

# Affirmative resolve of Riemann Hypothesis

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## Abstract

First, we prove the relation of the sum of the mobius function and Riemann Hypothesis. This relationship is well known. I prove next section, without any tool we prove Riemann Hypothesis about mobius function. This is a very challenging experiment.

## 1

We write R.H. as the abbreviation of Riemann Hypothesis.  $\mu(n)$  is the mobius function,

$[x]$  is the Gauss symbol.

**Theorem 1.1.**

$$\sum_{n=1}^m \mu(n) = O(\sqrt{m} \log(m)) \Leftrightarrow R.H$$

**proof.** [1] We define  $M(x)$  that is called the Mertens function.

$$M(x) := \sum_{n=1}^{[x]} \mu(n)$$

$$\frac{1}{\zeta(s)} = \sum \frac{\mu(n)}{n^s}$$

$$\frac{1}{\zeta(s)} = \int_{x=1}^{\infty} \frac{1}{x^s} d(M(x))$$

$d(M(x))$  is Stieltjes integral.

$$= [M(x)x^{-s}] + s \int_{x=1}^{\infty} M(x)x^{-s-1} dx$$

$M(x) < O(\sqrt{x} \log(x)) \Rightarrow$  This integral may not converge on  $Re(s) = \frac{1}{2}$  and must converge on  $Re(s) \neq \frac{1}{2}$

□

## 2

We prove Riemann Hypothesis in this chapter.

**Lemma 2.1.**

$$\sum_{n|m} \mu(n) = 1(m = 1)$$

$$\sum_{n|m} \mu(n) = 0(m \neq 1)$$

**proof.** The case  $m = 1$ ,  $\sum_{n|m} \mu(n) = \mu(1) = 1$  is clear.  $m \neq 1$ , We factorize  $m = p_1^{n_1} p_2^{n_2} p_3^{n_3} \dots p_k^{n_k}$ . We ignore zero terms,  $\sum_{n|m} \mu(n) = {}_k C_0 - {}_k C_1 + {}_k C_2 - {}_k C_3 + \dots + {}_k C_k = (1 - 1)^k = 0$ . □

**Theorem 2.1.**

$$\sum_{n \leq m} \mu(n) \left[ \frac{m}{n} \right] = 1$$

**proof.** By lemma2.1,  $\sum_{m'=1}^m \sum_{n|m'} \mu(n) = 1$ .

$$\sum_{m'=1}^m \sum_{n|m'} \mu(n) = (\mu(1)) + (\mu(1) + \mu(2)) + (\mu(1) + \mu(3)) + (\mu(1) + \mu(2) + \mu(4)) + \dots$$

In this formula, we watch  $\mu(n)$  as a character.  $\mu(1)$  appears  $m$  times.  $\mu(2)$  appears the number of the numbers of multiple of 2 lower than  $m$  that is  $\left[ \frac{m}{2} \right]$  times.  $\mu(3)$  appears the number of the numbers of multiple of 3 lower than  $m$  that is  $\left[ \frac{m}{3} \right]$  times.  $\mu(4)$  appears the number of the numbers of multiple of

4 lower than  $m$  that is  $\lfloor \frac{m}{4} \rfloor$  times. Generally, in this formula,  $\mu(n)$  ( $n \leq m$ ) appears the number of the numbers of multiple of  $n$  lower than  $m$  that is  $\lfloor \frac{m}{n} \rfloor$  times. So we get

$$1 = \sum_{m'=1}^m \sum_{n|m'} \mu(n) = (\mu(1)) + (\mu(1) + \mu(2)) + (\mu(1) + \mu(3)) + (\mu(1) + \mu(2) + \mu(4)) + \dots = \sum_{n \leq m} \mu(n) \lfloor \frac{m}{n} \rfloor$$

□

**Theorem 2.2.**

$$\text{If R.H. is true} \Rightarrow \sum_{n=1}^m \frac{m}{n} \times \mu(n) = O(\sqrt{m} \log m)$$

