}{p} \ (e_1 > 1, p \geq e^{14}). \tag{3.10}$$

3.5. An estimate of $G_0 := (\log p_0 \cdot R(\Im_{m-1}) - (N_0 - N_1))/N_0$

Here

$$R(\Im_{m-1}) := \frac{(\log\log \Im_{m-1})^2}{2 \cdot \sqrt{\log \Im_{m-1}}} \cdot \left(1 + \frac{4}{\log\log \Im_{m-1}}\right).$$

It is known that $p_{k+1}^2 \leq (p_1 \cdots p_k)$ for $p_k \geq 7$ by 246p of [6] and hence

$$\frac{\log p_0}{\log \Im_{m-1}} < \frac{1}{2} \ \ (p \geq e^{14}).$$

Since $\log(1+t) \geq (t - t^2/2)$ for any $t \quad (0 < t < 1/2)$, we have

$$N_0 - N_1 = (\sqrt{\log \Im_m} - \sqrt{\Im_{m-1}}) \cdot (\log\log \Im_m)^2 +$$
$$+\sqrt{\log \Im_{m-1}} \cdot ((\log\log \Im_m)^2 - (\log\log \Im_{m-1})^2) \geq$$
$$\geq \frac{\log p_0}{2 \cdot \sqrt{\log \Im_m}} \cdot (\log\log \Im_{m-1})^2 +$$
$$+2 \cdot \sqrt{\log \Im_{m-1}} \cdot \log\log \Im_{m-1} \cdot \log\left(1 + \frac{\log p_0}{\log \Im_{m-1}}\right) \geq$$
$$\geq \frac{\log p_0}{2 \cdot \sqrt{\log \Im_m}} \cdot (\log\log \Im_{m-1})^2 +$$
$$+\log p_0 \cdot \frac{2 \cdot \log\log \Im_{m-1}}{\sqrt{\log \Im_{m-1}}} \cdot \left(1 - \frac{\log p_0}{2 \cdot \log \Im_{m-1}}\right)$$

and

$$G_0 \cdot N_0 \leq \frac{\log p_0}{2} \cdot \left(\frac{1}{\sqrt{\log \Im_{m-1}}} - \frac{1}{\sqrt{\log \Im_m}}\right) \cdot (\log\log \Im_{m-1})^2 +$$
$$+\frac{\log^2 p_0}{(\log \Im_{m-1})^{3/2}} \cdot \log\log \Im_{m-1} \leq$$
$$\leq \log^2 p_0 \cdot \frac{(\log\log \Im_{m-1})^2}{(\log \Im_{m-1})^{3/2}} \cdot \left(\frac{1}{4} + \frac{1}{\log\log \Im_{m-1}}\right).$$

And it is known that $p_{k+1}^2 \leq 2 \cdot p_k^2$ for $p_k \geq 7$ by 247p. of [6] and

$$(-1/\log t) < \theta(t) < (1/\log t) \quad (t \geq 41) \tag{3.11}$$

by (3.15) and (3.16) of [5]. So

$$\log p_0 \leq (\log p) \cdot \left(1 + \frac{\log \sqrt{2}}{\log p}\right).$$

Since $p \geq e^{14}$, we have $\alpha \geq (1 - 1/14)$ and the function $(\log^3 t)/t$ is decreasing on the interval $(e^3, \ +\infty)$. Therefore we get

$$G_0 \leq \frac{\log^2 p_0}{(\log \Im_{m-1})^2} \cdot \left(\frac{1}{4} + \frac{1}{\log\log \Im_{m-1}}\right) \leq$$
$$\leq \frac{\log^3 p}{p \cdot \alpha^2} \cdot \left(1 + \frac{\log \sqrt{2}}{\log p}\right)^2 \cdot \left(\frac{1}{4} + \frac{1}{\log p + \log \alpha}\right) \cdot \frac{1}{p \cdot \log p} \leq$$
$$\leq \frac{0.01}{p \cdot \log p} \quad (p \geq e^{14}). \tag{3.12}$$

3.6. An estimate of $S(p') := \sum_{p \geq p'} 1/(p \cdot \log p)$

Put

$$s(t) := \sum_{p \leq t} p^{-1} = \log\log t + b + E(t).$$

Then by the Abel's identity [1], we have

$$S(p') = \int_{p'}^{+\infty} \frac{1}{\log t} \cdot ds(t) = \int_{p'}^{+\infty} \frac{1}{\log t} \cdot \left(\frac{dt}{t \cdot \log t} + dE(t)\right) \leq$$

$$\leq \frac{1}{\log p'} - \frac{E(p')}{\log p'} + \int_{p'}^{+\infty} \frac{1}{t \cdot \log^4 t} \cdot dt \leq$$

$$\leq \frac{1}{\log p'} + \frac{1}{\log^3 p'} - \frac{1}{3 \cdot \log^3 t}|_{p'}^{+\infty} =$$

$$= \frac{1}{\log p'} + \frac{4}{3 \cdot \log^3 p'} \tag{3.13}$$

and

$$S(p') \geq \frac{1}{\log p'} - \frac{4}{3 \cdot \log^3 p'}.$$

If p' is a first prime $\geq e^{14}$, then $p' = 1202609$ and it is 93118-th prime. And we have

$$0.070 \leq S(p') \leq 0.072. \tag{3.14}$$

3.7. A Condition (d)
Put

$$f(t) = t \cdot \log t \cdot E(t) - t \cdot \theta(t),$$

$$g(t) = \sqrt{t} \cdot \log^2(t \cdot \alpha),$$

$$d(t) = \frac{f(t)}{g(t)} \quad (3 \leq p \leq t < p+2), \tag{3.15}$$

where t is a real number and $\alpha = 1 + \theta(p)$ is dependent on p and a positive constant such that

$$(1 - 1/\log p) \leq \alpha \leq (1 + 1/\log p).$$

Then both $f(t)$ and $g(t)$ are continuously differentiable function on the interval $(p,\ p+2)$.
In fact, since the functions

$$\sum_{p \leq t} p^{-1} - b, \quad \vartheta(t) = \sum_{p \leq t} \log p$$

are constants on $(p,\ p+2)$, we have

$$E'(t) = \frac{-1}{t \cdot \log t}, \quad \theta'(t) = -\frac{1}{t} - \frac{\theta(t)}{t}, \tag{3.16}$$

where $E'(t)$ is the derivative of $E(t)$ and so on. Hence we obtain

$$f'(t) = (1 + \log t) \cdot E(t). \tag{3.17}$$

Thus the function $d(t)$ is also continuously differentiable on $(p,\ p+2)$, since $g(t) > 0$.
Moreover, $d(t)$ has the right hand derivative at the point $t = p$, that is,

$$\partial_+ d(p) := \lim_{t \to p+0} \frac{d(t) - d(p)}{t - p}.$$

Put $d'(p) = \partial_+ d(p)$. We here assume that

$$d'(p) \cdot g(p) \cdot \sqrt{p} \leq 2 \quad (p \geq 3) \tag{3.18}$$

and we will call (3.18) the condition (d) below. If the condition (d) holds, then for any prime $p \geq 3$ we have

$$(1 + \log p) \cdot E(p) \leq d(p) \cdot \frac{\log^2(p \cdot \alpha)}{2 \cdot \sqrt{p}} \cdot \left(1 + \frac{4}{\log(p \cdot \alpha)}\right) + \frac{2}{\sqrt{p}}. \tag{3.19}$$

3.8. Proof of Lemma 1

Now we are ready for the proof of the lemma 1. Let

$$D_m := \frac{p_m \cdot (e'_0 - \alpha_0)}{\sqrt{p_m \cdot \alpha_0} \cdot \log^2(p_m \cdot \alpha_0)} \quad (m \geq 5). \tag{3.20}$$

Then $C_m < 1$ is equivalent to $D_m < 1$. And we here have $D_m < 1$ for any $11 \leq p_m \leq e^{14}$, and

$$D_m \leq a_m := 1 - 13 \cdot S(p_m) \tag{3.21}$$

for any $p_m \geq e^{14}$. In fact, $D_m < 1$ is also equivalent to $\mathcal{R}_m < 0$ and it is easy to see that

$$\mathcal{R}_m := \log(e^{-\gamma} \cdot F_m) - \log\log(\log \Im_m + \sqrt{\log \Im_m} \cdot (\log\log \Im_m)^2) < 0. \tag{3.22}$$

for $11 \leq p_m \leq e^{14}$ by MATLAB (see the table 1 and the table 2).
Next, we will prove $D_m \leq a_m$ for any $p_m \geq e^{14}$ by the mathematical induction with respect to m. If $p' = 1202609$ then we have

$$D_{93118} = 0.010 \cdots \leq 0.06 \leq 1 - 13 \cdot S(p') \leq 0.09 < 1. \tag{3.23}$$

Now assume $p \geq e^{14}$ and $D_{m-1} \leq a_{m-1}$. Then

$$D_m = \frac{1}{N_0} \cdot (p \cdot (e'_1 - \alpha) + V_0) = D_{m-1} \cdot \frac{N_1}{N_0} + \frac{V_0}{N_0} \leq$$

$$\leq a_{m-1} \cdot \frac{N_1}{N_0} + \frac{1}{N_0} \cdot \log p_0 \cdot (\mu \cdot e'_1 - 1) \leq a_{m-1} + b_{m-1}, \tag{3.24}$$

where

$$b_{m-1} = \frac{1}{N_0} \cdot (\log p_0 \cdot (\mu \cdot e'_1 - 1) - a_{m-1} \cdot (N_0 - N_1)). \tag{3.25}$$

