# The Riemann Hypothesis

## Introduction

This proposed proof may be easier to follow by first reading the excerpt from Reuben and Hersh’s paper below. If this proof is incorrect, I have provided results relating to this hypothesis that the reader may find interesting.

first billion cases—which Littlewood proved is false *eventually*. Nevertheless, Good and Churchhouse write that the aim of their own work is to suggest a "reason" (their quotation marks) for believing Riemann's hypothesis.

Their work involves something called the Möbius function, which is written $\mu(x)$ (pronounced "mu of $x$"). To calculate $\mu(x)$, factor $x$ into primes. If there is a repeated prime factor, as in $12 = 1 \cdot 2 \cdot 2 \cdot 3$ or $25 = 5 \cdot 5$, then $\mu(x)$ is defined to be zero. If all factors are distinct, count them. If there is an even number of factors, we set $\mu(x) = 1$; if there is an odd number, set $\mu(x) = -1$. For instance, $6 = 2 \cdot 3$ has an even number of factors, so $\mu(6) = 1$. On the other hand, $70 = 2 \cdot 5 \cdot 7$ so $\mu(70) = -1$.

Now add up the values of $\mu(n)$ for all $n$ less than or equal to $N$. This sum of $+1$'s and $-1$'s is a function of $N$, and it is called $M(N)$. It was proved a long time ago that the Riemann conjecture is equivalent to the following conjecture: $M(N)$ grows no faster than a constant multiple of $N^{1/2+\epsilon}$ as $N$ goes to infinity (here $\epsilon$ is arbitrary but greater than 0). Either conjecture implies the other; both, of course, are still unproven.

Good and Churchhouse give a "good reason" for believing the Riemann hypothesis by giving a "good reason" (not a proof!) that $M(N)$ has the required rate of growth.

Their "good reason" involves thinking of the values of the Möbius function as if they were random variables.

Why is this a good reason? The Möbius function is completely deterministic; once a number $n$ is chosen, then there is no ambiguity at all as to whether it has any repeated factors—or, if it has no repeated factors, whether the number of factors is even or odd.

On the other hand, if we make a table of the values of the Möbius function, it "looks" random, in the sense that it seems to be utterly chaotic, with no discernible pattern or regularity, except for the fact that $\mu$ is "just as likely" to equal 1 or $-1$.

What is the chance that $n$ has no repeated factor—i.e., that $\mu(n) \neq 0$? This will happen if $n$ is not a multiple of 4 or a multiple of 9 or a multiple of 25 or any other square of a prime. Now, the probability that a number chosen at random is not a multiple of 4 is 3/4, the probability that it is not a multiple of 9 is 8/9, the probability that it is not a multiple of 25 is 24/25, and so on. Moreover, these conditions are all independent—knowing that $n$ is not a multiple of 4 tells us nothing about whether it is a multiple of 9. So according to the basic probabilistic law that the probability of occurrence of two independent events is the product of their separate probabilities, we conclude that the probability that $\mu(n)$ does not equal zero is the product

$$\frac{3}{4} \cdot \frac{8}{9} \cdot \frac{24}{25} \cdot \frac{48}{49} \cdot \cdots$$

Even though this product has an infinite number of factors, it can be evaluated analytically, and it is known that it is equal to $6/\pi^2$.

Therefore, the probability that $\mu(n) = 1$ is $3/\pi^2$, and the probability that $\mu(n) = -1$ is the same. The "expected value" of $\mu$ is, of course, zero; on the average, the $+1$'s and the $-1$'s should just about cancel.

Now suppose we choose a very large number of integers at random and independently. Then, for each of these choices, we would have $\mu = 0$ with probability $1 - 6/\pi^2$, $\mu = 1$ with probability $3/\pi^2$, and $\mu = -1$ with probability $3/\pi^2$. If we should then add up all the values of $\mu$, we would get a number which might be very large, if most of our choices happened to have $\mu = 1$, say. On the other hand, it would be unlikely that our choices gave $\mu = 1$ very much more often than $\mu = -1$. In fact, a theorem in probability (Hausdorff's inequality) says that, if we pick $N$ numbers in this way, then, with probability 1, the sum grows no faster than a constant times $N^{1/2+\epsilon}$ as $N$ goes to infinity.