**proof.**

$$\begin{aligned} \sum_{n=1}^m \frac{m}{n} \times \mu(n) &= \sum_{n=1}^{\lfloor \frac{m}{2} \rfloor} \mu(n) + \sum_{n=\lfloor \frac{m}{2} \rfloor}^m (\frac{m}{n} - \lfloor \frac{m}{n} \rfloor) \mu(n) \\ &+ \sum_{n=1}^{\lfloor \frac{m}{3} \rfloor} \mu(n) + \sum_{n=\lfloor \frac{m}{3} \rfloor}^{\lfloor \frac{m}{2} \rfloor} (\frac{m}{n} - \lfloor \frac{m}{n} \rfloor) \mu(n) + \dots + \sum_{n=1} \mu(n) \\ \sum_{n=1}^{\lfloor \frac{m}{2} \rfloor} \mu(n) + \sum_{n=\lfloor \frac{m}{2} \rfloor}^m (\frac{m}{n} - \lfloor \frac{m}{n} \rfloor) \mu(n) &\approx \sum_{n=1}^m \mu(n) = O(\sqrt{m} \log m) \\ \sum_{n=1}^{\lfloor \frac{m}{3} \rfloor} \mu(n) + \sum_{n=\lfloor \frac{m}{3} \rfloor}^{\lfloor \frac{m}{2} \rfloor} (\frac{m}{n} - \lfloor \frac{m}{n} \rfloor) \mu(n) &\approx \sum_{n=1}^{\lfloor \frac{m}{2} \rfloor} \mu(n) = O(\sqrt{\frac{m}{2}} \log \frac{m}{2}) \end{aligned}$$

Repeat this operation, we get

$$\sum_{n=1}^m \frac{m}{n} \times \mu(n) < O((1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \dots + \frac{1}{\sqrt{m}}) \times \sqrt{m} \log m) = O(\sqrt{m} \log m)$$

□

$\sum_{n=1}^m \mu(n)$  and  $\sum_{n=1}^m (\frac{m}{n} \times \mu(n) - [\frac{m}{n}] \mu(n))$  are term's absolute value is less than 1. So  $\sum_{n=1}^m \mu(n) \approx \sum_{n=1}^m \frac{m}{n} \times \mu(n) - 1$ . This argument is rough. We can not prove this. Next proposition is a "conjecture".

**Conjecture 2.1.**

$$\sum_{n=1}^m \mu(n) \approx \sum_{n=1}^m \frac{m}{n} \times \mu(n)$$

Eventually, we prove "Riemann Hypothesis".

**Theorem 2.3.**

$$\sum_{n \leq m} \mu(n) = O(\sqrt{m} \log(m))$$

**proof.** First, we use induction. We assume "For  $m_0 < M < m$ , the absolute value of  $\sum_{1 \leq n \leq M} \mu(n)$  is less than constant  $K$ -multiple of  $\sqrt{M} \times (\log \sqrt{M} + 1)$ . We take  $m_0$  is big enough. And  $K$  is more than 1.

By Theorem 2.1,

$$\sum_{n \leq \sqrt{m}} \mu(n) \left[ \frac{m}{n} \right] + \sum_{\sqrt{m} < n \leq m} \mu(n) \left[ \frac{m}{n} \right] = 1$$

If we use induction for  $\sqrt{m}$ . Use Theorem 2.2. Next 2 formula is got.

$$\left| \sum_{n \leq \sqrt{m}} \mu(n) \left[ \frac{m}{n} \right] \right| < K \sqrt{m} \times (m^{\frac{1}{4}} \log m^{\frac{1}{4}} + 1) \quad (1)$$

$$\left| \sum_{\sqrt{m} < n \leq m} \mu(n) \left[ \frac{m}{n} \right] \right| < K \sqrt{m} \times (m^{\frac{1}{4}} \log m^{\frac{1}{4}} + 1) - 1 \quad (2)$$