By the assumption $D_{m-1} \leq a_{m-1}$, we get

$$e'_1 \leq \alpha + a_{m-1} \cdot \frac{\sqrt{p \cdot \alpha} \cdot \log^2(p \cdot \alpha)}{p} = \alpha \cdot \left(1 + a_{m-1} \frac{\log^2(p\,\alpha)}{\sqrt{p\,\alpha}}\right)$$

and by taking logarithm of both sides

$$\log e'_1 = (\log p) \cdot (e_1 - 1) \leq \theta(p) + a_{m-1} \cdot \frac{\log^2(p \cdot \alpha)}{\sqrt{p \cdot \alpha}}.$$

From this

$$e_1 \leq 1 + \frac{1}{\log p} \cdot \left(\theta(p) + a_{m-1} \cdot \frac{\log^2(p \cdot \alpha)}{\sqrt{p \cdot \alpha}}\right),$$

$$E(p) + \varepsilon(p) \leq \frac{1}{\log p} \left(\theta(p) + a_{m-1} \cdot \frac{\log^2(p \cdot \alpha)}{\sqrt{p \cdot \alpha}}\right).$$

Thus

$$\log p \cdot E(p) - \theta(p) \leq a_{m-1} \cdot \frac{\log^2(p \cdot \alpha)}{\sqrt{p \cdot \alpha}} - \log p \cdot \varepsilon(p)$$

and the both sides multiply by

$$\frac{p}{\sqrt{p}\cdot\log^2(p\cdot\alpha)},$$

then

$$d(p)=\frac{p\cdot\log p\cdot E(p)-p\cdot\theta(p)}{\sqrt{p}\cdot\log^2(p\cdot\alpha)}\le\frac{a_{m-1}}{\sqrt{\alpha}}-\frac{p\cdot\log p\cdot\varepsilon(p)}{\sqrt{p}\cdot\log^2(p\cdot\alpha)}. \tag{3.26}$$

If the condition (d) holds, then from (3.19) we get

$$(1+\log p)\cdot E(p)\le a_{m-1}\cdot\frac{\log^2(p\cdot\alpha)}{2\cdot\sqrt{p\cdot\alpha}}\cdot\left(1+\frac{4}{\log(p\cdot\alpha)}\right)-$$

$$-(1+\log p)\cdot\varepsilon(p)+\frac{2}{\sqrt{p}},$$

because $\varepsilon(p)<0$ and

$$\frac{\log p}{2}\cdot\left(1+\frac{4}{\log(p\cdot\alpha)}\right)\le(1+\log p)\qquad(p\ge e^{14},\ \alpha\ge 1-1/14).$$

Thus we see

$$(1+\log p)\cdot(E(p)+\varepsilon(p))\le a_{m-1}\cdot\frac{\log^2(p\cdot\alpha)}{2\cdot\sqrt{p\cdot\alpha}}\cdot\left(1+\frac{4}{\log(p\cdot\alpha)}\right)+\frac{2}{\sqrt{p}}. \tag{3.27}$$

If $e_1>1$, then, since $0<a_{m-1}\le 1$ and

$$(1-1/14)\le\alpha\le(1+1/14),$$

we also have

$$(1+\log p)^2\cdot(E(p)+\varepsilon(p))^2\le\frac{\log^4(p\cdot\alpha)}{p\cdot\alpha}\cdot\left(\frac{1}{2}+\frac{2}{\log(p\cdot\alpha)}+\frac{2\cdot\sqrt{\alpha}}{\log^2(p\cdot\alpha)}\right)^2\le$$

$$\le 0.4287\cdot\frac{\log^4(p\cdot\alpha)}{p\cdot\alpha}\quad(p\ge e^{14}) \tag{3.28}$$

and

$$\log p_0\cdot(\mu\cdot e_1'-1)-a_{m-1}\cdot(N_0-N_1)\le\log p_0\cdot(1+\log p)\cdot(E(p)+\varepsilon(p))-$$

$$-a_{m-1}\cdot(N_0-N_1)+0.55\cdot\frac{\log^2 p_0}{p}+0.6\cdot\log p_0\cdot(1+\log p)^2\cdot(E(p)+\varepsilon(p))^2\le$$

$$\le a_{m-1}\cdot G_0\cdot N_0+0.55\cdot\frac{\log^2 p_0}{p}+0.2572\cdot\log p_0\cdot\frac{\log^4(p\cdot\alpha)}{p\cdot\alpha}+\frac{2\cdot\log p_0}{\sqrt{p}}.$$

Finally, by the function $(\log^4 t)/\sqrt{t}$ is decreasing on the interval $(e^8,\ +\infty)$ we have

$$b_{m-1}\le a_{m-1}\cdot G_0+0.55\cdot\frac{\log p}{\sqrt{p\cdot\alpha}}\cdot\left(1+\frac{\log\sqrt{2}-\log\alpha}{\log p+\log\alpha}\right)^2\cdot\frac{1}{p\cdot\log p}+$$

$$+0.2572\cdot\frac{\log^4 p}{\sqrt{p}}\cdot\frac{1+\log\sqrt{2}/\log p}{\alpha^{3/2}}\cdot\frac{(1+\log\alpha/\log p)^2}{p\cdot\log p}+$$

$$+\frac{2}{\sqrt{\alpha}}\cdot\left(1+\frac{\log\sqrt{2}-\log\alpha}{\log p+\log\alpha}\right)^2\cdot\frac{1}{p\cdot\log p}\le$$

$$\le\frac{0.01}{p\cdot\log p}+\frac{0.01}{p\cdot\log p}+\frac{10.421}{p\cdot\log p}+\frac{2.203}{p\cdot\log p}\le\frac{13}{p\cdot\log p}\ (p\ge e^{14}). \tag{3.29}$$

Next, if $e_1\le 1$ then we have

$$b_{m-1}\le 0.55\cdot\frac{\log^2 p_0}{p\cdot N_1}\le\frac{0.01}{p\cdot\log p}\ \ (p\ge e^{14}). \tag{3.30}$$

3.9. Algorithm and Tables for Sequence $\{C_m\}$ and $\{\mathcal{R}_m\}$

The table 1 shows the values of $C_m=\Phi_0(\Im_m)$ and $\mathcal{R}_m$ to $\omega(n)=m$ for $n\in N$. There are only values of C_m and $\mathcal{R}_m$ for $1\le m\le 10$ here. But it is not difficult to verify them for $31\le p_m\le e^{14}$. Note, if more informations, then it should be taken $\mathcal{R}_m<0$, not $C_m<1$, for $263\le p_m\le e^{14}$, by reason of the limited values of matlab 6.5. The table 2 shows the values $\mathcal{R}_m$ for $93109\le m\le 93118$. Of course, all the values in the table 1 and the table 2 are approximate.

The algorithm for $\mathcal{R}_m$ to $\omega(n)=m$ by matlab is as follows:
Function ETF-Index, clc, gamma=0.57721566490153286060; format long
$P=[2,3,5,7,\cdots,1202609]$; M=length(P);
for $m=1:M$; $p=P(1:m)$; $q=1-1./p$; $F=-gamma+\log(prod(1./q))$;
$N1=sum(\log(p.))$; $N2=(N1)^{1/2}$; $N3=(\log(N1))^2$; $N4=N2*N3$; $N5=N1+N4$;
$m,\ p(m),\ C_m=\exp(\exp(\exp(F)))/\exp(N1)/\exp(N4),\ \mathcal{R}_m=F-(\log(\log(N5)))$, end

Table 1

m	$p(m)$	C_m	$\mathcal{R}_m$
1	2	9.66806133818849	$-$
2	3	23.15168798263150	0.73259862957209
3	5	7.73864609733096	0.14633620860732
4	7	0.83171792006862	-0.00636141995881
5	11	0.01114282713904	-0.09308687002330
6	13	$1.102119966548700e-004$	-0.12730939385590
7	17	$3.834259945131073e-007$	-0.15077316854133
8	19	$1.397561045763582e-009$	-0.15960912308179
9	23	$2.821898264763264e-012$	-0.16612788105591
10	29	$2.081541289212468e-015$	-0.17415284347098

Table 2

m	$p(m)$	$\mathcal{R}_m$
93109	1202477	-0.01154791933871
93110	1202483	-0.01154786567870
93111	1202497	-0.01154781201949
93112	1202501	-0.01154775835370
93113	1202507	-0.01154770468282
93114	1202549	-0.01154765103339
93115	1202561	-0.01154759738330
93116	1202569	-0.01154754372957
93117	1202603	-0.01154749009141
93118	1202609	-0.01154743644815

IV. Lemma 2 for Proof of Condition (d)

In the middle of the proof of the Lemma 1, we used the condition (d). So we here would prove the following Lemma 2, which shows that the condition (d) holds.

[**Lemma 2**] For any prime number $p \geq 3$ we have

$$d'(p) \cdot \ g(p) \cdot \sqrt{p} \leq 2.$$

We make ready for the proof of the Lemma 2.