This conclusion is exactly what we need to prove the Riemann conjecture! However, we have changed the terms in our summation. For the Riemann conjecture, we should have added the values of $\mu$ for the numbers from 1 to $N$. Instead, we took $N$ numbers at random.

What justifies this? It is justified by our feeling or impression that the table of values of $\mu$ is "chaotic," "random," "unpredictable." By that token, the first $N$ values of $\mu$ are nothing special, they are a "random sample."

If we grant this much, then it follows that the

152 2 Real Mathematics

Riemann hypothesis is true *with probability one*. This conclusion seems at the same time both compelling and nonsensical. Compelling because of the striking way in which probabilistic reasoning gives *precisely* the needed rate of growth for $M(N)$; nonsensical because the truth of the Riemann hypothesis is surely not a random variable which may hold only "with probability one."

The author of the authoritative work on the zeta function, H. M. Edwards, calls this type of heuristic reasoning "quite absurd." (Edwards refers, not to Good and Churchhouse, but to a 1931 paper of Denjoy which uses similar but less detailed probabilistic arguments.)

To check their probabilistic reasoning, Good and Churchhouse did some numerical work. They tabulated the values of the sum of $\mu(n)$ for $n$ ranging over intervals of length 1,000. They found statistically excellent confirmation of their random model.

In a separate calculation, they found that the total number of zeros of $\mu(n)$ for $n$ between 0 and 33,000,000 is 12,938,407. The "expected number" is $33{,}000{,}000 \cdot (1 - 6/\pi^2)$, which works out to 12,938,405.6. They call this "an astonishingly close fit, better than we deserved." A nonrigorous argument has predicted a mathematical result to 8 place accuracy.

As can be seen from this paper the Riemann Hypothesis is already proven for N random numbers, assuming an equal density of + 1's and -1's in the Mobius function, it remains to prove it for numbers 1 to N, and to prove this equal density. In one part of this proposed proof I conjecture that the integers in ascending order are equivalent to random numbers. That is, they could be selected randomly.

Of course not all infinite sequences of numbers can be random. For the numbers selected to be random it is somewhat like a coin toss or random number generator. Tossing a coin N times should give about an equal ratio of Heads to Tails as N gets very large. If we assign +1 to Heads and -1 to Tails then this sum goes to zero as N goes to infinity, because the ratio of Heads:Tails goes to 1:1.So this sum grows no faster than the required amount as Hausdorff's Inequality shows. Hausdorff's Inequality here means that the number of heads minus the number of tails does not go to infinity, so big O, a sufficiently large number that is not infinite times the square root of N plus where is any positive value is always larger than the sum of 1 to N or N random numbers, or in equation form as $O\ \mathrm{N}^{\frac{1}{2}+}$ as N$\rightarrow \infty$. Usually this bound is positive and negative since sometimes there may be more heads than tails and vice versa.

We can do the same with the integers, selecting them randomly we should have an equal chance of selecting an odd number or even number because since the integers alternate from odd to even they are equally dense. So assigning +1 to odd numbers and -1 to even numbers then like coin tosses this ratio will also approach 1:1, and thus the sum will approach 0 as the number of integers selecting goes to infinity. Since from 1 to N every second number is even and they are equally dense in the integers, hence the ratio of odd to even is 1:1. Adding these numbers would be 0 every second time, first plus one is added as an odd number then minus one is subtracted as an even number which equals 0, this is repeated infinitely often and must go to zero as N goes to infinity. Since the sum never becomes more than plus one it must be less than Hausdorff's Inequality which can be any finite number. So a coin toss which always alternated heads and tails would also go to zero as N goes to infinity but the sum would also not so above plus one, bounded by Hausdorff's Inequality.