We do not use (1), we use

$$\left| \sum_{n \leq \sqrt{m}} \mu(n) \right| < \sqrt{m}$$

$$\sum_{\sqrt{m} < n \leq m} \mu(n) \left[ \frac{m}{n} \right] = ([\sqrt{m}] - 1) \times \sum_{\sqrt{m} < n \leq m/\sqrt{m}-1} \mu(n) + ([\sqrt{m}] - 2) \times$$

$$\sum_{m/\sqrt{m}-1 < n \leq m/\sqrt{m}-2} \mu(n) + \cdots + 1 \times \sum_{m/2 < n \leq m} \mu(n)$$

We calculate well,  $K \times \sqrt{m}$  is larger than right side all term's order. plus term and minus term exist so we can do this. The element of plus term delete with coefficient either, and the element of minus term delete with coefficient either. But  $(|\sqrt{m}| - 1), (|\sqrt{m}| - 2) \cdots, 1$  must not change. And 1 and -1 use same time. Each time deleting, the difference value move to left side. We can take left side's absolute value goes to near 0. By assumption,  $K \sqrt{m} \times \sqrt{m} > \sqrt{m} \times Km^{\frac{1}{4}}(\log m^{\frac{1}{4}} + 1) - 1 > \left| \sum_{\sqrt{m} < n \leq m} \mu(n) \left[ \frac{m}{n} \right] \right|$ . The most worst case, if left side is plus the middle of more terms are minus the last of more terms are plus. Let first terms are 0. Some delete after, left side's sign is change. Again delete process continue. So all terms less than  $K \times \sqrt{m}$ . Finally, left side large term move to right side. All terms is less than  $K \times \sqrt{m}$ . Left side is 0.

$\left| \sum_{n \leq \sqrt{m}} \mu(n) \right|$  is less than  $\sqrt{m}$ .

First induction is correct by later calculation. The proposition is "For  $m_0 < M \leq m$  (specially  $M = m$ ), the absolute value of  $\sum_{1 \leq n \leq M} \mu(n)$  is less than constant  $K$ -multiple of  $\sqrt{M} \times (\log \sqrt{M} + 1)$ ". Next order's property is important. If  $f(x), g(x)$  are function of  $x$ , then

$$O(f(x) + g(x)) \leq O(\max|f(x)|, |g(x)|)$$

$$([\sqrt{m}] - 1) \times \sum \mu(n) + ([\sqrt{m}] - 2) \times \sum \mu(n) + \cdots + 1 \times \sum \mu(n)$$

This formula is right side of formula that already done to delete. First we calculate

$$([\sqrt{m}] - 1) \times \sum \mu(n)$$

(This term may be 0)  $\sum \mu(n)$  has smaller order than  $K \times \sqrt{m} \times \frac{1}{([\sqrt{m}] - 1)}$

Repeat simillar argument, by  $\log \sqrt{m} \approx \frac{1}{([\sqrt{m}])} + \frac{1}{([\sqrt{m}] - 1)} + \cdots + 1$

$$\left| \sum_{n \leq \sqrt{m}} \mu(n) \right| + \left| \sum_{\sqrt{m} < n \leq m} \mu(n) \right| < K(\sqrt{m} \times (\log \sqrt{m} + 1))$$

(Here, We only calculate one of plus term's sum and minus term's sum) Induction method is proved.

$$\sum_{n \leq m} \mu(n) = \sum_{n \leq \sqrt{m}} \mu(n) + \sum_{\sqrt{m} < n \leq m} \mu(n) = O(\sqrt{m}(\log \sqrt{m} + 1))$$

$$= O(\sqrt{m} \log m)$$

□

### **Riferences**

[1]Riemann's zeta function,H.M.Edwards,Dover Books on Mathematics (28 Mar 2003)

[2]Über die Anzahl der Primzahim unter einer gegebenen Grosse, Riemann. B, Monatsberichie der Berliner Akademie, November 1859.