4.1. A Condition ($\mathbf{d}^{'}$)

If the Lemma 2 does not hold, then there exists a prime number $p \geq 3$ such that

$$d'(p) \cdot \ g(p) \cdot \sqrt{p} > 2. \tag{4.1}$$

We fix such prime p. Then from the table 3 and the table 4 we see $p \geq e^{14}$, because $H_m < 0$ for any $3 \leq p_m \leq e^{14}$ (see (4.36) below). Now we define the function

$$G(t) := d'(t) \cdot g(t) \cdot \sqrt{t}, \qquad t \in (p, \ p+2).$$

Then

$$G'(t) = \frac{1}{\sqrt{t}} \cdot \left(\partial_0(t) - 1 - \frac{1}{\log t} + \left(1 + \frac{2}{\log(t \cdot \alpha)}\right) \cdot D_1(t) \right), \tag{4.2}$$

where

$$\partial_0(t) = E(t) + \frac{f(t)}{4 \cdot t} \cdot \left(1 - \frac{8}{\log^2(t \cdot \alpha)}\right) -$$

$$-f'(t) \cdot \left(1 + \frac{4}{\log(t \cdot \alpha)}\right) + \frac{f(t)}{2 \cdot t} \cdot \left(1 + \frac{4}{\log(t \cdot \alpha)}\right)^2 \tag{4.3}$$

and

$$D_1(t) = d'(t) \cdot g(t) = f'(t) - d(t) \cdot g'(t).$$

Hence $G'(t) < 0$ is equivalent to

$$\partial_0(t) + \left(1 + \frac{2}{\log(t \cdot \alpha)}\right) \cdot D_1(t) < 1 + \frac{1}{\log t}.$$

Since $t \geq e^{14}$ and $\alpha \geq 1 - 1/14$ by (3.11), we get

$$\log(t \cdot \alpha) = \log t + \log \alpha \geq 13.925 > 0$$

and so by (3.7),

$$|\partial_0(t)| + \left(1 + \frac{2}{\log(t \cdot \alpha)}\right) \cdot |D_1(t)| < \frac{1}{\log^2 t} + \frac{1}{2 \cdot \log t} +$$

$$\left(1 + \frac{4}{\log(t \cdot \alpha)}\right) \cdot \left(\frac{1}{\log t} + \frac{1}{\log^2 t}\right) + \left(1 + \frac{4}{\log(t \cdot \alpha)}\right)^2 \cdot \frac{1}{\log t} +$$

$$+ \left(1 + \frac{2}{\log(t \cdot \alpha)}\right) \cdot \left(\frac{1}{\log t} + \frac{1}{\log^2 t} + \left(1 + \frac{4}{\log(t \cdot \alpha)}\right) \cdot \frac{1}{\log t}\right) < 0.46 \, (t \geq e^{14}). \tag{4.4}$$

This shows that the function $G(t)$ is decreasing on the interval $(p,\ p+2)$. Thus there exists a point t_1 such that $p < t_1 < p+1$ and

$$G(p+1) = G(p) + G(p+1) - G(p) =$$

$$= G(p) + G'(t_1) \cdot (p+1-p) > 2 - \frac{2}{\sqrt{p}},$$

where

$$G(p) := G(p+0) = \lim_{t \to p+0} G(t) = d'(p) \cdot g(p) \cdot \sqrt{p}.$$

For the convenient discussion, we put $x_1 = p,\ x_2 = p+1$. Then, since $G(t) \geq G(p+1)$, for any $t \in (x_1,\ x_2)$ we have

$$d'(t) \cdot\ g(t) \cdot \sqrt{t} > 2 \cdot (1 - 1/\sqrt{t}). \tag{4.5}$$

We will call (4.5) the condition (d'). For the proof of the lemma 2, we must obtain a certain contradiction from the condition (d').

4.2. Proof of $d''(t) < 0$

For any $t \in (x_1,\ x_2)$ we here have $d''(t) < 0$. In fact, since

$$d''(t) = \frac{1}{t \cdot g(t)} \cdot \left(\partial_0(t) - 1 - \frac{1}{\log t} \right),$$

we easily see that $d''(t) < 0$ is equivalent to $\partial_0(t) < 1 + 1/\log t$, and

$$|\partial_0(t)| \leq 0.2577 < 1 \quad (t \geq e^{14}).$$

4.3. Function $F(t)$ and $F'(t)$

Put

$$F(t) := (d_2 - d(t)) \cdot g'(t) - (g(t) - g_1) \cdot d'(t), \qquad t \in (x_1,\ x_2).$$

Then it is clear

$$\int_{x_1}^{x_2} F(t) \cdot dt = 0, \tag{4.6}$$

where $g_1 = g(x_1),\ d_2 = d(x_2)$. Hence there exists a point ξ_0 such that $x_1 < \xi_0 < x_2$ and

$$\int_{x_1}^{x_2} F(t) \cdot dt = F(\xi_0) \cdot (x_2 - x_1) = 0$$

and so

$$(d_2 - d(\xi_0)) \cdot g'(\xi_0) = (g(\xi_0) - g_1) \cdot d'(\xi_0). \tag{4.7}$$

We here have $F'(t) < 0$ for any $t \in (x_1,\ x_2)$ under the condition (d'). In fact, since $d'(t) > 0$ from the condition (d'), for $F'(t) < 0$ it is sufficient to show

$$(g(t) - g_1) \cdot (-d''(t)) < 2 \cdot d'(t) \cdot g'(t).$$

And there exists a point t_1 such that $x_1 < t_1 < t$ and

$$g(t) - g(x_1) = g'(t_1) \cdot (t - x_1) \leq g'(t_1) \leq$$

$$\leq g'(t)\cdot\left(1-\frac{g''(x_1)}{g'(x_2)}\right)\leq 1.01\cdot g'(t) \quad (t\geq e^{14}).$$

Hence

$$(g(t)-g_1)\cdot(-d''(t))\leq 1.01\cdot\frac{g'(t)}{t\cdot g(t)}\cdot\left(1+\frac{1}{\log t}-\partial_0(t)\right)\leq$$

$$\leq\frac{2\cdot(1-1/\sqrt{t})\cdot g'(t)}{\sqrt{t}\cdot g(t)}\leq 2\cdot d'(t)\cdot g'(t) \ \ (t\geq e^{14}). \tag{4.8}$$

Moreover, we note $F''(t)>0$ holds for any $t\in(x_1,\ x_2)$.

4.4. An estimate of the point ξ_0

From

$$\int_{x_1}^{x_2}F(t)\cdot dt=\int_{x_1}^{\xi_0}F(t)\cdot dt+\int_{\xi_0}^{x_2}F(t)\cdot dt=0, \tag{4.9}$$

there exist $\lambda_1,\ \lambda_2$ such that $x_1<\lambda_1<\xi_0<\lambda_2<x_2$ and

$$F(\lambda_1)\cdot(\xi_0-x_1)+F(\lambda_2)\cdot(x_2-\xi_0)=0. \tag{4.10}$$

Then, since $F'(t)<0 \ \ (t\in(x_1,\ x_2))$, we have

$$F(\lambda_1)>F(\xi_0)=0>F(\lambda_2).$$

Put $x_0:=x_1+x_2-\xi_0$. Then from (4.10) we have

$$F(\lambda_1)\cdot(x_2-x_0)+F(\lambda_2)\cdot(x_0-x_1)=0. \tag{4.11}$$

This (4.11) shows that the line passing the points $(x_1,\ F(\lambda_1))$ and $(x_2,\ F(\lambda_2))$ passes the point $(x_0,\ 0)$. On the other hand, by the mean value theorem, there exist the points η_1 and η_2 such that $x_1<\eta_1<\xi_0<\eta_2<x_2$ and

$$d(x_2)-d(\xi_0)=d'(\eta_2)\cdot(x_2-\xi_0),$$

$$g(\xi_0)-g(x_1)=g'(\eta_1)\cdot(\xi_0-x_1).$$

Since the function $g'(t)$ is decreasing on $(x_1,\ x_2)$ and from the condition (d'), we have

$$g'(\xi_0)\leq g'(\eta_1), \quad d'(t)>0 \quad (t\in(x_1,\ x_2)).$$

Here if $x_0\leq\xi_0$, then $x_2-\xi_0\leq\xi_0-x_1$ and by (4.7) we get

$$d'(\xi_0)=d'(\eta_2)\cdot\frac{x_2-\xi_0}{\xi_0-x_1}\cdot\frac{g'(\xi_0)}{g'(\eta_1)}\leq d'(\eta_2),$$

but it is a contradiction to $d''(t)<0$. Thus we have $x_0-\xi_0>0$ under the condition (d').

4.5. An estimate of $\varepsilon_0:=x_0-\xi_0$

Since $F(\xi_0)=0$, we have

$$F(x_1)=(d_2-d_1)\cdot g'(x_1)=F(x_1)-F(\xi_0)=-F'(\beta_0)\cdot(\xi_0-x_1),$$

where $x_1<\beta_0<\xi_0$. Also since $\xi_0-x_1=(1-\varepsilon_0)/2$ and

$$F'(t)=(d_2-d(t))\cdot g''(t)-(g(t)-g_1)\cdot d''(t)-2\cdot d'(t)\cdot g'(t), \tag{4.12}$$

we have

$$d'(\beta_0)\cdot g'(\beta_0)\cdot\varepsilon_0 = T_1 + T_2,$$

where

$$T_1 = d'(\beta_0)\cdot g'(\beta_0) - (d_2 - d_1)\cdot g'(x_1),$$

$$T_2 = -\Big((d_2 - d(\beta_0))\cdot g''(\beta_0) - (g(\beta_0) - g_1)\cdot d''(\beta_0)\Big)\cdot(\xi_0 - x_1).$$

By the condition (d'), we get

$$d'(\beta_0)\cdot g'(\beta_0)\cdot\varepsilon_0 \geq \frac{2\cdot(1-1/\sqrt{\beta_0})\cdot g'(\beta_0)}{g(\beta_0)\cdot\sqrt{\beta_0}}\cdot\varepsilon_0 \geq \frac{\varepsilon_0}{x_2\cdot\sqrt{x_2}}$$

and

$$\begin{aligned} T_1 &= d'(\beta_0)\cdot g'(\beta_0) - (d_2 - d_1)\cdot g'(x_1) = \\ &= d'(\beta_0)\cdot g'(\beta_0) - d'(\beta_1)\cdot g'(x_1) = \\ &= d''(\beta_1')\cdot g'(\beta_0)\cdot(\beta_0 - \beta_1) + d'(\beta_1)\cdot g''(\beta_0')\cdot(\beta_0 - x_1) \leq \\ &\leq a_2 + b_2 \leq \frac{0.9005}{x_1^2} \quad (x_1 \geq e^{14}), \end{aligned}$$

where $x_1 < \beta_1 < x_2,\quad \beta_0 < \beta_1' < \beta_1,\quad x_1 < \beta_0' < \beta_0$ and

$$a_2 := |d'(t_1)\cdot g''(t_2)|,\quad b_2 := |d''(t_1)\cdot g'(t_2)| \quad (\textit{see below}).$$