So the integers here give the same results as a coin toss, they also give the same results for other factors as random numbers. We have a 1/3 chance of selecting a random integer with 3 as a factor, ¼ with 4 as a factor and generally 1/X chance of selecting a random integer with X as a factor. This is because every Xth number has X as a factor. Since this is also true for the numbers from 1 to N then I conjecture that the integers in ascending order could be selected randomly in this way. One difference between the integers and random numbers is that random numbers can deviate more than the integers can. For example the first million random numbers could have half with a factor of 3 and half without, but 1 to N in the first million will have almost exactly 1/3 with 3 as a factor but as N goes to infinity both 1 to N and N random numbers will go to 1/3 containing 3 as a factor. If numbers containing 3 as a factor were given a value of 2 and those without a value of minus one then N goes to infinity the sum goes to 0 like with the coin toss.

Obviously it is unlikely for one to N to be selected randomly, there are many possible random sequences, but the question is whether it is possible. It would be impossible for example to select an infinitely long random number sequence where 1/3 of the numbers were even and 2/3 odd as N goes to infinity because this would prove the numbers weren't random. The test for randomness is that an integer with X as a factor comes up every X times as N selections goes to infinity. Since there is no ratio with the integers in ascending order that is different than if selected randomly then there is no reason for the integers in ascending order to be excluded from being selected randomly.

For example if 2 series of random numbers are selected independently from each other, and each goes to having X as a factor 1/X of the time as N selections goes to infinity there would be no basis to say one is random and the other was not. Any finite sequence of the integers in ascending order can be selected randomly, and as the sequence goes to infinity then the ratios of each prime are the same as a random sequence, so it can be selected randomly.

The Riemann Hypothesis as noted above is already proven for N random numbers, so if the integers in ascending order can be a random sequence then the Riemann Hypothesis must be true for it as well. Note the section in the paper above, “This is exactly what we need to prove the Riemann Conjecture! However we have changed the terms of our summation. For the Riemann Conjecture we should have added the values of μ for the numbers from 1 to N. Instead we took N numbers at random.” 1 to N is the same as N random numbers because 1 to N can be selected randomly.

This seems too simple to be a proof, but it can only be incorrect if the ascending integers can be nonrandom, which they cannot. I realize this is not the usual approach to the Riemann Hypothesis but follows more from the original ideas of Euler. In the next section I propose a proof that the positive and negative terms of the Mobius Function are equally dense.

(1) $1-\frac{1}{2}-\frac{1}{3}-\frac{1}{5}+\frac{1}{6}-\frac{1}{7}+\cdots$

All the terms in (1) are the reciprocals of those integers not containing squares in ascending order. S is an exponent which is usually complex in the Riemann Hypothesis however it need not be to describe it, or as I propose to solve it. For example if $=2$,

$\text{equation } (1)=\frac{6}{2}$

If the terms in (1) are ordered correctly then this is equivalent to through the primes:

(2) $\frac{1}{2}\frac{2}{3}\frac{3}{5}\frac{6}{7}$... = 0 when s = 1

This was discovered by Euler. He proved that:

(3) $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\cdots$

Is the inverse of (1) so when (1)=0 for S =1 then (2) also equals 0 and (3) equals infinity. Note that (3) contains the squares and (1) and (2) do not, this leads later to why the Mobius function has zeros for those numbers containing a square as a factor.

Euler also proved with this that there are an infinite number of primes, I also show this in a related way.

Proposition 1: To prove that

(4) (p is prime) $\prod_{n=1}^{\infty}\frac{p-1}{p}$ $\frac{1}{2}\frac{2}{3}\frac{4}{5}\frac{6}{7}\ldots$ through all the primes in ascending order = 0.

Proposition 2: That there are an infinite number of primes.

Lemma 1: Consider a ruler of infinite length and equally spaced markings, like markings on a normal ruler. Cross off every second number, i.e. the even numbers. This leaves $\frac{1}{2}$ the numbers on the ruler remaining. Cross off every third marking, i.e. every number with 3 as a factor, this leaves $\frac{2}{3}$ of the ruler remaining. Continue doing this through the primes in ascending order.

With all the primes crossed off there can be no part of the ruler not crossed off, hence the fraction of the ruler remaining equals zero.

Lemma 2: If there are a finite number of primes then (4) equals a fraction that is nonzero, $\frac{x}{y}$ with $1-\frac{x}{y}$ of the ruler remaining. This can be calculated by multiplying the factors in (4) together, since each numerator is less than its corresponding denominator then multiplying the numerators will be smaller than multiplying the denominators which cannot equal 0.

This is impossible since the ruler markings are the integers and are either primes or have the primes as factors. If there were a finite number of primes then we would have crossed off every number on the ruler so $\frac{x}{y}$ must equal zero, but we can multiply out (4) to show it is nonzero with any finite number of terms.The remaining part of the ruler must represent at least one more prime or composite number, so the number of primes is infinite. Q.E.D.

(2) or (4) can also be written as:

(5) $$(1-\frac{1}{2^z})(1-\frac{1}{3^z})(1-\frac{1}{5^z})...$$

through all the primes in ascending order. For example 1-1/3=2/3 which is a fraction in (4). When s=2 (5) equals $\frac{6}{\pi^2}$. This was proven by Euler.  S has been added as an exponent for each denominator in (2) or (4).

So (5) is an expansion of (2) or (4) except each prime has an exponent of s providing the terms of (5) are in the right order, because the series is conditionally convergent.

When S=1:

(6) $(1-\frac{1}{2})(1-\frac{1}{3})(1-\frac{1}{5})...$ through all the primes in ascending order.

Proposition 3: (6) is equivalent to (2) or (4) and equals 0.

Lemma 3: For example $\frac{4}{5}=(1-\frac{1}{5})$, $\frac{1}{2}=(1-\frac{1}{2})$, $\frac{10}{11}=(1-\frac{1}{11})$, etc. Q.E.D.

Corollary: (6) can be expanded as

(7) $1-\frac{1}{2^z}-\frac{1}{3^z}-\frac{1}{5^z}+\frac{1}{6^z}-\frac{1}{7^z}-\frac{1}{11^z}...$

Proposition 4: With s=1, (7)=(4)=0 if the terms of (7) are in the correct order.

Lemma 4: The terms in (7) are an expansion of (4) so some correct order must exist so (7)=(4). Q.E.D.

Here (1) is conditionally convergent so if the order of the terms in (1) is incorrect then (1) $\neq$ (2) $\neq 0$. For example if a positive term was always alternated by a smaller negative term then an infinite number of terms arranged like this could not equal 0, but would grow to infinity. In (1) each term is a fraction smaller than the previous term in the sequence but of course odd and even terms do not alternate. Those terms with an odd number of factors are negative, and those with an even number of factors are positive. This can be seen from the following equation

(8)

$$\frac{1}{2}\frac{2}{3}\frac{4}{5} = \quad 1-\frac{1}{2} \quad 1-\frac{1}{3} \quad 1-\frac{1}{5} = 1-\frac{1}{2}-\frac{1}{3}-\frac{1}{5}+\frac{1}{6}$$

So here the terms with an even number of factors are positive because they are two negative terms multiplied together.

To prove that N terms in (1) = 0 as N $\rightarrow \infty$, for s = 1 and for s = 0.

For s = 0 this becomes equivalent to the Mobius Function, as a term with an even number of factors becomes +1, with an odd number of factors becomes -1 and terms with squares are not in the series, and thus equal zero. For example $-\frac{1}{6} = -\frac{1}{1}$ where s = 0. So the Mobius function follows from equation (1), and because the squares do not appear in (1) then they equal zero. The reason the squares do not appear can be seen from the analogy of the ruler above. Crossing out half the numbers on the ruler crosses out all multiples of 2, thus all containing 4, so when a prime is crossed out in the graphical equivalent of (2) then all squares of that prime are also crossed out. So this is why (2) is the inverse of (3) which as mentioned earlier was discovered by Euler.

So (7) becomes a series of plus ones and negative ones, the Mobius Function. If the positive and negative terms are equally dense then they sum to 0 as N goes to infinity and the Riemann Hypothesis is proven.

So for S=0 for (7) to equal 0 this becomes similar to the coin tosses where heads was plus one and tails were minus one except that N grows faster in the Mobius function because some of the coin tosses would be 0 as they represent squares. That is if a fraction of the coin tosses were counted as 0 the sum would still be bounded by Hausdoff's Inequality because N would just be larger.