We also have

$$\begin{aligned} T_2 &= -\Big((d_2 - d(\beta_0))\cdot g''(\beta_0) - (g(\beta_0) - g_1)\cdot d''(\beta_0)\Big)\cdot(\xi_0 - x_1) = \\ &= -\Big(d'(\beta_2)\cdot g''(\beta_0)\cdot(x_2 - \beta_0) - g'(\beta_2')\cdot d''(\beta_0)\cdot(\beta_0 - x_1)\Big)\cdot(\xi_0 - x_1) \leq \\ &\leq (a_2 + b_2)/2 \leq \frac{0.4503}{x_2^2} \quad (x_2 \geq e^{14}), \end{aligned}$$

where $\beta_0 < \beta_2 < x_2,\quad x_1 < \beta_2' < \beta_0$. Thus we have

$$0 < \varepsilon_0 \leq \frac{1.3508}{\sqrt{x_2}} \leq 0.0015 \quad (x_2 \geq e^{14}). \tag{4.13}$$

4.6. An estimate of $\delta_0 := \lambda_0 - x_0$

Here a point λ_0 is determined as follows. If the line passing the points $(\lambda_1, F(\lambda_1)$ and $(\lambda_2, F(\lambda_2)$ intersects the line $y = 0$ at the point λ_0, then we obtain

$$F(\lambda_1)\cdot(\lambda_2 - \lambda_0) + F(\lambda_2)\cdot(\lambda_0 - \lambda_1) = 0. \tag{4.14}$$

Since the function $F(t)$ is decreasing and convex on $(x_1,\, x_2)$ under the condition (d'), it is clear $\xi_0 < \lambda_0$. And the equation of the line passing $(\xi_0,\ F(\lambda_1))$ and $(\lambda_0,\ 0)$ is

$$F(\lambda_1)\cdot x + (\lambda_0 - \xi_0)\cdot y - F(\lambda_1)\cdot\lambda_0 = 0 \tag{4.15}$$

and one passing $(\xi_0,\ -F(\lambda_2))$ and $(x_0,\ 0)$ is

$$-F(\lambda_2)\cdot x + (x_0 - \xi_0)\cdot y + F(\lambda_2)\cdot x_0 = 0. \tag{4.16}$$

We put

$$\triangle_0 := F(\lambda_1) \cdot (x_0 - \xi_0) + F(\lambda_2) \cdot (\lambda_0 - \xi_0) \neq 0,$$
$$\triangle_1 := F(\lambda_1) \cdot F(\lambda_2) \cdot \delta_0.$$

Then y-coordinate of the cross point of above two lines is $\triangle_1/\triangle_0$. Here if $\delta_0 > 0$ then $\triangle_1/\triangle_0 < 0$ and $\triangle_1 < 0$, if $\delta_0 < 0$ then $\triangle_1/\triangle_0 > 0$. and $\triangle_1 > 0$. Hence we always have $\triangle_0 > 0$ and

$$F(\lambda_1) \cdot (x_0 - \xi_0) + F(\lambda_2) \cdot (\lambda_0 - \xi_0) > 0. \tag{4.17}$$

From this

$$-F(\lambda_2) \cdot \delta_0 < (F(\lambda_1) + F(\lambda_2)) \cdot (x_0 - \xi_0).$$

And since

$$x_2 - x_0 = \frac{1 - \varepsilon_0}{2},$$
$$x_0 - x_1 = \frac{1 + \varepsilon_0}{2},$$

we have

$$\frac{F(\lambda_1) + F(\lambda_2)}{-F(\lambda_2)} = \frac{x_0 - x_1}{x_2 - x_0} - 1 = \frac{2 \cdot \varepsilon_0}{1 - \varepsilon_0}$$

and so

$$\delta_0 < 2 \cdot \varepsilon_0^2/(1 - \varepsilon_0).$$

Similarly, for the lines passing the points $(\xi_0, F(\lambda_1))$, $(x_0, 0)$ and the points $(\xi_0, -F(\lambda_2))$, $(\lambda_0, 0)$, we have

$$F(\lambda_1) \cdot (\lambda_0 - \xi_0) + F(\lambda_2) \cdot (x_0 - \xi_0) > 0$$

and from this

$$\delta_0 > (-1) \cdot 2 \cdot \varepsilon_0^2/(1 + \varepsilon_0).$$

Therefore we have

$$(-1) \cdot \frac{2 \cdot \varepsilon_0^2}{1 + \varepsilon_0} < \delta_0 < \frac{2 \cdot \varepsilon_0^2}{1 - \varepsilon_0}. \tag{4.18}$$

4.7. An estimate of $\delta_1 := \lambda_1 - \lambda_1'$

Here $\lambda_1' := x_1 + \xi_0 - \lambda_1$. By the same method as in δ_0, for the lines passing $(x_1,\ F(\lambda_1))$, $(\lambda_1,\ 0)$ and $(x_1,\ -F(\lambda_2))$, $(\lambda_1',\ 0)$ we have

$$F(\lambda_1) \cdot (\lambda_1' - x_1) + F(\lambda_2) \cdot (\lambda_1 - x_1) > 0.$$

From this, since

$$\frac{F(\lambda_1) + F(\lambda_2)}{F(\lambda_1)} = \frac{2 \cdot \varepsilon_0}{1 + \varepsilon_0},$$
$$\lambda_1 - x_1 = \frac{1}{4} \cdot (1 - \varepsilon_0) + \frac{1}{2} \cdot \delta_1,$$

we have $\delta_1 < \varepsilon_0/2$. Also for the lines passing the points $(x_1, F(\lambda_1))$, $(\lambda_1', 0)$ and $(x_1, -F(\lambda_2))$, $(\lambda_1, 0)$ we have

$$F(\lambda_1) \cdot (\lambda_1 - x_1) + F(\lambda_2) \cdot (\lambda_1' - x_1) > 0.$$

and so $\delta_1 > (-\varepsilon_0)/2$. Therefore we get

$$(-1) \cdot \frac{\varepsilon_0}{2} < \delta_1 < \frac{\varepsilon_0}{2}. \tag{4.19}$$

4.8. An estimate of $\omega_0 := (\lambda_1 + \lambda_2) - 2 \cdot \xi_0$

Let $\lambda_0' := \lambda_1 + \lambda_2 - \lambda_0$. Then from (4.10) and (4.14) we get

$$-\frac{F(\lambda_1)}{F(\lambda_2)} = \frac{x_2 - \xi_0}{\xi_0 - x_1} = \frac{\lambda_0 - \lambda_1}{\lambda_2 - \lambda_0} = \frac{\lambda_2 - \lambda_0'}{\lambda_0' - \lambda_1}$$

and

$$\lambda_0 = \lambda_1 \cdot (\xi_0 - x_1) + \lambda_2 \cdot (x_2 - \xi_0),$$
$$\lambda_0' = \lambda_1 \cdot (x_2 - \xi_0) + \lambda_2 \cdot (\xi_0 - x_1).$$

Hence

$$\lambda_0 - \lambda_0' = (\lambda_2 - \lambda_1) \cdot \varepsilon_0$$

Put $\omega_1 := (\lambda_1 + \lambda_2) - (x_1 + x_2)$. Then, since

$$\lambda_1 + \lambda_2 = \lambda_0 + \lambda_0' = 2 \cdot \lambda_0 - (\lambda_0 - \lambda_0'),$$

$$x_1 + x_2 = x_0 + \xi_0 = 2 \cdot x_0 - (x_0 - \xi_0),$$

we have

$$\omega_1 = 2 \cdot \delta_0 + (1 - (\lambda_2 - \lambda_1)) \cdot \varepsilon_0.$$

Thus

$$\omega_0 = (\lambda_1 + \lambda_2) - 2 \cdot \xi_0 = \omega_1 + \varepsilon_0 =$$
$$= 2 \cdot \delta_0 + 2 \cdot \varepsilon_0 - (\lambda_2 - \lambda_1) \cdot \varepsilon_0. \qquad (4.20)$$

4.9. An estimate of $\Lambda_0 := (\lambda_2 - \lambda_1)$

Since

$$\xi_0 - \lambda_1 = \frac{1}{4} \cdot (1 - \varepsilon_0) - \frac{1}{2} \cdot \delta_1,$$

we have

$$(\lambda_2 - \lambda_1) = 2 \cdot (\lambda_0 - \lambda_1) - (\lambda_0 - \lambda_0') =$$
$$= 2 \cdot \delta_0 + 2 \cdot \varepsilon_0 + \frac{1}{2} \cdot (1 - \varepsilon_0) - \delta_1 - (\lambda_2 - \lambda_1) \cdot \varepsilon_0,$$

and

$$(\lambda_2 - \lambda_1) = \frac{1}{2} + \frac{\varepsilon_0}{1 + \varepsilon_0} - \frac{\delta_1}{1 + \varepsilon_0} + \frac{2 \cdot \delta_0}{1 + \varepsilon_0}$$

and hence

$$\frac{1}{2} < (\lambda_2 - \lambda_1) < \frac{1}{2} + 2 \cdot \varepsilon_0. \qquad (4.21)$$

4.10. An estimate of $\delta_2 := \lambda_1 - \eta_1$

By the same method as above, for the lines passing the points $(x_1,\ F(\lambda_1))$, $(\lambda_1,\ 0)$ and $(x_1,\ -F(\lambda_2))$, $(\eta_1,\ 0)$ we have

$$F(\lambda_1) \cdot (\eta_1 - x_1) + F(\lambda_2) \cdot (\lambda_1 - x_1) > 0.$$