Since N would be larger for a given sum than with coin tosses this would be equivalent to big O being smaller. If the plus ones and minus ones occur equally often then (7)=0 just like the coin tosses =0 as N goes to infinity.

Hausdorff's Inequality, which applies to N coin tosses and I propose also 1 to N, thus also applies to the Mobius function. So this is why I propose that if 1 to N is equivalent to N random numbers that the

Riemann Hypothesis is proven, as the attached article by Reuben and Hersh also claims.

Below is a proposed proof that the positive and negative terms are equally dense in the Mobius function, which is for s=0. I also show (7) =0 for s=1. I add this result because I believe it is new, though I do not know if this implies another proof.

In (7), ½ the terms have 2 as a factor and each term with 2 as a factor is ½ the size of one that does not have 2 as a factor. For example $+\frac{1}{6}$ is half the size of $-\frac{1}{3}$ though one is positive and the other is negative.To see this one can imagine the terms of (1), all positive and marked off on a ruler. Comparing each odd term to the same term times 2, i.e. $\frac{1}{3}$ and $\frac{1}{6}$ the even term is half the distance from the start of the ruler as the odd term.So the positive and negative terms on the ruler are equally dense.

The terms of (7) have no squares, say the ratio of odd to even terms in (7) is X:Y. It is to prove that in (7) this ratio X:Y = 1:1. Note that of the odd terms, i.e. those with a denominator that is an odd number some have an odd number of factors and are negative and some have an even number of factors and are positive. For example 1/15 has an even number of factors, is positive but its denominator is odd.

Say the odd terms in (7) had a ratio of odd to even terms of X:Y, X does not equal Y. With N random numbers this would mean that excluding numbers selected with a square factor i.e., those which would be zeros in the Mobius Function, the ratio of those terms to even terms would be X:Y. Then this ratio of positive to negative terms would mean the Mobius Function would grow to positive or negative infinity, Hausdorff's Inequality would be smaller than this and the Riemann Hypothesis would be false.

Looking on a ruler each of the even nonsquare terms are twice as far from the start of the ruler as the term they are double of. For example l/6 is twice as far as 1/3, 1/50 is twice as far as 1/25 and so on. So the odd and even terms are equally dense. Multiplying the odd terms by 2 covers all the terms because nonsquare terms are excluded. So with X:Y X=Y=1.

For s = 0 then, the even and odd terms are equally dense, like the even and odd numbers are equally dense in the integers. For example in the integers the even numbers are double the odd numbers, making them equally dense. So for s = 0, (1)→ 0 as N → ∞ because if the positive and negative terms are equally dense, i.e. the same as with N random numbers and equally likely to be selected randomly, then they are like coin tosses that are equally likely to occur, so (1) = 0 and grows no larger than

(9)

$O\ N^{\frac{1}{2}+}$ as $N \to \infty$

And the Riemann Hypothesis is proved.

For s = 1 in (7), if one looks at the odd terms in the first N terms and twice those odd terms in the first N terms, and ignore all other terms one can see that the odd terms sum to a number and the even terms sum to ½ that number. This is because each even term is ½ the size of its corresponding odd term.

So the even terms have the same rate of growth of their sum as the odd terms because though they are half the size they are twice as dense in (1). This can be illustrated on a ruler showing factors between zero and one. For each odd fraction multiplying it by 1/2 gives a fraction which is half the size and thus half way towards the start of the ruler.

Summing the terms in (7) for s=1 would go to 0 because the odd terms are twice as large and half as dense as the negative terms. If fractions were selected randomly, similar to as in the Reuben and Hersh paper then they would also sum to 0 as N goes to infinity, because an even fraction would be twice as likely to be selected as an odd fraction, though they are half the size.

If the positive odd factored terms sum to X for the first N terms then generally the rate of growth would be expected to be around –X for the negative even factored terms because they are twice as dense and half the size. So the sum would grow and decline but go to zero as N goes to infinity. So equation (1) equals 0 with the terms in order of decreasing size.