From this

$$\delta_2 < \frac{F(\lambda_1) + F(\lambda_2)}{F(\lambda_1)} \cdot (\lambda_1 - x_1) =$$

$$= \frac{2 \cdot \varepsilon_0}{1+\varepsilon_0} \cdot \left(\frac{1}{4} \cdot (1-\varepsilon_0) + \frac{1}{2} \cdot \delta_1 \right) < \frac{1}{2} \cdot \frac{\varepsilon_0}{1+\varepsilon_0}.$$

Similarly, for the lines passing the points $(x_1,\, F(\lambda_1))$, $(\eta_1,\, 0)$ and $(x_1,\, -F(\lambda_2))$, $(\lambda_1,\, 0)$ we have

$$F(\lambda_1) \cdot (\lambda_1 - x_1) + F(\lambda_2) \cdot (\eta_1 - x_1) > 0.$$

From this

$$\delta_2 > \frac{F(\lambda_1) + F(\lambda_2)}{F(\lambda_2)} \cdot (\lambda_1 - x_1) > -\frac{1}{2} \cdot \frac{\varepsilon_0}{1-\varepsilon_0}.$$

Consequently, we have

$$(-1) \cdot \frac{\varepsilon_0}{2 \cdot (1-\varepsilon_0)} < \delta_2 < \frac{\varepsilon_0}{2 \cdot (1+\varepsilon_0)}. \tag{4.22}$$

4.11. An estimate of $Q_0 := (\eta_2 + \eta_1 - 2 \cdot \xi_0) \cdot (\xi_0 - x_1)$

First, we will find the lower bound of Q_0.
If the line passing $(\eta_1,\, F(\eta_1))$, $(\eta_2,\, F(\eta_2))$ intersects $y = 0$ at η_0, then

$$F(\eta_1) \cdot (\eta_2 - \eta_0) + F(\eta_2) \cdot (\eta_0 - \eta_1) = 0$$

and from this

$$F(\lambda_1) \cdot (\eta_2 - \eta_0) + F(\lambda_2) \cdot (\eta_0 - \eta_1) + W_0 = 0,$$

where

$$W_0 = (F(\eta_1) - F(\lambda_1)) \cdot (\eta_2 - \eta_0) + (F(\eta_2) - F(\lambda_2)) \cdot (\eta_0 - \eta_1).$$

From (4.10) we have

$$\eta_0 = \eta_1 + (\eta_2 - \eta_1) \cdot (x_2 - \xi_0) + W_1,$$

where

$$W_1 = \frac{x_2 - \xi_0}{F(\lambda_1)} \cdot W_0.$$

Since $F(t)$ is convex on $(x_1,\, x_2)$ and $\eta_1 < \xi_0 < \eta_2$, we have $\eta_0 > \xi_0$ and

$$\eta_1 + (\eta_2 - \eta_1) \cdot (x_2 - \xi_0) + W_1 > \xi_0.$$

From this

$$Q_0 > -(\eta_2 - \xi_0) \cdot \varepsilon_0 - W_1.$$

Here if $W_1 < 0$ then

$$Q_0 > -\varepsilon_0$$

and if $W_1 > 0$ then we put $\eta_0' := \eta_1 + \eta_2 - \eta_0$. Then

$$F(\lambda_1) \cdot (\eta_0' - \eta_1) + F(\lambda_2) \cdot (\eta_2 - \eta_0') + W_0 = 0$$

and by same way as above

$$\eta_0' = \eta_1 + (\eta_2 - \eta_1) \cdot (\xi_0 - x_1) - W_1.$$

Thus we have

$$\eta_0 - \eta_0' = (\eta_2 - \eta_1) \cdot \varepsilon_0 + 2 \cdot W_1$$

and so $(\eta_0 - \eta_0') > 0$. Similarly, for the lines passing the points $(\lambda_1,\ F(\lambda_1))$, $(\eta_0,\ 0)$ and $(\lambda_1,\ -F(\lambda_2))$, $(\eta_0',\ 0)$ we have

$$F(\lambda_1) \cdot (\eta_0' - \lambda_1) + F(\lambda_2) \cdot (\eta_0 - \lambda_1) > 0.$$

On the other hand, since $\eta_2 > \eta_0$ and

$$(\xi_0 - \lambda_1) < \frac{1}{4},$$

we have

$$(\eta_0 - \eta_0') < \frac{F(\lambda_1) + F(\lambda_2)}{F(\lambda_1)} \cdot (\eta_0 - \lambda_1) <$$

$$< \frac{2 \cdot \varepsilon_0}{1 + \varepsilon_0} \cdot ((\eta_0 - \xi_0) + (\xi_0 - \lambda_1)) <$$

$$< 2 \cdot \varepsilon_0 \cdot (\eta_2 - \xi_0) + \frac{\varepsilon_0}{2}.$$

Hence

$$2 \cdot W_1 = (\eta_0 - \eta_0') - (\eta_2 - \eta_1) \cdot \varepsilon_0 <$$

$$< 2 \cdot \varepsilon_0 \cdot (\eta_2 - \xi_0) + \frac{\varepsilon_0}{2} - (\eta_2 - \xi_0) \cdot \varepsilon_0 - (\xi_0 - \eta_1) \cdot \varepsilon_0 \leq$$

$$\leq \varepsilon_0 \cdot (\eta_2 - \xi_0) + \frac{\varepsilon_0}{2} - (\xi_0 - \lambda_1) \cdot \varepsilon_0 - \delta_2 \cdot \varepsilon_0 =$$

$$= \varepsilon_0 \cdot (\eta_2 - \xi_0) + \frac{\varepsilon_0}{2} - \varepsilon_0 \cdot \left(\frac{1}{4} \cdot (1 - \varepsilon_0) - \frac{1}{2} \cdot \delta_1 \right) - \delta_2 \cdot \varepsilon_0 <$$

$$< \varepsilon_0 \cdot (\eta_2 - \xi_0) + \frac{\varepsilon_0}{4} + 2 \cdot \varepsilon_0^2$$

and, since $x_2 > \eta_2$ and $x_2 - \xi_0 = (1 + \varepsilon_0)/2$,

$$Q_0 > -(\eta_2 - \xi_0) \cdot \varepsilon_0 - W_1 >$$

$$> -(\eta_2 - \xi_0) \cdot \varepsilon_0 - \frac{\varepsilon_0}{2} \cdot (\eta_2 - \xi_0) - \frac{\varepsilon_0}{8} - \varepsilon_0^2 >$$

$$> -\frac{3}{2} \cdot \varepsilon_0 \cdot (x_2 - \xi_0) - \frac{\varepsilon_0}{8} - \varepsilon_0^2 =$$

$$= -\frac{3}{4} \cdot \varepsilon_0 \cdot (1 + \varepsilon_0) - \frac{\varepsilon_0}{8} - \varepsilon_0^2 > -\varepsilon_0.$$

Therefore, generally, we have

$$Q_0 = (\eta_2 + \eta_1 - 2 \cdot \xi_0) \cdot (\xi_0 - x_1) > -\varepsilon_0. \tag{4.23}$$

Next, we will find the upper bound of Q_0. It is easy to see that

$$(\eta_2 + \eta_1 - 2 \cdot \xi_0) = (\eta_2 - \xi_0) - (\xi_0 - \eta_1) <$$

$$< (x_2 - \xi_0) - (\xi_0 - \eta_1) =$$

$$= \frac{1}{2} \cdot (1 + \varepsilon_0) - \left(\frac{1}{4} \cdot (1 - \varepsilon_0) - \frac{1}{2} \cdot \delta_1 + \delta_2 \right) <$$

$$< \frac{1}{4} + 2 \cdot \varepsilon_0$$

and

$$(2 \cdot \xi_0 - \eta_2 - \eta_1) = (\xi_0 - \eta_1) - (\eta_2 - \xi_0) <$$
$$< (\xi_0 - \eta_1) < \frac{1}{4} + 2 \cdot \varepsilon_0.$$

Therefore, since

$$|(\eta_2 + \eta_1 - 2 \cdot \xi_0)| \cdot |\xi_0 - x_1| <$$
$$< \left(\frac{1}{4} + 2 \cdot \varepsilon_0\right) \cdot \frac{1}{2} \cdot (1 - \varepsilon_0) < \frac{1}{8} + \varepsilon_0,$$

we have

$$|Q_0| < \frac{1}{8} + \varepsilon_0. \tag{4.24}$$

4.12. An estimate of $H_0 := (\xi_0 - \eta_1) \cdot (\xi_0 - x_1)$

Since

$$(\xi_0 - \eta_1) = \frac{1}{4} \cdot (1 - \varepsilon_0) - \frac{1}{2} \cdot \delta_1 + \delta_2,$$
$$(\xi_0 - x_1) = \frac{1}{2} \cdot (1 - \varepsilon_0),$$

we easily have

$$\frac{1}{8} - \varepsilon_0 < H_0 < \frac{1}{8} + \varepsilon_0. \tag{4.25}$$

4.13. A new equality

Now we add (4.10) and (4.11), then we have

$$F(\lambda_1) + F(\lambda_2) = (F(\lambda_1) - F(\lambda_2)) \cdot \varepsilon_0$$

and both sides multiply by $d'(\eta_2) \cdot g'(\xi_0)$, then

$$(F(\lambda_1) + F(\lambda_2)) \cdot d'(\eta_2) \cdot g'(\xi_0) =$$
$$= (F(\lambda_1) - F(\lambda_2)) \cdot d'(\eta_2) \cdot g'(\xi_0) \cdot \varepsilon_0. \tag{4.26}$$

On the other hand, since $F(\xi_0) = 0$, we get

$$F(\lambda_1) + F(\lambda_2) = (F(\lambda_2) - F(\xi_0)) - (F(\xi_0) - F(\lambda_1)) =$$
$$= F'(\alpha_2) \cdot (\lambda_2 - \xi_0) - F'(\alpha_1) \cdot (\xi_0 - \lambda_1) =$$
$$= (F'(\alpha_2) - F'(\alpha_1)) \cdot (\xi_0 - \lambda_1) + F'(\alpha_2) \cdot ((\lambda_2 - \xi_0) - (\xi_0 - \lambda_1)) =$$
$$= F''(\tau_0) \cdot (\alpha_2 - \alpha_1) \cdot (\xi_0 - \lambda_1) + F'(\alpha_2) \cdot (\lambda_1 + \lambda_2 - 2 \cdot \xi_0)$$

and

$$F(\lambda_1) - F(\lambda_2) = (F(\lambda_1) - F(\xi_0)) - (F(\lambda_2) - F(\xi_0)) =$$
$$= -F'(\alpha_1) \cdot (\xi_0 - \lambda_1) - F'(\alpha_2) \cdot (\lambda_2 - \xi_0) =$$
$$= (F'(\alpha_2) - F'(\alpha_1)) \cdot (\xi_0 - \lambda_1) - F'(\alpha_2) \cdot ((\lambda_2 - \xi_0) + (\xi_0 - \lambda_1)) =$$
$$= F''(\tau_0) \cdot (\alpha_2 - \alpha_1) \cdot (\xi_0 - \lambda_1) - F'(\alpha_2) \cdot (\lambda_2 - \lambda_1),$$

where $\lambda_1 < \alpha_1 < \xi_0 < \alpha_2 < \lambda_2$ and $\alpha_1 < \tau_0 < \alpha_2$.
As mentioned in the section 3.4, there exist the points η_1 and η_2 such that $x_1 < \eta_1 < \xi_0 < \eta_2 < x_2$ and

$$\begin{aligned} &(d_2 - d(\xi_0)) \cdot g'(\xi_0) - (g(\xi_0) - g_1) \cdot d'(\xi_0) = \\ &= d'(\eta_2) \cdot g'(\xi_0) \cdot (x_2 - \xi_0) - g'(\eta_1) \cdot d'(\xi_0) \cdot (\xi_0 - x_1) = \\ &= \Big(d'(\eta_2) \cdot g'(\xi_0) - g'(\eta_1) \cdot d'(\xi_0) \Big) \cdot (\xi_0 - x_1) + \\ &+ d'(\eta_2) \cdot g'(\xi_0) \cdot (x_0 - \xi_0) = 0 \end{aligned} \tag{4.27}$$

from (4.7). Here also there exist μ_1, μ_2 such that $\eta_1 < \mu_1 < \xi_0 < \mu_2 < \eta_2$ and

$$\begin{aligned} &d'(\eta_2) \cdot g'(\xi_0)) - g'(\eta_1) \cdot d'(\xi_0) = \\ &= (d'(\eta_2) - d'(\xi_0)) \cdot g'(\xi_0)) + (g'(\xi_0) - g'(\eta_1)) \cdot d'(\xi_0) = \\ &= d''(\mu_2) \cdot g'(\xi_0)) \cdot (\eta_2 - \xi_0) + g''(\mu_1) \cdot d'(\xi_0) \cdot (\xi_0 - \eta_1) = \\ &= \Big(d''(\mu_2) \cdot g'(\xi_0) + g''(\mu_1) \cdot d'(\xi_0) \Big) \cdot (\xi_0 - \eta_1) + \\ &+ d''(\mu_2) \cdot g'(\xi_0) \cdot (\eta_2 + \eta_1 - 2 \cdot \xi_0). \end{aligned} \tag{4.28}$$

We put

$$\begin{aligned} A_0 &:= d'(\eta_2) \cdot g'(\xi_0), \\ A_1 &:= (-d''(\mu_2)) \cdot g'(\xi_0) + (-g''(\mu_1)) \cdot d'(\xi_0), \\ A_2 &:= (-d''(\mu_2)) \cdot g'(\xi_0), \\ B_0 &:= d'(\alpha_2) \cdot g'(\alpha_2), \\ B_1 &:= (d_2 - d(\alpha_2)) \cdot g''(\alpha_2) - (g(\alpha_2) - g_1) \cdot d''(\alpha_2), \\ U_0 &:= d''(\tau_0) \cdot g'(\tau_0) + d'(\tau_0) \cdot g''(\tau_0), \\ U_1 &:= (d_2 - d(\tau_0)) \cdot g'''(\tau_0) - (g(\tau_0) - g_1) \cdot d'''(\tau_0). \end{aligned}$$

Then since

$$\begin{aligned} F''(t) = (d_2 - d(t)) \cdot g'''(t) - (g(t) - g_1) \cdot d'''(t) - \\ -3 \cdot (d''(t) \cdot g'(t) + d'(t) \cdot g''(t)), \end{aligned} \tag{4.29}$$

by (4.12) we have

$$\begin{aligned} &F'(\alpha_2) = B_1 - 2 \cdot B_0, \\ &F''(\tau_0) = U_1 - 3 \cdot U_0, \\ &\mathfrak{M}_{\mathfrak{o}} := A_0 \cdot \varepsilon_0 = A_1 \cdot (\xi_0 - \eta_1) \cdot (\xi_0 - x_1) + \\ &+ A_2 \cdot (\eta_2 + \eta_1 - 2 \cdot \xi_0) \cdot (\xi_0 - x_1) = O(1/t^2). \end{aligned} \tag{4.30}$$

Thus the left side of (4.26) is

$$L_0 := (F(\lambda_1) + F(\lambda_2)) \cdot d'(\eta_2) \cdot g'(\xi_0) = L_1 + L_2,$$

where

$$\begin{aligned} L_1 &= F''(\tau_0) \cdot d'(\eta_2) \cdot g'(\xi_0) \cdot (\alpha_2 - \alpha_1) \cdot (\xi_0 - \lambda_1), \\ L_2 &= F'(\alpha_2) \cdot d'(\eta_2) \cdot g'(\xi_0) \cdot (\lambda_1 + \lambda_2 - 2 \cdot \xi_0). \end{aligned}$$

And $L_1 = L_{11} - L_{12}$, where

$$L_{11} = U_1 \cdot A_0 \cdot (\alpha_2 - \alpha_1) \cdot (\xi_0 - \lambda_1) = O(1/t^4),$$

$$L_{12} = 3 \cdot U_0 \cdot A_0 \cdot (\alpha_2 - \alpha_1) \cdot (\xi_0 - \lambda_1) = O(1/t^3).$$

Also $L_2 = L_{21} - L_{22}$, where

$$L_{21} = 2 \cdot A_0 \cdot B_1 \cdot \delta_0 - 4 \cdot A_0 \cdot B_0 \cdot \delta_0 +$$

$$+ B_1 \cdot (2 - (\lambda_2 - \lambda_1)) \cdot \mathfrak{M}_{\mathfrak{o}} = O(1/t^4),$$

$$L_{22} = 2 \cdot (2 - (\lambda_2 - \lambda_1)) \cdot B_0 \cdot \mathfrak{M}_{\mathfrak{o}} = O(1/t^3)$$

Similarly, the right side of (4.26) is

$$R_0 := (F(\lambda_1) - F(\lambda_2)) \cdot \mathfrak{M}_{\mathfrak{o}} = R_1 - R_2,$$

where

$$R_1 = F''(\tau_0) \cdot \mathfrak{M}_{\mathfrak{o}} \cdot (\alpha_2 - \alpha_1) \cdot (\xi_0 - \lambda_1) =$$

$$= (U_1 - 3 \cdot U_0) \cdot \mathfrak{M}_{\mathfrak{o}} \cdot (\alpha_2 - \alpha_1) \cdot (\xi_0 - \lambda_1) = O(1/t^4)$$

And

$$R_2 := F'(\alpha_2) \cdot \mathfrak{M}_{\mathfrak{o}} \cdot (\lambda_2 - \lambda_1) = R_{21} - R_{22},$$

where

$$R_{21} = B_1 \cdot \mathfrak{M}_{\mathfrak{o}} \cdot (\lambda_2 - \lambda_1) = O(1/t^4),$$

$$R_{22} = 2 \cdot \ B_0 \cdot \mathfrak{M}_{\mathfrak{o}} \cdot (\lambda_2 - \lambda_1) = O(1/t^3).$$

Thus (4.26) is equivalent to $L_0 = R_0$, that is, we have a new equality

$$(L_{11} + L_{21}) - (L_{12} + L_{22}) = R_1 - (R_{21} - R_{22}). \tag{4.31}$$

4.14. A new inequality

Put

$$K_1 := (L_{12} + L_{22}) + R_{22},$$

$$K_2 := (L_{11} + L_{21}) - R_1 + R_{21},$$

then, from the equation (4.26), we have $K_1 = K_2$. And by $\mathfrak{M}_{\mathfrak{o}}$ we have

$$K_1 = \mathfrak{A}_{\mathfrak{o}} + K_{11} + K_{12},$$

where

$$\mathfrak{A}_{\mathfrak{o}} = A_1 \cdot B_0 \cdot (\xi_0 - \eta_1) \cdot (\xi_0 - x_1),$$

$$K_{11} = 3 \cdot U_0 \cdot A_0 \cdot (\alpha_2 - \alpha_1) \cdot (\xi_0 - \lambda_1) +$$

$$+ 3 \cdot A_1 \cdot B_0 \cdot (\xi_0 - \eta_1) \cdot (\xi_0 - x_1),$$

$$K_{12} = 4 \cdot A_2 \cdot B_0 \cdot (\eta_2 + \eta_1 - 2 \cdot \xi_0) \cdot (\xi_0 - x_1).$$

By the condition (d') we have

$$\left| \frac{d'(\alpha_2) \cdot g'(\alpha_2)}{d'(\eta_2) \cdot g'(\xi_0)} \right| \le 1 + x_2 \cdot \sqrt{x_2} \cdot (a_2 + b_2) \le$$

$$\leq 1 + \frac{0.9005}{\sqrt{x_1}} \quad (x_1 \geq e^{14})$$

and

$$\mathfrak{M}_1 := B_0 \cdot \varepsilon_0 = \frac{d'(\alpha_2) \cdot g'(\alpha_2)}{d'(\eta_2) \cdot g'(\xi_0)} \cdot \mathfrak{M}_0 = O(1/t^2).$$

And also we see $A_2 > 0$, $\mathfrak{M}_1 > 0$. Thus we get

$$K_{12} \geq K'_{12} := -4 \cdot A_2 \cdot \mathfrak{M}_1 = O(1/t^4).$$

It is clear that $d'(t) > 0$ by the condition (d') and $g'(t) > 0$, $g''(t) < 0$, $d''(t) < 0$ for any $t \in (x_1, \ x_2)$, so we have $A_0 > 0$, $A_1 > 0$, $B_0 > 0$, $U_0 < 0$. On the other hand,

$$(\alpha_2 - \alpha_1) \cdot (\xi_0 - \lambda_1) < (\lambda_2 - \lambda_1) \cdot (\xi_0 - \lambda_1) <$$

$$< \Lambda_0 \cdot \frac{1}{4} < \frac{1}{8} + \varepsilon_0$$

and hence by (4.25),

$$K_{11} \geq K'_{11} + K''_{11},$$

where

$$K'_{11} = \frac{3}{8} \cdot (U_0 \cdot A_0 + A_1 \cdot B_0) = O(1/t^4) \quad (\textit{see below}),$$

$$K''_{11} = 3 \cdot \varepsilon_0 \cdot (U_0 \cdot A_0 - A_1 \cdot B_0) = O(1/t^4),$$

We also have $\mathfrak{A}_0 \geq \Omega_0 - \mathfrak{A}_1$, where

$$\Omega_0 = \frac{1}{8} \cdot A_1 \cdot B_0,$$

$$\mathfrak{A}_1 = A_1 \cdot \mathfrak{M}_1 = O(1/t^4).$$

From this we have a new inequality

$$\Omega_0 \leq \Omega_1 := (L_{11} + L_{21}) + (R_{21} - R_1) - (K'_{12} + K'_{11} + K''_{11}) + \mathfrak{A}_1. \tag{4.32}$$

Now we intend to obtain the estimates for the lower bound of Ω_0 and the upper bound of Ω_1 respectively.

4.15. Lower bound of $\Omega_0 := (A_1 \cdot B_0)/8$

Using the condition (d'), we get

$$B_0 = d'(\alpha_2) \cdot g'(\alpha_2) \geq \frac{1}{x_2 \cdot \sqrt{x_2}}$$

and

$$(-g''(\mu_1)) \cdot d'(\xi_0) \geq \frac{0.4789}{x_2^2 \cdot \sqrt{x_2}} \quad (x_2 \geq e^{14})$$

and

$$(-d''(\mu_2)) \cdot g'(\xi_0) = \frac{g'(\xi_0)}{\mu_2 \cdot g(\mu_2)} \cdot \left(1 + \frac{1}{\log \mu_2} - \partial_0(\mu_2)\right) \geq$$

$$\geq \frac{g'(x_2)}{x_2 \cdot g(x_2)} \cdot (1 - 0.2577) \geq \frac{0.3711}{x_2^2} \quad (x_2 \geq e^{14}).$$

Thus we have

$$\Omega_0 \ge \frac{1}{8}\cdot\left(\frac{0.4789}{x_2^4} + \frac{0.3711}{x_2^3\cdot\sqrt{x_2}}\right) \ge \frac{0.0462}{x_1^3\cdot\sqrt{x_1}} \quad (x_1 \ge e^{14}).$$

Put

$$\Omega_0' := \frac{0.0462}{x_1^3\cdot\sqrt{x_1}} \quad (x_1 \ge e^{14}). \tag{4.33}$$

4.16. Some estimates

For any $t \in (x_1,\, x_2)$ it is easy to see that

$$g'(t) = \frac{\log^2(t\cdot\alpha)}{2\cdot\sqrt{t}}\cdot\left(1+\frac{4}{\log(t\cdot\alpha)}\right),$$

$$g''(t) = -\frac{\log^2(t\cdot\alpha)}{4\cdot t\cdot\sqrt{t}}\cdot\left(1-\frac{8}{\log^2(t\cdot\alpha)}\right),$$

$$g'''(t) = \frac{\log^2(t\cdot\alpha)}{t^2\cdot\sqrt{t}}\cdot\left(\frac{3}{8}-\frac{1}{2\cdot\log(t\cdot\alpha)}-\frac{3}{\log^2(t\cdot\alpha)}\right)$$

and

$$d'(t)\cdot g(t) = f'(t) - d(t)\cdot g'(t),$$

$$d''(t)\cdot g(t) = f''(t) - d(t)\cdot g''(t) - 2\cdot d'(t)\cdot g'(t),$$

$$d'''(t)\cdot g(t) = f'''(t) - d(t)\cdot g'''(t) -$$

$$-3\cdot\left(d''(t)\cdot g'(t) + d'(t)\cdot g''(t)\right).$$

And it is also clear that

$$f'(t) = (1+\log t)\cdot E(t),$$

$$f''(t) = \frac{1}{t}\cdot\left(E(t)-1-\frac{1}{\log t}\right),$$

$$f'''(t) = \frac{1}{t^2}\cdot\left(1+\frac{1}{\log^2 t}-E(t)\right).$$

By (3.7) and (3.11), we respectively have

$$D_0(t) := |\log t\cdot E(t) - \theta(t)| \le \frac{2}{\log t} \quad (t \ge e^{14}),$$

$$G_1(t) := \left|\frac{g'(t)}{g(t)}\right| = \frac{1}{2\cdot t}\cdot\left|1+\frac{4}{\log(t\cdot\alpha)}\right| \le \frac{0.6437}{x_1} \quad (x_1 \ge e^{14}),$$

$$D_1(t) := |d'(t)\cdot g(t)| \le |(1+\log t)\cdot E(t)| + t\cdot D_0(t)\cdot G_1(t) \le$$

$$\le 0.1685 \quad (t \ge e^{14}),$$

$$G_2(t) := \left|\frac{g''(t)}{g(t)}\right| = \frac{1}{4\cdot t^2}\cdot\left|1-\frac{8}{\log^2(t\cdot\alpha)}\right| \le \frac{0.26}{x_1^2} \quad (x_1 \ge e^{14}),$$

$$D_2(t) := |d''(t)\cdot g(t)| \le |f''(t)| + t\cdot D_0(t)\cdot G_2(t) + 2\cdot D_1(t)\cdot G_1(t) \le$$

$$\le \frac{1.3307}{x_1} \quad (x_1 \ge e^{14}),$$

$$G_3(t) := \left|\frac{g'''(t)}{g(t)}\right| = \frac{1}{t^3}\cdot\left|\frac{3}{8} - \frac{1}{2\cdot\log(t\cdot\alpha)} - \frac{3}{\log^2(t\cdot\alpha)}\right| \le$$
$$\le \frac{0.3751}{x_1^3} \quad (x_1 \ge e^{14})$$
$$D_3(t) := |d'''(t)\cdot g(t)| \le |f'''(t)| + t\cdot D_0(t)\cdot G_3(t) +$$
$$+3\cdot(D_2(t)\cdot G_1(t) + D_1(t)\cdot G_2(t)) \le$$
$$\le \frac{3.7650}{x_1^2} \quad (x_1 \ge e^{14}).$$

From this for any $t,\ t_1 \in (x_1,\ x_2)$ we have

$$r_0 := \left|\frac{g(t_1)}{g(t)}\right| \le 1 + \frac{g'(t_0)}{g(t)} \le 1.000001 \quad (t < t_0 < t_1),$$

$$a_1 := |d'(t)\cdot g'(t_1)| = r_0\cdot D_1(t)\cdot G_1(t_1) \le \frac{0.1085}{x_1} \quad (x_1 \ge e^{14}),$$

$$a_2 := |d'(t)\cdot g''(t_1)| = r_0\cdot D_1(t)\cdot G_2(t_1) \le \frac{0.0439}{x_1^2} \quad (x_1 \ge e^{14}),$$

$$a_3 := |d'(t)\cdot g'''(t_1)| = r_0\cdot D_1(t)\cdot G_3(t_1) \le \frac{0.0633}{x_1^3} \quad (x_1 \ge e^{14}),$$

$$b_2 := |d''(t)\cdot g'(t_1)| = r_0\cdot D_2(t)\cdot G_1(t_1) \le \frac{0.8566}{x_1^2} \quad (x_1 \ge e^{14}),$$

$$b_3 := |d'''(t)\cdot g'(t_1)| = r_0\cdot D_3(t)\cdot G_1(t_1) \le \frac{2.4236}{x_1^3} \quad (x_1 \ge e^{14}),$$

$$b_1 := |d''(t)\cdot g''(t_1)| = r_0\cdot D_2(t)\cdot G_2(t_1) \le \frac{0.3460}{x_1^3} \quad (x_1 \ge e^{14}).$$

By (4.24), we get

$$|\mathfrak{M}_\mathfrak{o}| \le (|A_1| + |A_2|)\cdot|Q_0| \le (a_2 + 2\cdot b_2)\cdot|Q_0| \le$$
$$\le \frac{0.2223}{x_1^2} \quad (x_1 \ge e^{14})$$
$$|\mathfrak{M}_\mathfrak{1}| \le |\mathfrak{M}_\mathfrak{o}|\cdot\left(1 + \frac{0.9005}{\sqrt{x_1}}\right) \le \frac{0.2225}{x_1^2} \quad (x_1 \ge e^{14}).$$

4.17. Upper bound of Ω_1

The upper bound of Ω_1 is obtained as follows. First we will show the estimate of

$$K'_{11} = \frac{3}{8}\cdot(U_0\cdot A_0 + A_1\cdot B_0).$$

Since

$$U_0\cdot A_0 + A_1\cdot B_0 = (U_0 + A_1)\cdot A_0 - (A_0 - B_0)\cdot A_1,$$

we have

$$U_0 + A_1 = \Big(d''(\tau_0)\cdot g'(\tau_0) - d''(\mu_2)\cdot g'(\xi_0)\Big) +$$
$$+\Big(d'(\tau_0)\cdot g''(\tau_0) - g''(\mu_1)\cdot d'(\xi_0)\Big) =$$

$$=\Big(d''(\tau_0)\cdot(g'(\tau_0)-g'(\xi_0))+(d''(\tau_0)-d''(\mu_2))\cdot g'(\xi_0)\Big)+$$
$$+\Big((d'(\tau_0)-d'(\xi_0))\cdot g''(\tau_0)+(g''(\tau_0)-g''(\mu_1))\cdot d'(\xi_0)\Big)\le$$
$$\le(b_1+b_3)+(b_1+a_3)$$

and

$$A_0-B_0=d'(\eta_2)\cdot g'(\xi_0)-d'(\alpha_2)\cdot g'(\alpha_2)=$$
$$=\Big((d'(\eta_2)-d'(\alpha_2))\cdot g'(\xi_0)\Big)+$$
$$+\Big((g'(\xi_0)-g'(\alpha_2))\cdot d'(\alpha_2)\Big)<$$
$$<a_2+b_2$$

and more

$$A_0=d'(\eta_2)\cdot g'(\xi_0)<a_1,$$
$$A_1=(-g''(\mu_1))\cdot d'(\xi_0)+(-d''(\mu_2))\cdot g'(\xi_0)<$$
$$<a_2+b_2.$$

Thus we have

$$|K'_{11}|\le\frac{3}{8}\cdot\Big((a_2+b_2)^2+a_1\cdot(a_3+b_3)+2\cdot a_1\cdot b_1\Big)\le$$
$$\le\frac{0.4334}{x_1^4}\quad(x_1\ge e^{14}).$$

Next, we would show the estimate of

$$L_{21}=2\cdot A_0\cdot B_1\cdot\delta_0-4\cdot A_0\cdot B_0\cdot\delta_0+$$
$$+B_1\cdot(2-(\lambda_2-\lambda_1))\cdot\mathfrak{M}_{\mathfrak{o}}=O(1/t^4).$$

Since

$$A_0\cdot\varepsilon_0=\mathfrak{M}_{\mathfrak{o}},\quad B_0\cdot\varepsilon_0=\mathfrak{M}_{\mathfrak{1}}$$

and

$$B_1=(d_2-d(\alpha_2))\cdot g''(\alpha_2)-(g(\alpha_2)-g_1)\cdot d''(\alpha_2)<$$
$$<a_2+b_2$$

and

$$\frac{1}{2}<\Lambda_0=(\lambda_2-\lambda_1)<\frac{1}{2}+2\cdot\varepsilon_0<0.5034,$$

we have

$$|L_{21}|\le\frac{4\cdot\varepsilon_0}{1-\varepsilon_0}\cdot(a_2+b_2)\cdot|\mathfrak{M}_{\mathfrak{o}}|+\frac{8}{1-\varepsilon_0}\cdot|\mathfrak{M}_{\mathfrak{o}}|\cdot|\mathfrak{M}_{\mathfrak{1}}|+$$
$$+\frac{3}{2}\cdot(a_2+b_2)\cdot|\mathfrak{M}_{\mathfrak{o}}|\le\frac{0.6978}{x_1^4}\quad(x_1\ge e^{14}).$$

And since

$$\mathcal{V}_0:=(\alpha_2-\alpha_1)\cdot(\xi_0-\lambda_1)<\frac{1}{8}+\varepsilon_0<0.1267$$

and

$$U_1 - 3 \cdot U_0 \le (a_3 + b_3) + 3 \cdot (a_2 + b_2),$$

we also have

$$|R_1| \le \Big((a_3 + b_3) + 3 \cdot (a_2 + b_2) \Big) \cdot |\mathfrak{M}_\mathfrak{o}| \cdot \mathcal{V}_0 \le$$
$$\le \frac{0.1462}{x_1^4} \quad (x_1 \ge e^{14}).$$

Similarly, we have respectively

$$|K''_{11}| \le 3 \cdot (a_2 + b_2) \cdot (|\mathfrak{M}_\mathfrak{o}| + |\mathfrak{M}_\mathfrak{1}|) \le \frac{1.2017}{x_1^4} \quad (x_1 \ge e^{14}),$$

$$|L_{11}| \le (a_3 + b_3) \cdot a_1 \cdot \mathcal{V}_0 \le \frac{0.0342}{x_1^4} \quad (x_1 \ge e^{14}),$$

$$|R_{21}| \le (a_2 + b_2) \cdot |\mathfrak{M}_\mathfrak{o}| \cdot (\lambda_2 - \lambda_1) \le \frac{0.1008}{x_1^4} \quad (x_1 \ge e^{14}),$$

$$|\mathfrak{A}_\mathfrak{1}| \le (a_2 + b_2) \cdot |\mathfrak{M}_\mathfrak{1}| \le \frac{0.2004}{x_1^4} \quad (x_1 \ge e^{14}),$$

$$|K'_{12}| \le 4 \cdot b_2 \cdot |\mathfrak{M}_\mathfrak{1}| \le \frac{0.7625}{x_1^4} \quad (x_1 \ge e^{14}).$$

Consequently we obtain

$$\Omega_1 \le |K'_{11}| + |L_{21}| + |R_1| + |K''_{11}| + |L_{11}| + |R_{21}| + |\mathfrak{A}_\mathfrak{1}| + |K'_{12}| \le$$
$$\le \frac{0.4334}{x_1^4} + \frac{0.6978}{x_1^4} + \frac{0.1462}{x_1^4} + \frac{1.2017}{x_1^4} +$$
$$+ \frac{0.0342}{x_1^4} + \frac{0.1008}{x_1^4} + \frac{0.2004}{x_1^4} + \frac{0.7625}{x_1^4} \le$$
$$\le \frac{3.5770}{x_1^4} \quad (x_1 \ge e^{14}). \tag{4.34}$$

4.18. Proof of Lemma 2

If $3 \le p_m \le e^{14}$, then we could confirm that the condition (d) holds from the table 3 and the table 4. Hence if the Lemma 2 does not hold, then there exists a prime number $p_m \ge e^{14}$ such that the condition (d') holds. For such prime p_m, we have $\Omega'_0 \le \Omega_1$ and, finally, we get

$$1 \le \frac{3.5770}{0.0462 \cdot \sqrt{x_1}} \le 0.08 \quad (x_1 \ge e^{14}), \tag{4.35}$$

but it is a contradiction. This shows that the condition (d') is not valid. Consequently, the Lemma 2 holds for any prime number $p_m \ge 3$.

4.19. Algorithm and Tables for Sequence $\{H_m\}$

Here

$$H_m := (1 + \log p_m) \cdot E(p_m) -$$
$$- d(p_m) \cdot \frac{\log^2(p_m \cdot \alpha)}{2 \cdot \sqrt{p_m}} \cdot \left(1 + \frac{4}{\log(p_m \cdot \alpha)}\right) - \frac{2}{\sqrt{p_m}} \tag{4.36}$$

and $\alpha = 1 + \theta(p_m)$. The table 3 and 4 show the values of H_m for $2 \le p_m \le 29$ and $93109 \le p_m \le 93118$. Note that the condition (d) holds if and only if $H_m < 0$ for any $m \ge 2$. It is easy to see that $H_m < 0$ for any $29 \le p_m \le 93109$.

The algorithm for H_m by MATLAB is as follows:
Function EMF-Index, clc, b=0.261497212847643; format long,
$P = [2, 3, 5, 7, \cdots, 1202609]$; M=length(P);
for $m = 1 : M$; $p = P(1 : m)$; $E = sum(1./p) - b - \log(\log(p(m)))$; $E1 = (1 + \log(p(m))) * E$;
$V1 = sum(\log(p.))$; $Q = (V1/p(m)) - 1$; $R = (p(m))^{1/2}$; $V = \log(V1)$; $g = R * V^2$;
$f = p(m) * (\log(p(m)) * E - Q)$; $d = f/g$; $B = (V^2) * (1 + 4/V)/2/R$;
m, $p(m)$, $H_m = E1 - d * B - 2/R$, end.

Table 3

m	$p(m)$	H_m
1	2	4.92781518770647
2	3	−3.79708871931795
3	5	−1.82084025624172
4	7	−1.24240415973621
5	11	−1.05892911097784
6	13	−0.82377421885520
7	17	−0.75298886049588
8	19	−0.60562813217931
9	23	−0.56797602737022
10	29	−0.59342397038654

Table 4

m	$p(m)$	H_m
93109	1202477	−0.00169503567169
93110	1202483	−0.00168790073361
93111	1202497	−0.00168788503420
93112	1202501	−0.00167897043350
93113	1202507	−0.00167183566691
93114	1202549	−0.00169673633799
93115	1202561	−0.00169494084824
93116	1202569	−0.00168958602998
93117	1202603	−0.00170736660136
93118	1202609	−0.00170023228562

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2010 Mathematics Subject Classification; 11M26, 11N05.